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THE RELATIONSHIP BETWEEN THE ZEROS OF BEST APPROXIMATIONS AND DIFFERENTIABILITY

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ABSTRACT. We examine the relationship between the analytic properties of continuous functions on $[-1,1]$ and the location of the roots of the sequence of best polynomial approximations. We show that if the approximants have no zeros in a certain ellipse then the function being approximated must be analytic in this ellipse. We also show that the rate at which the zeros of the nth approximant tend to the interval $[-1,1]$ determines the global differentiability of the function under consideration.

1. Introduction. The theorems of Bernstein and Jackson establish an exact relationship between the rate of convergence of the sequence of best polynomial approximants to a function $f \in C[-1,1]$ and the global differentiability of f on $[-1,1]$ (see [**1** or **3**]). We intend to show that such a relationship also holds between the location of the zeros of the sequence of best approximants and the global differentiability of f.

Let $C[-1,1]$ be the set of continuous functions on $[-1,1]$. Let π_n denote the collection of algebraic polynomials of degree at most n with real coefficients. For $f \in C[-1,1]$ let

$$P_n^*(f) := \min_{p_n \in \pi_n} \|f - p_n\|_{[-1,1]} \tag{1}$$

where $\|\cdot\|_{[a,b]}$ denotes the supremum norm on $[a,b]$. Let $P_n(f)$ be the best uniform approximation to f from π_n. Then $P_n(f)$ is, of course, the unique polynomial of degree at most n which attains the minimum in (1).

Let E_ρ, $\rho \geq 1$, be the open ellipse (in the complex plane $\mathbf{C}$) with foci at ± 1 and with axes $\frac{1}{2}(\rho \pm \rho^{-1})$.

We will prove the following theorems:

THEOREM 1. *Let N be an integer. Suppose $f \in C[-1,1]$ and, for all $n \geq N$, $P_n(f)$ has no zeros in E_ρ. Then f is analytic in E_ρ (that is, f is the restriction to $[-1,1]$ of a function analytic in E_ρ).*

THEOREM 2. *Let N be an integer. Suppose $f \in C[-1,1]$ and there exist integers k and N and a $\delta \in (0,1)$ so that, for each $n \geq N$, $P_n(f)$ has no zeros in $E\rho_n$, where $\rho_n = n^{(1+k+\delta)/n}$. Then, for any $\varepsilon > 0$, f is k times continuously differentiable on $[-1+\varepsilon, 1-\varepsilon]$ and $f^{(k)}$ satisfies a Lipschitz condition of order δ on $[-1+\varepsilon, 1-\varepsilon]$.*

Theorem 1 is an analogue of the observation that if a formal power series $\sum_{n=0}^{\infty} a_n z^n$ has the property that, for each N, $\sum_{n=0}^{N} a_n z^n$ has no zeros in

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$C_\rho = \{|z| < \rho\}$, then $\sum_{n=0}^{\infty} a_n z^n$ is convergent in C_ρ. This is straightforward. We can write, for $a_N \neq 0$,

$$\sum_{n=0}^{N} a_n z^n = a_N \prod_{i=1}^{N} (z - z_i), \qquad |z_i| > \rho,$$

and thus,

$$|a_N| \prod_{i=1}^{N} |z_i| = |a_0|.$$

In particular,

$$|a_N|^{1/N} \leq |a_0|^{1/N}/\rho,$$

from whence the observation follows.

Jentzsch's Theorem says that the zeros of the partial sums of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ are dense in $\{|z| = \rho\}$, where ρ is the radius of convergence of f. Walsh [**5**] offers the following analogue of Jentzsch's Theorem.

THEOREM 3. *If E_ρ is the maximal ellipse of analyticity for f, then the zeros of $P_n(f)$ are dense in the boundary of E_ρ.*

If we had assumed, in Theorem 1, that f was analytic on $[-1, 1]$, instead of just continuous, then we could have deduced the result from this analogue of Jentzsch's Theorem. Professor Gončar, in private communication, informs me that he can prove the following interesting generalization: If there exists a domain D, $D \cap [-1, 1] \neq \emptyset$, so that no $P_n(f)$ has any zero in D, then f is analytic in D. This generalization and other extensions of Theorem 1 to general compact sets were discussed by H. P. Blatt and E. B. Saff at the Tampa Conference on Rational Approximation and Interpolation in December 1983. Related problems for Padé approximants are considered by Gončar in [**2**].

2. An inequality for polynomials with no roots in E_ρ. The proofs of Theorems 1 and 2 are consequences of the following inequality:

INEQUALITY 1. Suppose $p_n(x) = \sum_{k=0}^{n} a_k x^k \in \pi_n$ and p_n has no zeros in E_ρ. Then

$$|a_n| \leq (2^n/\rho^n)\|p_n\|_{[-1,1]}.$$

PROOF. Let

$$q_{2n}(z) = z^n p_n((z + z^{-1})/2).$$

We note that $q_{2n} \in \pi_{2n}$ has lead coefficient $a_n/2^n$ and that if z_0 is a root of q_{2n} then so are $1/z_0$ and $\bar{z}_0$. Since $w = (z + z^{-1})/2$ maps $\mathbf{C} - \{z|1/\rho < |z| < \rho\}$ into $\mathbf{C} - E_\rho$, it follows that

$$q_{2n}(z) = \frac{a_n}{2^n} \prod_{i=1}^{n} (z - \alpha_i) \prod_{i=1}^{n} \left(z - \frac{1}{\alpha_i}\right),$$

where $|\alpha_i| \geq \rho$ for each i and where the α_i come in conjugate pairs. Thus, for $|z| = 1$,

$$\begin{aligned} |q_{2n}(z)| &= \left| \frac{z^n a_n}{2^n} \prod_{i=1}^{n} (z - \alpha_i) \prod_{i=1}^{n} \frac{(\alpha_i - 1/z)}{\alpha_i} \right| \\ &= \frac{|a_n|}{2^n} \frac{|\prod_{i=1}^{n}(z - \alpha_i)\prod_{i=1}^{n}(\bar{z} - \alpha_i)|}{|\prod_{i=1}^{n} \alpha_i|} \\ &= \frac{|a_n|}{2^n} \frac{|\prod_{i=1}^{n}(z - \alpha_i)|^2}{|\prod_{i=1}^{n} \alpha_i|}. \end{aligned}$$

By the maximum modulus principle

$$\left\| \prod_{i=1}^{n} (z - \alpha_i) \right\|_{\{|z|=1\}} \geq \left| \prod_{i=1}^{n} \alpha_i \right|,$$

and, hence,

$$\|p_n\|_{[-1,1]} = \|q_{2n}\|_{\{|z|=1\}} \geq \frac{|a_n|}{2^n} \left| \prod_{i=1}^{n} \alpha_i \right| \geq |a_n| \frac{\rho^n}{2^n}. \qquad \square$$

There exists $p_n \in \pi_n$, $p_n = 2^n x^n + \cdots$, so that, in the variable $x = (z + z^{-1})/2$,

$$p_n\left(\frac{z + z^{-1}}{2}\right) = (z^n - \rho^n)(z^{-n} - \rho^n)/(-\rho^n).$$

For this polynomial $a_n = 2^n$ and

$$\|p_n\|_{[-1,1]} \leq \|z^n - \rho^n\|^2_{\{|z|=1\}}/\rho^n \leq \rho^n + 2 + \rho^{-n}$$

or

$$|a_n| \geq (2^n/(\rho^n + 2 + \rho^{-n}))\|p_n\|_{[-1,1]}.$$

In particular, Inequality 1 is asymptotically best possible.

Minor modifications to the proof of Inequality 1 yield:

INEQUALITY 2. Suppose $p_n(x) = \sum_{k=0}^{n} a_n x^k \in \pi_n$ and p_n has k or fewer zeros in E_ρ. Then

$$|a_n| \leq (2^n/\rho^{n-k})\|p_n\|_{[-1,1]}.$$

PROOF OF THEOREMS 1 AND 2. Let $n \geq N$ and let a_n be the lead coefficient of $P_n(f)$. From Inequality 1 and the assumptions on the roots of p_n we have

$$|a_n| \leq (2^n/\rho^n)\|P_n(f)\|_{[-1,1]} \leq (2^n/\rho^n)M, \tag{2}$$

where $M = 2\|f\|_{[-1,1]}$. It follows that there exists $S_{n-1} \in \pi_{n-1}$ so that

$$\|S_{n-1} - P_n(f)\|_{[-1,1]} \leq 2M/\rho^n. \tag{3}$$

One need only set $P_n(f) - S_{n-1} = a_n T_n/2^{n-1}$, where T_n is the nth Čebyčev polynomial (see [3, p. 31]). Thus,

$$\begin{aligned} P_n^*(f) &\leq \|f - S_n\|_{[-1,1]} \leq \|f - P_{n+1}\|_{[-1,1]} + \|S_n - P_{n+1}\|_{[-1,1]} \\ &\leq P_{n+1}^*(f) + 2M/\rho^{n+1} \end{aligned} \tag{4}$$

and

$$P_n^*(f) \le 2M \sum_{m=1}^{\infty} \frac{1}{\rho^{n+m}} \le \frac{C}{\rho^n}, \tag{5}$$

where C is independent of n. This, by Bernstein's characterization of analytic functions in terms of the rate of convergence of P_n^*, guarantees that f is analytic in E_ρ. (See, for example, [**3**, p. 76].)

The proof of Theorem 2 proceeds along similar lines. Equations (2)–(5) become

$$|a_n| \le 2^n M/n^{1+k+\delta}, \tag{2'}$$

$$\|S_{n-1} - P_n(f)\|_{[-1,1]} \le 2M/n^{1+k+\delta}, \tag{3'}$$

$$P_n^*(f) \le P_{n+1}^*(f) + 2M/n^{1+k+\delta}, \tag{4'}$$

and

$$P_n^*(f) \le C'/n^{k+\delta}. \tag{5'}$$

Equation (5′), once again by Bernstein's results [**3**, p. 61], guarantees that f has k continuous derivatives on $[-1+\varepsilon, 1-\varepsilon]$ and $f^{(k)} \in \mathrm{Lip}_\delta$ on $[-1+\varepsilon, 1-\varepsilon]$. □

If we use Inequality 2 instead of Inequality 1 in the above proofs, we can deduce that both Theorems 1 and 2 hold under weaker assumptions. For example, we need only assume that the number of zeros of $P_n(f)$ in E_ρ (or E_{ρ_n}) is $o(n)$.

3. How sharp are Theorems 1 and 2? If f is analytic and nonzero in E_ρ then the sequence $\{P_n\}$ converges uniformly to f on compact subsets of E_ρ [**3**, p. 76]. Hence, for $\rho' < \rho$ and for n sufficiently large, the zeros of P_n will lie outside $E_{\rho'}$. In particular, Theorem 1 can be used to characterize the largest ellipse in which f is analytic.

For nonanalytic f we have the following example: Let k be a positive integer and let

$$f(x) := 5 + \sum_{m=1}^{\infty} (3^{-m})^k T_{3^m}(x),$$

where T_i is the ith Čebyčev polynomial on $[-1, 1]$. Then, for $3^h \le n < 3^{h+1}$,

$$P_n(f) = 5 + \sum_{m=1}^{h} (3^{-m})^k T_{3^m}(x).$$

(See [**1**, p. 132].) Also,

$$\|T_m\|_{E_\rho} \le \rho^m.$$

(See [**3**, p. 42].) For h sufficiently large, if $\rho < 3^{kh/3^h}$ then

$$\left\| \sum_{m=1}^{h} (3^{-m})^k T_{3^m} \right\|_{E_\rho} \le \sum_{m=1}^{h} \left(\frac{3^{3^{(m-h)}h}}{3^m} \right)^k < 5.$$

Thus, for n sufficiently large, $P_n(f)$ has no zeros in E_ρ, where $\rho = n^{k/n}$. However, by examining $f(\cos\theta)$ one can verify that f is not k times continuously differentiable on $[-1+\varepsilon, 1-\varepsilon]$.

The previous example leaves a gap of $1 + \delta$ between the assumptions on k in Theorem 2 and the "best possible" assumptions.

The results of this paper are quintessentially results about best approximants. Given $f \in C[-1, 1]$ and any compact sets K in the complex plane that separates $[-1, 1]$ from infinity, it is always possible to find a sequence of polynomials with all roots in K that converges to f. (See, for example, [**4**].)

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