On the Denseness of Weighted Incomplete Approximations

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ABSTRACT For a given weight function $w(x)$ on an interval $[a,b]$, we study the generalized Weierstrass problem of determining the class of functions $f \in C[a,b]$ that are uniform limits of weighted polynomials of the form $\{w^n(x)p_n(x)\}_1^\infty$, where p_n is a polynomial of degree at most n. For a special class of weights, we show that the problem can be solved by knowing the denseness interval of the alternation points for the associated Chebyshev polynomials.

1 Introduction

In the asymptotic analysis of orthogonal polynomials with respect to an exponential weight of the form $w(x) = \exp(-|x|^\alpha)$, $\alpha > 1$, on $\mathrm{R} = (-\infty, \infty)$, an important step is to determine the class of functions f continuous on R that are uniform limits of weighted polynomials $\{w^n p_n\}$, where $p_n \in \Pi_n$ (the class of polynomials of degree $\leq n$), and the power n of w matches the (maximum) degree of the polynomial. For these so-called Freud weights, this problem was solved by Lubinsky and Saff [9] using techniques from potential theory. The analogous problem for weighted polynomials of the form $\{x^{n\lambda}p_n(x)\}$ on $[0,1]$, which are called *incomplete polynomials*, was raised by G.G. Lorentz and was resolved independently by Saff and Varga [12] and by M. v. Golitschek [3]. Further extensions to Jacobi type weights were obtained by He and Li [6] and He [5].

The above investigations are special cases of the following:

Generalized Weierstrass Problem: Given a closed set $E \subset \mathrm{R}$ and a weight $w : E \to [0,\infty)$, determine necessary and sufficient conditions on f such that f is the uniform limit on E of a sequence of weighted polynomials $\{w^n p_n\}$, $p_n \in \Pi_n$, as $n \to \infty$.

For the case when E is an interval and $w(x) = e^{-Q(x)}$, with $Q(x)$ con-

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vex on E, a plausible solution can be described as follows. From potential theoretic considerations, it is known (cf. [10]), that there exists a unique smallest compact interval S_w such that for every $n \geq 1$ and every $p_n \in \Pi_n$,

$$\|w^n p_n\|_E = \|w^n p_n\|_{S_w},$$

where $\|\cdot\|_A$ denotes the sup norm over the set A. Moreover, if one considers *weighted Chebyshev polynomials* $\hat{T}_n(x) = w^n(x)(x^n + \cdots)$ that are defined by the extremal property

$$\|\hat{T}_n\|_E = \inf_{p_{n-1} \in \Pi_{n-1}} \|w^n(x)(x^n + p_{n-1}(x))\|_E,$$

then the alternation (extreme) points of $\hat{T}_n$ are dense in S_w. Based on the above mentioned special cases, the second author has previously made the following

Conjecture. If $E \subset \mathbf{R}$ is a compact interval and $w(x) = e^{-Q(x)}$, with $Q(x)$ convex on E, then $f \in C(E)$ is the uniform limit on E of a sequence of the form $\{w^n p_n\}_1^\infty$, $p_n \in \Pi_n$, if and only if f vanishes identically on $E \setminus S_w$. (In case E is unbounded, additional assumptions need to be imposed on $Q(x)$ as $|x| \to \infty$, $x \in E$.)

The aim of the present paper is to show that for a special class of weights w, a proof of the above conjecture follows from the denseness property of the alternation points of the weighted Chebyshev polynomials. Thus we avoid much of the "hard analysis" involved with the potential theoretic arguments used in [9]. However, our technique requires strong assumptions on the weight w and so falls short of proving the general conjecture.

2 An Approximation Lemma

Let

$$H_n := \operatorname{span}\{g_0, \ldots, g_n\}, \quad g_i \in C[a,b]$$

be a Chebyshev system on $[a,b]$. Define T_n, the normalized Chebyshev polynomial for H_n on $[a,b]$, by

$$T_n := T_{n,[a,b]} = \sum_{i=0}^{n} c_i g_i,$$

where the c_i are chosen so that $\|T_n\|_{[a,b]} = 1$ and so that T_n has exactly n zeros $x_1 < \cdots < x_n$ in (a,b) and oscillates $n+1$ times between ± 1 on $[a,b]$. So defined, T_n exists and is unique up to multiplication by -1. (See [7, p. 72].) With $x_0 := a$ and $x_{n+1} := b$ we define the *mesh* of T_n by

$$M_n := M_n(T_n : [a,b]) := \max_{1 \leq i \leq n+1} |x_i - x_{i-1}|$$

and the *mesh* of T_n *restricted* to an interval $I := [\alpha, \beta] \subset [a, b]$ by

$$M_n|_I := \max_{k \le i \le j+1} |x_i - x_{i-1}|,$$

where $x_{k-1} := \alpha$, $x_{j+1} := \beta$ and $x_k < \cdots < x_j$ are all the zeros of T_n in (α, β).

Lemma 1. *Assume that*

$$H_n := \operatorname{span}\{g_0, \ldots, g_n\}$$

is a Chebyshev system on $[a, b]$ *with associated Chebyshev polynomials* T_n.

a) *Suppose that each* $g_i \in C^1[\alpha, \beta]$ *and* $H_n' := \operatorname{span}\{g_0', \ldots, g_n'\}$ *is a Chebyshev system on* $[\alpha, \beta] \subset [a, b]$. *If* $f \in C[\alpha, \beta]$, *then there exists* $h_n^* \in H_n^* := \operatorname{span}\{1, g_0, \ldots, g_n\}$ *such that*

$$\|h_n^* - f\|_{[\alpha,\beta]} \le D\omega_f(\sqrt{\delta_n}),$$

where $\delta_n := M_n(T_n : [a, b])|_{[\alpha,\beta]}$. (*Here* D *is a constant that depends only on* a *and* b, *and* ω_f *is the modulus of continuity*).

b) *Suppose further that* $f \in C[\alpha, \beta]$, *that* I *is a closed interval contained in* $[\alpha, \beta]$, *and that* f *is constant on* $[\alpha, \beta] \setminus I$. *Then there exists* $h_n^* \in H_n^*$ *such that*

$$\|h_n^* - f\|_{[\alpha,\beta]} \le D'\omega_f(\sqrt{\delta_n'})$$

where $\delta_n' := M_n(T_n : [a, b])|_I$ *and* D' *depends only on* a *and* b.

Proof. The proof of Lemma 1 follows [1] closely where a similar result is proved for Markov systems, but is reworked for current purposes. Note that H_n' a Chebyshev system on $[\alpha, \beta]$ implies that H_n^* is a Chebyshev system on $[\alpha, \beta]$. Suppose $S_n \in H_n^*$ is the best uniform approximation from H_n^* to F on $[\alpha, c] \cup [d, \beta]$, where

$$F(x) := \begin{cases} 0, & x \in [\alpha, c] \\ 1, & x \in [d, \beta]. \end{cases}$$

Then we claim the following:

A) S_n is monotone on $[c, d]$.

B) $\|S_n - F\|_{[\alpha,c]\cup[d,\beta]} \le 10\delta_n/(d - c)$.

Let $\eta := n+2$ be the size of the Chebyshev system H_n^*. Since S_n is a best approximant to F, there exist $\eta + 1$ points where the maximum error, ϵ_n, occurs with alternating sign. Suppose $m+1$ of these points $y_0 < \cdots < y_m$ lie in $[\alpha, c]$ and $\eta - m$ of these points $y_{m+1} < \cdots < y_\eta$ lie in $[d, \beta]$. Then S_n' has at least $m - 1$ zeros in $[\alpha, c]$ (one at each alternation point in $[\alpha, c]$ except possibly at the endpoints α and c). Likewise S_n' has at least $\eta - m - 2$ zeros

in (d,β). So S_n' has at least $\eta-3$ zeros in $(\alpha,c)\cup(d,\beta)$. Note that this count excludes y_m and y_{m+1}. Thus S_n' has at most one more zero in (α,β) unless S_n' vanishes identically (which is not possible for $\eta>1$). Now suppose S_n' has a zero (with sign change) on (c,d). Then since there is at most one zero of S_n' in (c,d) it cannot be the case that both $y_m=c$ and $y_{m+1}=d$ with both $S_n'(c)\neq 0$ and $S_n'(d)\neq 0$. (Otherwise $\operatorname{sign}(S_n(c)-f(c))=\operatorname{sign}(S_n(d)-f(d))$ as a consideration of the two cases shows.) But if $y_m\neq c$ or $y_{m+1}\neq d$ or $S_n'(c)=0$ or $S_n'(d)=0$, we have accounted for all the zeros of S_n' by accounting for the (possibly) one additional zero (either S_n' vanishes at c or d or one of y_m or y_{m+1} is an interior alternation point where S_n' vanishes). Thus S_n' has no zeros with sign change in (c,d) and claim (A) is proved.

For claim (B) we make the following observation. Let

$$\epsilon_n := \|F-S_n\|_{[\alpha,c]\cup[d,\beta]}.$$

Then

$$D_n := \epsilon_n T_n - S_n$$

has at least m zeros on $[\alpha,c]$ and

$$D_n^* := D_n+1 = 1+\epsilon_n T_n - S_n$$

has at least $\eta-m-1$ zeros on $[d,\beta]$ (counting the possibility of double zeros). Thus D_n' has at least $\eta-3$ zeros on $[\alpha,c]\cup[d,\beta]$. Suppose T_n has at least 4 alternations on an interval $[\delta,\gamma]\subset(c,d)$ and suppose that

$$S_n(\gamma)-S_n(\delta)\leq\epsilon_n.$$

Then, because of part (A) and the oscillation of T_n on $[\delta,\gamma]$,

$$D_n+\frac{S_n(\gamma)+S_n(\delta)}{2}=\epsilon_n T_n-\left[S_n-\frac{S_n(\gamma)+S_n(\delta)}{2}\right]$$

has at least 3 zeros on $[\alpha,\beta]$ and hence

$$D_n'=\left(D_n+\frac{S_n(\gamma)+S_n(\delta)}{2}\right)'$$

has at least 2 zeros on $[\delta,\gamma]$. This, however, gives $D_n'\in H_n'$ a total of at least $\eta-1=n+1$ zeros which is impossible. In particular,

$$S_n(\gamma)-S_n(\delta)>\epsilon_n$$

on any interval $[\delta,\gamma]\subset(c,d)$ where T_n has at least 4 alternations. Thus

$$S_n(d)-S_n(c)\geq\frac{(d-c)}{5\delta_n}\epsilon_n.$$

However, since S_n is a best approximation,

$$S_n(d) - S_n(c) \leq 1 + 2\epsilon_n$$

and we deduce claim (B) on comparing these last two inequalities and noting that $\epsilon_n \leq 1/2$.

The proof of (a) is now a routine argument which for simplicity we present only on the interval $[\alpha, \beta] := [0, 1]$. Let

$$V(x) := \sum_{i=1}^{m-1} \left(f\left(\frac{i+1}{m}\right) - f\left(\frac{i}{m}\right) \right) S_i(x) + f(0),$$

where $S_i(x) \in H_n^*$ is the best approximant to

$$F_i(x) := \begin{cases} 0, & x \in [0, \frac{i}{m}] \\ 1, & x \in [\frac{i+1}{m}, 1] \end{cases}$$

(as in claims (A) and (B)). Then with $(d - c) = 1/m$ we deduce that

$$|V(x) - f(x)| \leq m^2 10 \delta_n \omega_f \left(\frac{1}{m}\right) + \omega_f \left(\frac{1}{m}\right)$$

and with $m := 1/\sqrt{\delta_n}$

$$|V(x) - f(x)| \leq D\omega_f(\sqrt{\delta_n}).$$

The proof of part (b) is an obvious modification of the proof of part (a). ∎

3 Weighted Incomplete Approximants for Special Weights

We restrict our attention to systems of the form

$$H_n := \text{span}\{w^n \cdot 1, w^n x, \ldots, w^n x^n\}$$

where $w := w(x) \geq 0$, $x \in [a, b]$. Then for a large class of weights w we are guaranteed the existence of a support set $\mathcal{S}_w$ where all the zeros of all the associated Chebyshev polynomials lie. Moreover, whenever H_n satisfies the conditions of Lemma 1 we will be able to conclude that $\{H_n\}$ is dense in the continuous functions that vanish off of $\mathcal{S}_w$. Denseness, for such f, in this context means that there exists $f_n \in H_n, \lim_{n\to\infty} f_n = f$. The basic result we need is (essentially) Corollary 2.5 due to Mhaskar and Saff [10].

Theorem 1. *Let $\Sigma := [a,b]$ with a,b possibly infinite. Let $w(x) = e^{-Q(x)}$ where, $Q(x)$ is continuous on $[a,b]$, convex on (a,b) and where $w(x)\cdot|x| \to 0$ as $|x| \to \infty$ (when Σ is unbounded). There exists a smallest compact interval $S_w \subset \Sigma$ with the following properties.*
a) *The Chebyshev polynomials for H_n have all their zeros in S_w.*
b) *The zeros are dense in S_w in the sense that $M_n|_{S_w} \to 0$ as $n \to \infty$.*
c) *If $p_n \in \Pi_n$, then $||w^n p_n||_{S_w} = ||w^n p_n||_\Sigma$, $n = 0,1,\ldots$.*
d) *If $p_n \in \Pi_n$ and A is a compact subset of $\Sigma \setminus S_w$, then*

$$||w^n p_n||_A = o(||w^n p_n||_{S_w}), \quad \text{as} \quad n \to \infty.$$

The interval S_w is known (cf. [10]) to be the support of the unique probability measure μ_w that minimizes the generalized energy integral

$$I[\mu] := \iint \log[|z-t|w(z)w(t)]^{-1} d\mu(z) d\mu(t)$$

over all probability measures supported on $[a,b]$. Moreover, for the case when $Q(x)$ is convex on $[a,b]$, the endpoints of the support set $S_w = [c^*, d^*]$ can be obtained by maximizing the so-called *F-functional*

$$F(c,d) := \log\left(\frac{d-c}{4}\right) - \frac{1}{\pi}\int_c^d \frac{Q(x)dx}{\sqrt{(d-x)(x-c)}},$$

over all pairs (c,d) with $a \le c < d \le b$. This maximum will be attained precisely when $c = c^*$ and $d = d^*$, i.e. at the endpoints of S_w.

Lemma 2. a) *Suppose w satisfies the conditions of Theorem 1 on $[a,b]$ and $S_w^c = [a,b] \setminus S_w$ is nonempty. Let $H_n := \operatorname{span}\{w^n \cdot 1, w^n x, \ldots, w^n x^n\}$. Suppose that*

$$H_n^* := \operatorname{span}\{1, w^n \cdot 1, w^n x, \ldots, w^n x^n\}$$

and

$$H_n' := \operatorname{span}\{(w^n \cdot 1)', (w^n x)', \ldots, (w^n x^n)'\}$$

are both Chebyshev systems on the interval $[a,b]$, for all n. Then for every $f \in C[a,b]$ that vanishes identically on S_w^c (a collection we denote by $C_0[S_w]$), there exists a sequence $p_n \in \Pi_n$, with

$$\lim_{n\to\infty} ||w^n p_n - f||_{[a,b]} = 0.$$

(This is referred to as $\{H_n\}$ being dense in $C_0[S_w]$).

b) *Suppose H_n^* and H_n' are Chebyshev systems on S_w (but not necessarily on $[a,b]$). Suppose the other assumptions of* (a) *hold. Then there exists a sequence $p_n \in \Pi_n$ with*

$$\lim_{n\to\infty} ||w^n p_n - f||_{S_w} = 0$$

and

$$\lim_{n\to\infty} \|w^n p_n - f\|_A = 0,$$

where A is any compact subset of $[a,b] \setminus S_w$.

Proof. By Lemma 1 and Theorem 1, $f \in C_0[S_w]$ is uniformly the limit of elements $h_n^* \in H_n^*$ on $[a,b]$. We now show that f is actually the limit of elements $q_n \in H_n$. If $h_n^* \to f$, $h_n^* \in H_n^*$, then we may write

$$h_n^*(x) := a_n + q_n(x), \quad q_n \in H_n, \quad a_n \in \mathbf{R}.$$

If $|a_n| \to \infty$, then $q_n(x)/a_n \to -1$ uniformly on $[a,b]$ and we may approximate constants from $\{H_n\}$. If $|a_n| \not\to \infty$, then there exists $\{a_{n_k}\}$ with $a_{n_k} \to c \neq \pm\infty$. In this case $\|q_{n_k}\|_{S_w}$ is uniformly bounded and by Theorem 1, part (d), if A is a compact subset of $\Sigma \setminus S_w$, then $\|q_{n_k}\|_A \to 0$. From this and the assumption that $f \equiv 0$ on S_w^c we deduce that $a_{n_k} \to 0$ and we are done. ∎

We wish now to record classes of weights which satisfy the conditions of Lemma 2, part (a), because for these weights we can conclude that the weighted incomplete approximants are dense exactly in $C_0[S_w]$.

Lemma 3. *Suppose $w \in C^\infty[a,b]$, $w(x) \geq 0$. If $\operatorname{span}\{1, w^n \cdot 1, \ldots, w^n x^m\}$ is a Chebyshev system for all positive integers n and m, then $\operatorname{span}\{(w^n 1)', \ldots, (w^n x^m)'\}$ is also a Chebyshev system.*

Proof. See [7, p. 378].

Lemma 4. *Suppose either*
a) *$1/w$ is totally monotone on $[a,b]$ or*
b) *$1/w(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$, $a_n \geq 0$, is convergent on $[a,b]$, where $a \geq 0$.*
Then w satisfies the conditions of Lemma 3.

Proof. To show that w satisfies the conditions of Lemma 2 it suffices to show that a non-vanishing linear form

$$L_m(x) := \frac{1}{w^n(x)} - \sum_{i=0}^{m} b_i x^i$$

has at most $m+1$ zeros. This follows, in both cases, on differentiating $m+1$ times and observing that $(L_m(x))^{(m+1)}$ has no sign changes in $[a,b]$. ∎

This gives us the following result.

Theorem 2. *Suppose w satisfies the conditions of Theorem 1 and that either*
a) *w^{-1} is totally monotone on $[a,b]$ or*
b) *w^{-1} has a power series expansion at a, convergent on $[a,b]$, with non-negative coefficients or (equivalently to (b))*

b') w^{-1} *has all derivatives strictly positive on* $(a,b]$.

Then $\{w^n p_n\}$, $p_n \in \Pi_n$, *is dense in* $C_0[S_w]$.

Observe that weights of the following form work on any interval $[a,b] \subset [0,\infty)$.

a) $\exp(-x^p)$, p a positive integer;
b) x^θ , $\theta > 0$;
c) $\exp(-x^\delta)$, $\delta \in (0,1)$.

For (c) above, the convexity condition of Theorem 1, doesn't hold. However in the case $\Sigma := [a,b] \subset [0,\infty)$, we can replace the convexity of $Q = \log(1/w)$ by the condition that $xQ'(x)$ is strictly increasing on (a,b).

We remark that for the generalized Weierstrass problem von Golitschek, Lorentz, and Makovoz (cf. [4]) have simultaneously but independently obtained results similar to Lemma 2.

4 The Sublinear and Superlinear Cases

For a sequence of positive numbers $\{\lambda_n\}_1^\infty$, we consider weighted spaces

$$H_n(w,\lambda_n) := \{w^{n\lambda_n} p_n \mid p_n \in \Pi_n\}.$$

We expect the following to happen for "decent" nonconstant weights. If $\lambda_n \to \infty$, the approximation should be impossible. If $\lambda_n \to 0$, then the whole interval becomes the interval of approximation. If $\lambda_n \to c > 0$, then the approximants should live on the set S_{w^c} associated with w^c. We prove the following.

Theorem 3. *Assume* $w \in C[a,b]$, $w \geq 0$ *and* w *is nonconstant on* $[a,b]$. *Suppose that* $\lambda_n \to \infty$ *as* $n \to \infty$. *If there exist* $w^{n\lambda_n} p_n \in H_n(w,\lambda_n)$ *such that* $w^{n\lambda_n} p_n \to f$ *as* $n \to \infty$ *uniformly on* $[a,b]$, *then* $f \equiv 0$.

Proof. Suppose $w^{n\lambda_n} p_n \in H_n(w,\lambda_n)$ converges uniformly to $f > 0$ on $[\alpha,\beta] \subset [a,b]$. Since w is not constant on $[\alpha,\beta]$, there exist intervals I_1 and I_2 contained in $[\alpha,\beta]$ with

$$0 < c_1 \leq w(x) \leq c_2, \quad x \in I_1,$$
$$0 < c_3 \leq w(x) \leq c_4 < c_1, \quad x \in I_2,$$

for some positive constants c_1, c_2, c_3, c_4. Now from the convergence of $w^{n\lambda_n} p_n$ to a strictly positive limit on $[\alpha,\beta]$ we deduce the existence of positive constants d_1, d_2, d_3, d_4 so that

$$\frac{d_2}{c_2^{n\lambda_n}} \leq \|p_n(x)\|_{I_1} \leq \frac{d_1}{c_1^{n\lambda_n}}, \quad n \geq N_1 \tag{4.1}$$

and

$$\frac{d_4}{c_4^{n\lambda_n}} \leq \|p_n(x)\|_{I_2} \leq \frac{d_3}{c_3^{n\lambda_n}}, \quad n \geq N_2. \tag{4.2}$$

From (3.1) and Bernstein's inequality we have

$$\|p_n(x)\|_{[a,b]} \leq \frac{d_5^n}{c_1^{n\lambda_n}}, \quad n \geq N_1, \tag{4.3}$$

for some constant d_5. However with (3.2) and the facts that $\lambda_n \to \infty$ and $c_4 < c_1$ this leads to the contradiction that

$$\|p_n(x)\|_{[a,b]} \leq \frac{d_5^n}{c_1^{n\lambda_n}} < \frac{d_4}{c_4^{n\lambda_n}} \leq \|p_n(x)\|_{I_2}$$

for some large n. ∎

Theorem 4. *Suppose that for n large w^{λ_n} as well as w satisfy the conditions of Lemma 2.*
a) *If $\lim_{n\to\infty} \lambda_n = \theta > 0$, then $\{w^{n\lambda_n} p_n\}$ is dense in $C_0[S^\theta]$, where S^θ is the support associated with w^θ.*
b) *If $\lim_{n\to\infty} \lambda_n = 0$, then $\{1, w^{n\lambda_n} p_n\}$ is dense in $C[a,b]$.*

Proof. Part (a) requires knowing that S_w only depends on the n^{th} root asymptotic (cf. [11]) and the rest follows as before.

For part (b) we show that the zeros of the Chebyshev polynomials fill out $[a,b]$ and apply Lemma 1. For this purpose consider functions of the form

$$F_n(x) := \left[w^{(m+k)\lambda_n}(x)(b-x)^m(x-a)^k\right]^{n/(m+k)},$$

where the integer $m+k$ divides n. Observe that since $(F_n)^{(m+k)/n}$ converges to $(b-x)^m(x-a)^k$ uniformly, F_n behaves like a δ–function. In particular, given $I \subset [a,b]$ it is possible to construct $q_n \in \{w^{n\lambda_n} p_n\}$, $\forall\ n \geq N_\epsilon$ so that

$$\|q_n(x)\|_{[a,b]-I} \leq \epsilon, \quad \max_{x\in I} q_n(x) \geq 2 \quad \text{and}$$

$$\min_{x\in I} q_n(x) \leq -2.$$

In fact, such q_n can be constructed having many oscillations of magnitude ≥ 2. It now follows, that for $n \geq N_\epsilon$ the Chebyshev polynomial, T_n, for $\{w^{n\lambda_n} p_n\}$ has a zero in I; otherwise T_n would have too many zeros. Thus, $M_n|_{[a,b]} \to 0$ and we can apply Lemma 1, to get denseness. ∎

5 Remarks

1. The condition in Lemma 2 that H_n' be Chebyshev can be weakened to the following condition (as is apparent from the proof of Lemma 1).

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