More Ramanujan-type Series for $1/\pi$

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Abstract: We present various classes of rapidly convergent power series for $1/\pi$. This allows us to give all of Ramanujan's mysterious series for $1/\pi$ and to produce some interesting additional examples. Many of these additional examples add more than 10 or 20 digits accuracy per term.

1. **Introduction:** In [7, §13] Ramanujan sketches the derivation of 3 remarkable series for $1/\pi$. In §14, with essentially no explanation, he gives 14 more remarkable series. Hardy [3], quoting Mordell, observes that "it is unfortunate that Ramanujan has not developed in detail the corresponding theories." In [1] we constructed seven general classes of hypergeometric-like power series for $1/\pi$. In each case the power is an invariant from elliptic function theory and the coefficients involve similar quantities. In particular, we recovered all of Ramanujan's series. In this note we concentrate on the three forms which prove the most flexible.

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We begin by listing some additional examples. First,

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} (-1)^n \frac{(6n)!}{(n!)^3 (3n)!} \frac{13591409 + n545140134}{(640320^3)^{n+1/2}},$$

which arises with $N := 163$ in TYPE 3a of the tables of our final section. (This series seems to have been first observed by the Chudnovskys.) Second, with $N := 427$ in TYPE 3c of the tables we have

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!}{(n!)^3 (3n)!} \frac{(A + nB)}{C^{n+1/2}}$$

where

$$A := 212175710912\sqrt{61} + 1657145277365$$

$$B := 13773980892672\sqrt{61} + 107578229802750$$

$$C := [5280\,(236674 + 30303\sqrt{61})]^3 .$$

This series adds roughly twenty-five digits per term, $\sqrt{C}/(12A)$ already agrees with pi to twenty-five places. Surprisingly, one also has

$$\frac{1}{\pi} = 7\cdot 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!}{(n!)^3 (3n)!} \frac{\bar{A} + n\bar{B}}{\bar{C}^{\,n+1/2}}$$

where $\bar{A}$, $\bar{B}$, and $\bar{C}$ are the conjugate quadratic numbers

$$\bar{A} := 212175710912\sqrt{61} - 1657145277365$$

$$\bar{B} := 13773980892672\sqrt{61} - 107578229802750$$

$$\bar{C} := [5280\,(236674 - 30303\sqrt{61})]^3 .$$

In this case convergence is much slower - less than one digit per term.

The most recent record setting calculations of digits of pi all rely on methods that trace their genesis to related material. Details of the theory and of the calculations of Gosper, Bailey, Tamura and Kanada, and Kanada may be found in [1].

2. **Preliminary Results**: The complete elliptic integrals of the first and second kind may be defined in terms of hypergeometric functions by

$$K(k) := \frac{\pi}{2} \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \tag{2.1}$$

and

$$E(k) := \frac{\pi}{2} \cdot {}_2F_1\left(\frac{1}{2}, \frac{-1}{2}; 1; k^2\right), \tag{2.2}$$

for $0 \leqslant k < 1$. The complementary modulus is the quantity

$$k' := \sqrt{1-k^2} \text{ and we write } K'(k) := K(k'), E'(k) := E(k).$$

These are related by the differential equation

$$E = k'^2 K + kk'^2 \frac{dK}{dk} \tag{2.3}$$

They are also linked by the beautiful Legendre relation ([1], [11])

$$E'K + K'E - K'K = \pi/2 \tag{2.4}$$

We will use the following invariants employed by Ramanujan [7],

$$G := (2kk')^{-1/12}, \quad g := (2k/k')^{-1/12}$$

and (2.5)

$$2^{1/4} gG = (k^2/2k')^{-1/12}.$$

In Weber's terms [10] $2^{1/4}G = f$, $2^{1/4}g = f_1$. We also need Klein's absolute invariant J which is expressible as

$$J := \frac{(4G^{24}-1)^3}{27G^{24}} = \frac{(4g^{24}+1)^3}{27g^{24}} \tag{2.6}$$

With these invariants one can obtain the following hypergeometric equations

$$\left[\frac{2K}{\pi}(k)\right]^2 = (1+k^2)^{-1}\,{}_3F_2\left(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;\left[\frac{g^{12}+g^{-12}}{2}\right]^{-2}\right),$$
$$= (k'^2-k^2)^{-1}\,{}_3F_2\left(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;-\left[\frac{G^{12}-G^{-12}}{2}\right]^{-2}\right)$$

and (2.7)

$$\left[\frac{2K}{\pi}(k)\right]^2 = (1-(kk')^2)^{-1/2}\,{}_3F_2\left(\frac{1}{6},\frac{5}{6},\frac{1}{2};1,1;J^{-1}\right).$$

Finally, we will need the rising factorial $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$.

3. Series Identities of Ramanujan-type: The singular value function may be defined as the solution of

$$\frac{K'}{K}(k(N)) = \sqrt{N} \tag{3.1}$$

for positive real N. This uniquely defines k on $(0,\infty)$ as a decreasing function with $k(0) = \infty$, $k(1) = 1/\sqrt{2}$, $k(\infty) = 0$. Importantly, $k(N)$ is algebraic when N is rational [1]. Some values are given below (§4). Moreover, for some $k = k(N)$ one or more of our invariants becomes very simple. In terms of theta functions, $k(N) = (\theta_2(q)/\theta_3(q))^2$ with $q = e^{-\pi\sqrt{N}}$.

In [1] we introduced the function α (a singular value of the second kind). It connects elliptic integrals of the first and second kinds and is intimately related to Ramanujan's algebraic approximations to pi. It is defined by

$$\alpha(N) = \frac{E'}{K} - \frac{\pi}{4K^2} \qquad (k := k(N)) \tag{3.2}$$

and also is algebraic at rational values with $\alpha(1) = \frac{1}{2}$, $\alpha(\infty) = \frac{1}{\pi}$.

Additionally, α satisfies recursions which allow one to compute it at many values both numerically and explicitly. The simplest recursion is,

$$\alpha(4N) = \frac{4\alpha(N) - 2\sqrt{N}\,k^2(N)}{[1 + k'(N)]^2}\ , \tag{3.3}$$

and

$$k(4N) = \frac{1 - k'(N)}{1 + k'(N)}\ . \tag{3.4}$$

This is equivalent to the Gauss-Salamin-Brent iteration for pi. The recursion and its extensions lead to explicit high-order iterations for $1/\pi$ and to the recent record breaking computations of pi, [1]. Values of α are also given below (§4). The construction of α shows that

$$\frac{1}{\pi} = \sqrt{N}\ kk'^2\,\frac{4K\dot{K}}{\pi^2} + (\alpha(N) - \sqrt{N}\,k^2)\,\frac{4K^2}{\pi^2} \qquad (k := k(N))\ . \tag{3.5}$$

(Here and below the dot signifies differentiation.) This follows from using Legendre's identity (2.4) to write a one-sided Legendre identity

$$\alpha(N) = \pi/(4K^2) - \sqrt{N}(E/K - 1)$$

and then using (2.3) to replace E by the derivative $\dot{K}$. Similarly,

$$\frac{1}{K} = \sqrt{N}\ kk'^2\,\frac{4\dot{K}}{\pi} + (\alpha(N) - \sqrt{N}\ k^2)\,\frac{4K}{\pi} \qquad (k := k(N))\ . \tag{3.6}$$

Given $\alpha(N)$ and $k(N)$ we can combine (3.5) with (2.7) to produce power series for $1/\pi$ as follows. In each case we have $(\frac{2K}{\pi})^2(k) = m(k)F(\phi(k))$ for algebraic m and ϕ where $F(\phi)$ has a hypergeometric power-series expansion $\sum_{n=0}^{\infty} a_n \phi^n$.

Now

$$\frac{4K\dot{K}}{\pi^2} = \frac{1}{2}\dot{m}F + \frac{1}{2}m\dot{\phi}\,\dot{F}(\phi)$$

and substitution in (3.5) leads to

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} a_n \left[\frac{\sqrt{N}}{2} kk'^2 \dot{m} + (\alpha(N) - \sqrt{N}k^2)m + n\frac{\sqrt{N}}{2} m \frac{\dot{\phi}}{\phi} kk'^2 a\right]\phi^n . \qquad (3.7)$$

Note that, for each rational N, the bracketed term is of the form $A + nB$ with A and B algebraic. We now make explicit these considerations for the three invariants which give the most remarkable and elegant special cases.

1. Series in $x_N := \left[\frac{g_N^{12} + g_N^{-12}}{2}\right]^{-1} = \frac{4k(N)k'^2(N)}{[1+k^2(N)]^2}$: For $N > 2$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{2}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3} d_n(N) x_N^{2n+1} \qquad (3.8)$$

where

$$d_n(N) := \left[\frac{\alpha(N)x_N^{-1}}{1+k^2(N)} - \frac{\sqrt{N}}{4} g_N^{-12}\right] + n\sqrt{N}\left[\frac{g_N^{12} - g_N^{-12}}{2}\right] .$$

2. Series in $y_N := \left[\frac{G_N^{12}-G_N^{-12}}{2}\right]^{-1} = \frac{4k(N)k'(N)}{1-[2k(N)k'(N)]^2}$: For $N \geqslant 4$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{4}\right)_n \left(\frac{2}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3} \; e_n(N) y_N^{2n+1} \tag{3.9}$$

where

$$e_n(N) := \left[\frac{\alpha(N) y_N^{-1}}{k'^2(N)-k^2(N)} + \frac{\sqrt{N}}{2} k^2(N) G_N^{12}\right] + n\sqrt{N} \left[\frac{G_N^{12}+G_N^{-12}}{2}\right].$$

3. Series in $J_N^{-1} := \frac{27G_N^{24}}{(4G_N^{24}-1)^3} = \frac{27g_N^{24}}{(4g_N^{24}+1)^3}$: For $N > 1$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{3}{6}\right)_n \left(\frac{5}{6}\right)_n}{(n!)^3} f_n(N) \left(J_N^{-1/2}\right)^{2n+1} \tag{3.10}$$

where

$$f_n(N) := \frac{1}{3\sqrt{3}} \left[\sqrt{N} \sqrt{1-G_N^{-24}} + 2(\alpha(N) - \sqrt{N}\, k^2(N))(4G_N^{24}-1)\right]$$

$$+ n\sqrt{N}\, \frac{2}{3\sqrt{3}} \left[(8G_N^{24}+1) \sqrt{1-G_N^{-24}}\right].$$

There are many equivalent rearrangements of the formulae for $d_n(N)$, $e_n(N)$, $f_n(N)$. (See [1] especially page 186.)

4. Specific Examples: We begin by listing values of $\alpha(N)$ and of $k(N)$ (or equivalently G_N^{-12} or g_N^{-12} whichever is simpler).

N	$2k(N)k'(N) = G_N^{-12}$	$\alpha(N)$
1	1	$1/2$
9	$(2-\sqrt{3})^2$	$(3-3^{3/4}\sqrt{2}(\sqrt{3}-1))/2$
13	$5\sqrt{13}-18$	$(\sqrt{13}-\sqrt{74\sqrt{13}-258})/2$
27	$(2^{1/3}-1)^4/2$	$3(\sqrt{3}+1-2^{4/3})/2$
37	$(\sqrt{37}-6)^3$	$(\sqrt{37}-(171-25\sqrt{37})(\sqrt{37}-6)^{1/2})/2$

N	$2k(N)/k'(N)^2 = g_N^{-12}$	$\alpha(N)$
2	1	$(\sqrt{2}-1)$
10	$(\sqrt{5}-2)^2$	$(7+2\sqrt{5})(\sqrt{10}-3)(\sqrt{2}-1)^2$
18	$(\sqrt{3}-\sqrt{2})^4$	$3(\sqrt{3}+\sqrt{2})^4(\sqrt{6}-1)^2(7\sqrt{2}-5-2\sqrt{6})$
22	$(\sqrt{2}-1)^6$	$(\sqrt{2}+1)^6(33-17\sqrt{2})(3\sqrt{22}-7-5\sqrt{2})$
58	$\left(\frac{\sqrt{29}-5}{2}\right)^6$	$\left(\frac{\sqrt{29}+5}{2}\right)^6(99\sqrt{29}-444)(99\sqrt{2}-70-13\sqrt{29})$

k(N) Many other values of G_N , g_N may be found in [7], [10] or [1]. Certain values of k(N) are given in [12]. The computation of k(N) is discussed in [1], [8] and [13]. Many values of $\alpha(N)$ are derived in [1]. For $N := 2,3,4,5,7$ they are given in [12]. From information like that in these tables and the formulae given for $1/\pi$, we may explicitly compute all but two of Ramanujan's series. These two which rely on another ${}_3F_2$ are treated in [1].

Ramanujan gives series of form (3.8) for $N := 6, 10, 18, 22, 58$ and of form (3.9) for $N := 5, 9, 13, 25, 37$. He gives series of form (3.10) for $N := 3, 7$. In each case manipulation of the formulae yields the desired result. In fact $\alpha(37)$ and $\alpha(58)$ were calculated by obtaining $d_0(58)$ and $e_0(37)$ to high precision numerically and then solving for α . Given the algebraic nature of α this ultimately suffices to verify the values. In these cases, we have

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{4}\right)_n \left(\frac{2}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3} \left[\frac{1123 + n21460}{4}\right] \left(\frac{1}{882}\right)^{2n+1} \tag{4.1}$$

using (3.9) for $N := 37$; and using (3.8) for $N := 58$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{2}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3} \left[2\sqrt{2}\,(1103 + n26390)\right] \left(\frac{1}{99^2}\right)^{2n+1} \tag{4.2}$$

Series (4.2) is the most rapid and most celebrated of the series given by Ramanujan. It is the series with which Gosper performed his record breaking computation of more than 17,000,000 terms of the continued fraction for π in 1985.

Since $k^2(N)$ behaves like $16e^{-\pi\sqrt{N}}$ [1] it is very easy to estimate the number of digits added in each series. For N at all large, the convergence while linear is most impressive. Not surprisingly, Ramanujan has given most of the special cases of (3.8) and (3.9) for which the power is rational. We add some quadratic examples which come in conjugate pairs from invariants G_N (respectively g_N) corresponding to discriminants with one form per genus for which N is of the form PQ or P/Q (respectively 2PQ and 2P/Q) with P and Q prime. (See [1, page 293].) In these cases the invariant is a product of two algebraic units and so x_N or y_N is a real quadratic irrational. These examples are listed as TYPE 1 and TYPE 2 in the next section.

In a similar fashion we may apply (3.10) for rational or quadratic values of J_N. Ramanujan gives two of the four series for which J_N is rational and positive (N := 3 , 7) and in [1] we produced the two others (N := 2, 4). There are also eight negative rational values of J. In our terms they come from $(\sqrt{N} - i)/2$ for N := 3, 7, 11, 19, 27 ,43, 67, 163. These correspond (27 excepted) to the seven imaginary quadratic fields $Q(\sqrt{-N})$ of class number 1, with N congruent to 3 mod 4. These 7 give rise to the series listed as TYPE 3a. While we stated (3.10) for positive numbers, it continues to hold more generally by analytic continuation - as long as $J < -1$. The real part of the identity gives a series for $1/\pi$; the imaginary part an obscure formula for zero. Note that now the underlying q variable becomes $ie^{-\sqrt{N}\pi/2}$ rather than $e^{-\sqrt{N}\pi}$.

There are also many quadratic values of J both positive and negative. They again give rise to conjugate pairs of series. These are listed as TYPE 3b and TYPE 3c. It should be noted that each case the J value is a perfect cube, while the X and Y values are perfect squares. Granted the knowledge that the quantities A and B corresponding to the conjugate invariants are conjugate, the easiest way to determine their precise values is to compute the underlying q-series expansions from the information in Chapter 5 of [1] and to match off the rational coefficients. This is how the quadratic series were determined. In similar fashion, one can determine the corresponding series for many other values of the invariants: both quadratic and higher order.

Finally we should mention the interesting log series for pi derived in [5] and [8], and the more abstract approach to Ramanujan's approximations described by the Chudnovskys in their contribution to this volume.

TYPE 1

$$\frac{1}{\pi} = \Delta \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{2}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3} \frac{(A + nB)}{X^{2n+1}}$$

Here $\Delta := 1$ if the signs are "+" and $\Delta :=$ (smallest odd prime factor of N) if the signs are "−".

	(Approximate) digits correct per term (+ signs)	(− signs)
$N := 42 = 14 \cdot 3$ $A := 186\sqrt{2} \pm 151\sqrt{3}$ $B := 3780\sqrt{2} \pm 3080\sqrt{3}$ $X := 825 \pm 336\sqrt{6}$	6	<1

$N := 78 := 26\cdot 3$ $A := \sqrt{3}(4302\sqrt{2} \pm 12161/2)$ $B := \sqrt{3}(119340\sqrt{2} \pm 168740)$ $X := 33099 \pm 23400\sqrt{2}$	9	2
$N := 70 = 14\cdot 5$ $A := \sqrt{7}(1356\sqrt{2} \pm 1715/2\sqrt{5})$ $B := \sqrt{7}(35640\sqrt{2} \pm 22540\sqrt{5})$ $X := 15939 \pm 5040\sqrt{10}$	8	<1
$N := 130 := 26\cdot 5$ $A := \sqrt{2}(117046\sqrt{13} \pm 188730\sqrt{5})$ $B := \sqrt{2}(4192540\sqrt{13} \pm 6760260\sqrt{5})$ $X := 1874961 \pm 232560\sqrt{65}$	12	1
$N := 190 = 38\cdot 5$ $A := \sqrt{19}(11552301/2 \pm 4084354\sqrt{2})$ $B := \sqrt{19}(250129620 \pm 176868340\sqrt{2})$ $X := 79097931 \pm 55930680\sqrt{2}$	15	1

TYPE 2

$$\frac{1}{\pi} = \Delta \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{4}\right)_n \left(\frac{2}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3} \frac{(A + nB)}{Y^{2n+1}}$$

Here $\Delta := 1$ if the signs are "+" and $\Delta :=$ (smallest odd prime factor of N) if the signs are "-".

	(Approximate) digits correct per term	
	(+ signs)	(- signs)
$N := 177 = 59\cdot 3$ $A := 1781017/2\sqrt{177} \pm 47389527/4$ $B := 37219780\sqrt{177} \pm 495176085$ $Y := 21488850\sqrt{3} \pm 4845594\sqrt{59}$	15	4

N :=253 = 23·11 A := $212750712\sqrt{11} \pm 2822457127/4$ B := $10631172240\sqrt{11} \pm 35259609385$ Y := $2216752650 + 668376072\sqrt{11}$	19	(DIVERGES)

TYPE 3a (J<0, RATIONAL)

$$\frac{1}{\pi} = \frac{1}{\sqrt{-1728\,J}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{3}{6}\right)_n \left(\frac{5}{6}\right)_n}{(n!)^3} \frac{(A + nB)}{J^n}$$

	(Approximate) digits correct per term
N := 7 A := 24 B := 189 J := - 125/64	≤1
N := 11 A := 60 B := 616 J := - 512/27	1
N := 19 A := 300 B := 4104 J := -512	3
N := 27 A := 1116 B := 18216 J := -64000/9	4

N := 43 A := 9468 B := 195048 J := -512000	6
N := 67 A := 122124 B := 3140424 J := -85184000	8
N := 163 A := 163096908 B := 6541681608 J := -151931373056000	15

TYPE 3b (J>0, QUADRATIC)

$$\frac{1}{\pi} = \frac{\Delta}{\sqrt{3J}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{3}{6}\right)_n \left(\frac{5}{6}\right)_n}{(n!)^3} \frac{(A+nB)}{J^n}$$

Here $\Delta := 1$ if the signs are "+" and $\Delta := 2$ if the signs are "−".

	(Approximate) digits correct per term (+ signs)	(− signs)
$N := 6 = 3 \cdot 2$ $A := 15 \pm 10\sqrt{2}$ $B := 228 \pm 156\sqrt{2}$ $J := 1399 \pm 988\sqrt{2}$	3	<1
$N := 10 = 5 \cdot 2$ $A := 62\sqrt{5} \pm 135$ $B := 1224\sqrt{5} \pm 2700$ $J := 123175 \pm 55080\sqrt{5}$	5	1

N := 22 = 11·2 A := 16659 ± 11750 $\sqrt{2}$ B := 490644 ± 346500 $\sqrt{2}$ J := 1821424375 ± 1287940500 $\sqrt{2}$	9	3
N := 58 = 29·2 A := 30282810 $\sqrt{29}$ ± 163073763 B := 1449063000 $\sqrt{29}$ ± 7803343548 J := 174979733174158375 ± 32492920723263000 $\sqrt{29}$	17	7

TYPE 3c (J<0, QUADRATIC)

$$\frac{1}{\pi} = \frac{\Delta}{2\sqrt{-3J}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{3}{6}\right)_n \left(\frac{5}{6}\right)_n}{(n!)^3} \frac{(A + nB)}{J^n}$$

Here $\Delta := 1$ if the signs are "+" and $\Delta :=$ (smallest prime factor of N) if the signs are "–".

	(Approximate) digits correct per term	
	(+ signs)	(- signs)
N := 235 = 47·5 A := 380527125 ± 170176896 $\sqrt{5}$ B := 18326073150 ± 8195668992 $\sqrt{5}$ J := – (238187910720320000 ± 106520871957857280 $\sqrt{5}$)	17	1
N := 267 := 89·3 A := 197238000 $\sqrt{89}$ ± 1860739157 B := 10125024000 $\sqrt{89}$ ± 95519278302 J := – (569533907814800000 ± 60370473487542400 $\sqrt{89}$)	19	4
N := 427 = 61·7 A := 212175710912 $\sqrt{61}$ ± 1657145277365 B := 13773980892672 $\sqrt{61}$ ± 107578229802750 J := – (4517203562651557847168000 ± 578368650183667447104000 $\sqrt{61}$)	25	1

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