EXPLICIT ALGEBRAIC NTH ORDER APPROXIMATIONS TO PI

J. M. Borwein and P. B. Borwein

Dalhousie University

ABSTRACT

We present a family of algorithms for computing pi which converge with order m (m any integer larger than one). Details are given for two, three and seven.

INTRODUCTION

In the course of a general study of elliptic integral transforms and their applications in the construction of good algebraic approximations to transcendental functions and natural constants [2], the authors discovered the following general multiplication formula which gives algebraic approximations of order m to π (for m any integer greater than 1). The formula is constructed as follows. Let

$$K(k) := \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2}} \tag{1.1}$$

and

$$E(k) := \int_0^{\pi/2} \sqrt{1 - k^2\sin^2} \tag{1.2}$$

denote the complete elliptic integrals of first and second kind respectively, for $0 \le k \le 1$. For each integer m there is an

S. P. Singh et al. (eds.), Approximation Theory and Spline Functions, 247–256.

integral polynomial in two variables u and v, ϕ_n, called the modular equation (of order m) and a rational function, M_m, of u and v, called the multiplier such that for any u in $]0,1[$

$$K(u^4) = mM_m(u,v)K(v^4) \tag{1.3}$$

whenever v is the (unique) solution in $]0,u[$ to $\Phi_m(u,v) = 0$, [5]. When $u^8 + v^8 = 1$, $u := u(m)$ and $v := v(m)$ are said to be conjugate. Let $K(m)$ and $E(m)$ denote $K(v^4(m))$ and $E(v^4(m))$. In [2] the authors showed that there is a computable algebraic constant, $\alpha(m)$, such that

$$\frac{\pi}{4} = K^2(m)[\sqrt{m}\,(\frac{E(m)}{K(m)} - 1) + \alpha(m)] \tag{1.4}$$

and

$$0 \le \pi - \alpha(m)^{-1} = O(10^{-\sqrt{m}}). \tag{1.5}$$

In fact $\alpha(m)^{-1}$ converges monotonically [2] to π. Heuristically this goes as follows. As m tends to infinity $\lambda(m) := v^4(m)$ tends to zero and so $K(m)$ tends to $\pi/2$. Moreover, $E(m)/K(m)$ decreases to one sufficiently fast to validate (1.5). When $m = 1$, $\alpha(m) = 1/2$ and (1.4) is Legendre's identity [5]. The multiplication formula, which allows one to compute π rapidly, is now able to be stated. Let p be any positive integer. Then for integral m

$$\alpha(p^2m) = p^2M_p^2\,\alpha(m) + p\sqrt{m}\,[\frac{v(1-v^8)}{4M_p}\frac{dM_p}{dv} + v^8 - pM_p^2u^8] \tag{1.6}$$

where $u := v(m)$ and v is the unique solution to $\Phi_p(v(m),v) = 0$ in $]0,v(m)[$. Also $\frac{dM_p}{dv}$ is the complete differential of M_p with respect to v. To compute this quantity it helps to know Jacobi's identity

$$\frac{du}{dv} = \frac{u(1-u^8)}{v(1-v^8)}\,pM_p^2 \tag{1.7}$$

whenever $\Phi_p(u,v) = 0$ and $0 \le v \le u$. For convenience we denote v as $T_p(u)$, and let $k := u^4$, $\lambda := v^4$.

Algebraic details for p a prime less than twenty are given in [6]. The general theory is nicely laid out in [5].

A GENERAL ITERATION

By iterating (1.5) we are led to the following algorithm. Let m be integral and let

$$\text{(i)}\quad \alpha_0 := \alpha(m), \quad v_0 := v(m). \tag{2.1}$$

For n in $\mathbb{N}$ we compute

$$\text{(ii)}\quad v_{n+1} := T_p(v_n) \tag{2.2}$$

$$\text{(iii)}\quad s_n := pM_p^2(v_n, v_{n+1}) \tag{2.3}$$

$$\text{(iv)}\quad d_n := \frac{v_{n+1}}{4}\,\frac{(1-v_{n+1}^8)}{M_p(v_n,v_{n+1})}\,\frac{dM_p(v_n,v_{n+1})}{dv} \tag{2.4}$$

and have

$$\alpha_{n+1} := ps_n\alpha_n + p^{n+1}\sqrt{m}\,[d_n + v_{n+1}^8 - s_n v_n^8]. \tag{2.5}$$

Moreover,

$$\alpha_n^{-1} - \pi = 0(10^{-p^n\sqrt{m}}). \tag{2.6}$$

The larger m is, the better the initial approximation. This is illustrated below. For small m we have the following initial values [2].

Starting Values

m	$v^4(m) = \lambda(m)$	$\alpha(m)$
1	$2^{-1/2}$	$1/2$
2	$\sqrt{2} - 1$	$\sqrt{2} - 1$
3	$\sqrt{2}(\sqrt{3}-1)/4$	$(\sqrt{3}-1)/2$
4	$3 - 2\sqrt{2}$	$6 - 4\sqrt{2}$
5	$(\sqrt{\sqrt{5} - 1} - \sqrt{3 - \sqrt{5}})/2$	$(\sqrt{5} - \sqrt{2(\sqrt{5}-1)})/2$
7	$\sqrt{2}(3-\sqrt{7})/8$	$(\sqrt{7}-2)/2$

Other values are computed in [2]. We now specialize our algorithm for $m = 2, 3, 7$. The specializations are remarkably clean. Note also that (1.7) allows one to calculate (2.2), (2.3) and (2.4) as soon as

Φ_p is known since $\dfrac{du}{dv} = -\dfrac{\partial\Phi_p}{\partial v} / \dfrac{\partial\phi_p}{\partial u}$.

The Quadratic Case

In this case the multiplier is $M_2 := \frac{1+\lambda}{2}$ and the transformation T_2 is given by $\lambda := (1 - \sqrt{1-k^2})/(1 + \sqrt{1-k^2})$. The iteration becomes

$$\text{(i)} \quad x_{n+1} := (1 - \sqrt{1-x_n^2})/(1 + \sqrt{1-x_n^2}) \tag{2.1}$$

$$\text{(ii)} \quad \alpha_{n+1} := (1+x_{n+1})^2\alpha_n - 2^{n+1}\sqrt{m}\, x_{n+1}, \tag{2.2}$$

with $\alpha_0 := \alpha(m)$ and $x_0 := \lambda(m)$; and

$$\alpha_n^{-1} - \pi = 0(10^{-2^n\sqrt{m}}).$$

A more exact asymptotic is given in [2]. The first few iterations behave as follows:

Digits Correct in Quadratic Algorithms

	n=1	2	3	4	5	6	7	8
m = 1	0	3	8	19	41	84	171	344
m = 2	2	5	13	28	56	120	242	> 400
m = 7	5	12	26	55	112	227	> 400	

If one replaces x_n by c_n/a_n (where $a_{n+1} := (a_n+b_n)/2$; $b_{n+1} := \sqrt{a_n b_n}$; $c_n := \sqrt{a_n^2 - b_n^2}$ is the AGM iteration [4], [7]) then we may replace (2.2) by

$$a_{n+1}^2 \alpha_{n+1} = a_n^2 \alpha_n - 2^{n-1} \sqrt{m}\, c_n^2 , \tag{2.3}$$

which on summing yields

$$\pi = \frac{\lim a_{n+1}^2}{\alpha(m) - \sqrt{m} \sum_{n=0}^{\infty} 2^{n-1} c_n^2} . \tag{2.4}$$

When m=1 this is an identity known to Gauss [3] which forms the basis for the Salamin-Brent [4], [7] algorithm recently used by Tamura and Kanada to compute 2^{24} digits of π, ([8] and private communication). There is some advantage to (2.2) over (2.4) in that all root extractions in the former are of numbers converging rapidly to one.

THE CUBIC CASE

The modular equation is $u^4 - v^4 + 2uv(1-u^2v^2) = 0$. It is convenient, though, to use a form of the modular equation given in Cayley [5] which uses an auxiliary variable t. We have

$$M_3 := \frac{2t+1}{3} \; ; \; t := v^3/u$$

$$u^8 := t \left(\frac{t+2}{2t+1}\right)^3 ; v^8 := t^3 \left(\frac{t+2}{2t+1}\right) .$$

As in [1] we can explicitly compute the v_n. The algorithm becomes:

$$\text{(i)} \quad v_{n+1} := v_n^3 - \sqrt{v_n^6 + \sqrt[3]{4v_n^2(1-v_n^8)}} + v_{n-1} \qquad (3.1)$$

$$\text{(ii)} \quad t_n := \frac{v_{n+1}^3}{v_n} \qquad (3.2)$$

$$\text{(iii)} \quad \alpha_{n+1} := (2t_n+1)^2\alpha_n - 2\sqrt{m}\, 3^n(t_n+2)t_n, \qquad (3.3)$$

with $\alpha_0 := \alpha(m)$, $v_0 := v(m)$, and v_1 calculated from the quartic formula as in [1]. When $m = 1$, we have
$v_1 := ((\frac{1-\sqrt{3}}{\sqrt{2}} + 3^{1/4})\, 2^{-7/8}$. Then

$$\alpha_n^{-1} - \pi = O(10^{-3^n\sqrt{m}}).$$

Alternately, we can give (3.1) and (3.2) in terms of t_n directly. We get

$$t_n := \left(\frac{t_{n-1}+2}{2t_{n-1}+1}\right) [(t_{n-1}+1) - \sqrt{t_{n-1}^2 + (1+t_{n-1})\sqrt[3]{\frac{4(1-t_{n-1})(2t_{n-1}+1)}{(t_{n-1}+2)^2}}}\,]^3 .$$

For practical purposes it seems better to directly invert the modular equation. The first few iterations give:

Digits Correct in Cubic Algorithms

	n=1	2	3	4	5
m=1	2	10	34	107	327
m=7	8	30	93	288	873

In both the quadratic and cubic cases it is easy to directly establish the error estimate.

The Septic Case. The modular equation is

$(1-u^8)(1-v^8) = (1-uv)^8$. We use (1.7) and

$$7M_7^2 = v(u-v^7)/u(u^7-v)$$
$$= b/a$$

where $b := uv/(u^8 - uv)$; $a := uv/(uv-v^8)$ to derive the following algorithm.

(i) Generate (v_n) decreasingly from

$$(1-v_n^8)(1-v_{n+1}^8) = (1-v_n v_{n+1})^8 \quad (4.1)$$

(ii) $$a_n := v_n v_{n+1}/(v_n v_{n+1}-v_{n+1}^8);\ b_n := v_n v_{n+1}/(v_n^8-v_n v_{n+1}) \quad (4.2)$$

(iii) $s_n := b_n/a_n$

(iv) $$t_n := 1/8[(1-v_{n+1}^8)(49a_n-b_n) + (1-v_n^8)(s_n-1)b_n] \quad (4.3)$$

(v) $$\alpha_{n+1} := s_n\alpha_n + 7^n\sqrt{m}\,(7-s_n-t_n) \quad (4.4)$$

with $\alpha_0 := \alpha(m)$, $v_0 := v(m)$ as before. Then

$$\alpha_n^{-1} - \pi = 0(10^{-7^n\sqrt{m}}) .$$

The first few iterations are as follows:

Digits Correct in Septic Algorithms

	n=1	2	3
m=1	7	63	464
m=7	22	173	>1000

We finish by observing that while the rate of convergence improves as p increases the complexity remains unchanged [1] [3]. Also, it is possible, using the data given in [6], to write down an explicit iteration for p any odd number less than twenty. The case p = 5 can be handled almost as cleanly as 3 or 7.

REFERENCES

1. Borwein, J. M., and Borwein, P. B., "Cubic and higher order algorithms for π", to appear, Canad. Math. Bull.

2. Borwein, J. M., and Borwein, P. B., "Elliptic integral and approximation of π'. Preprint.

3. Borwein, J. M., and Borwein, P. B., "The arithmetic-geometric mean and fast computation of elementary functions", SIAM Review, 26, 1984.

4. Brent, R. P., "Fast multiple-precision evaluation of elementary functions", J. Assoc. Comput. Mach. 23, 1976, pp. 242-251.

5. Cayley, A., "An elementary treatise on elliptic functions", Bell and Sons, 1895, republished Dover 1961.

6. Cayley, A., "A memoir on the transformation of elliptic functions", Phil. Trans. T., 164, 1874, pp. 397-456.

7. Salamin, E., "Computation of π using arithmetic-geometric mean", Math. Comput. 135, 1976, pp. 565-570.

8. Tamura, Y., and Kanada, Y., "Calculation of π to 4,196,293 decimals based on Gauss-Legendre algorithm", preprint.

Research partially supported by NSERC Grants.

THREE ALGORITHMS FOR π

QUADRATIC: with $\alpha_0 := \alpha(m)$ $x_0 := \lambda(m)$; and

(i) $x_{n+1} := (1 - \sqrt{1-x_n^2})/(1 + \sqrt{1-x_n^2})$

(ii) $\alpha_{n+1} := (1+x_{n+1})^2\alpha_n - \sqrt{m}\, 2^{n+1}x_{n+1}$,

$$\alpha_n^{-1} - \pi = O(10^{-2^n\sqrt{m}}).$$

CUBIC: with $\alpha_0 := \alpha(m)$, $v_0 := v(m)$; and

(i) generate (v_n) decreasingly from

$$v_{n+1}^4 + 2v_nv_{n+1} = v_n^4 + 2(v_nv_{n+1})^3$$

(ii) $t_n := v_{n+1}^3/v_n$

(iii) $\alpha_{n+1} := (2t_n+1)^2\alpha_n - 2\sqrt{m}\, 3^n(t_n+2)t_n$,

$$\alpha_n^{-1} - \pi = O(10^{-3^n\sqrt{m}}).$$

SEPTIC: with $\alpha_0 := \alpha(m)$, $v_0 := v(m)$; and

(i) generate (v_n) decreasingly from

$$(1-v_n^8)(1-v_{n+1}^8) = (1-v_nv_{n+1})^8$$

(ii) $a_n := v_nv_{n+1}/(v_nv_{n+1}-v_{n+1}^8)$; $b_n := v_nv_{n+1}/(v_n^8-v_nv_{n+1})$

(iii) $s_n := b_n/a_n$

(iv) $t_n := 1/8[(1-v_{n+1}^8)(49a_n-b_n) + (1-v_n^8)(s_n-1)b_n]$

(v) $\alpha_{n+1} := s_n\alpha_n - \sqrt{m}\, 7^n(s_n+t_n-7)$

$$\alpha_n^{-1} - \pi = O(10^{-7^n\sqrt{m}}).$$

In cubic algorithms: (i) may be replaced by

$$v_{n+1} := v_n^3 - \sqrt{v_n^6 + \sqrt[3]{4v_n^2(1-v_n^8)}} + v_{n-1}$$

once v_1 is known. When $m = 1$

$$v_1 := \left(\left(\frac{1-\sqrt{3}}{\sqrt{2}}\right) + 3^{1/4}\right)2^{-7/8}.$$

Selected Starting Values

m	$v^4(m) = \lambda(m)$	$\alpha(m)$
1	$2^{-1/2}$	$1/2$
2	$\sqrt{2} - 1$	$\sqrt{2} - 1$
3	$\sqrt{2}(\sqrt{3}-1)/4$	$(\sqrt{3}-1)/2$
5	$(\sqrt{\sqrt{5}-1} - \sqrt{3-\sqrt{5}})/2$	$(\sqrt{5} - \sqrt{2(\sqrt{5}-1)})/2$
7	$\sqrt{2}(3-\sqrt{7})/8$	$(\sqrt{7}-2)/2$