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A Remarkable Cubic Mean Iteration

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1. Introduction

Consider the two term iteration defined by

$$a_{n+1} := \frac{a_n + 2b_n}{3}, \qquad a_0 := a, \tag{1.1}$$

and

$$b_{n+1} := \sqrt[3]{b_n\left(\frac{a_n^2 + a_n b_n + b_n^2}{3}\right)}, \qquad b_0 := b. \tag{1.2}$$

Then since

$$a_{n+1}^3 - b_{n+1}^3 = \frac{(a_n - b_n)^3}{27}, \tag{1.3}$$

it follows that, for $a, b \in (0, \infty)$, and for $n \geq 1$,

$$|a_{n+1} - b_{n+1}| \leq \frac{|a_n - b_n|}{27}$$

and

$$F(a, b) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \tag{1.4}$$

is well defined, and that on compact subsets of $(0, \infty)$ the convergence is cubic. It is also easy to see that $F(1, z)$ is analytic in some complex neighbourhood of 1. All of this is a straightforward exercise. What is less predictable is that we can identify the limit function explicitly, and that it is a non-algebraic hypergeometric function. Thus, it is one of a very few such examples; and it is certainly the simplest cubic example we know. The most familiar quadratic example is the arithmetic-geometric mean iteration of Gauss and Legendre. Namely the iteration

$$\begin{aligned} a_{n+1} &:= \frac{a_n + b_n}{2}, & a_0 &:= a, \\ b_{n+1} &:= \sqrt{a_n b_n}, & b_0 &:= x, \end{aligned}$$

where, for $0 < x < 1$ $a := 1$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{1}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x^2\right)}.$$

For a discussion of this and a few other examples see [2] and [3].

2. The main theorem

The point of this note is to provide a self-contained proof of the closed form of the limit of (1.1) and (1.2). This is the content of the next theorem.

Theorem 1. *Let* $0 < x < 1$. *Let*

$$\begin{aligned} a_{n+1} &:= \frac{a_n + 2b_n}{3} & a_0 &:= 1 \\ b_{n+1} &:= \sqrt[3]{\frac{b_n(a_n^2 + a_n b_n + b_n^2)}{3}} & b_0 &:= x. \end{aligned}$$

Then the common limit, $F(1, x)$, *is*

$$\frac{1}{F(1,x)} = \sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3\, 3^{3n}} \left(1 - x^3\right)^n = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - x^3\right).$$

Proof. The limit function $F(a, b)$ must satisfy

$$F(a_0, b_0) = F(a_1, b_1) = \cdots \tag{2.1}$$

and since the iteration is positively homogeneous so is F. In particular

$$F(a_0, b_0) = F(a_1, b_1) = F\left(\frac{a_0 + 2b_0}{3}, \sqrt[3]{\frac{b_0(a_0^2 + a_0 b_0 + b_0^2)}{3}}\right) \tag{2.2}$$

or

$$\begin{aligned} F(1, x) &= F\left(\frac{1 + 2x}{3}, \sqrt[3]{\frac{x(1 + x + x^2)}{3}}\right) \\ &= \frac{1 + 2x}{3} F\left(1, \sqrt[3]{\frac{9x(1 + x + x^2)}{(1 + 2x)^3}}\right). \end{aligned} \tag{2.3}$$

If we set $H(x) := \sqrt{x(1-x)}/F(1,(1-x)^{\frac{1}{3}})$ then the functional equation (2.3) becomes

$$H(x) = \sqrt{\frac{3}{t'(x)}}\, H(t(x)), \tag{2.4}$$

where

$$t(x) := 1 - \frac{9x^*(1+x^*+x^{*2})}{(1+2x^*)^3}, \qquad x^* := (1-x)^{\frac{1}{3}}. \tag{2.5}$$

Furthermore $\sqrt{x}H(x)$ is analytic at 0. The point of the proof is to show that

$$G(x) := \sqrt{x(1-x)}\ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x\right) \tag{2.6}$$

also satisfies the functional equation (2.4). From this it is easy to deduce that $G(x) = H(x)$; as follows from the functional equation for H/G, and the value at $x = 1$. The (hypergeometric) differential equation satisfied by G is

$$a(x) := \frac{G''(x)}{G(x)} = \left(\frac{-8x^2+8x-9}{36x^2(1-x)^2}\right). \tag{2.7}$$

Now it is a calculation (for details see [2]) that

$$G^*(x) := \sqrt{\frac{3}{t'(x)}}G(t(x)) \tag{2.8}$$

also satisfies (2.7) exactly when

$$a(x) = (t'(x))^2 a(t(x)) - \frac{1}{2}\left[\frac{t'''(x)}{t(x)} - \frac{3}{2}\left(\frac{t''(x)}{t'(x)}\right)^2\right]. \tag{2.9}$$

It is now another calculation, albeit a fairly tedious one, that a and t defined by (2.7) and (2.5) satisfy (2.9). We have now deduced that $G^*(x)$ and $G(x)$ both satisfy (2.7). Furthermore, since the roots of the indicial equation of (2.7) are $(1/2, 1/2)$ there is a fundamental logarithmic solution. Since both G^* and G are asymptotic to $\sqrt{x}$ at 0, they are in fact equal. Thus (2.8) shows that G satisfies (2.4). This finishes the proof. ∎

As a consequence we derive the following particularly beautiful cubic hypergeometric transformation.

Corollary 1. *For* $x \in (0,1)$

$${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-x^3\right) = \frac{3}{1+2x}\ {}_2F_1\left(\frac{1}{3};\frac{2}{3};1,\left(\frac{1-x}{1+2x}\right)^3\right).$$

Proof. This is just a rewriting of the functional equation (2.3). ∎

The above verification entirely obscures our discovery of Theorem 1. This arose from an examination of some quadratic modular equations of Ramanujan [1, Chapter 21]. Notably, Ramanujan observed that,

$$(1-u^3)(1-v^3)=(1-uv)^3 \tag{2.10}$$

is a quadratic modular equation, for ${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\cdot\right)$. We then observed, with the aid of considerable symbolic computation, that if

$$L(q):=\sum_{-\infty}^{\infty} q^{m^2+mn+n^2} \tag{2.11}$$

and

$$R(q):=\frac{3L(q^3)}{2L(q)}-\frac{1}{2} \tag{2.12}$$

then

$$u:=u(q):=R(q) \qquad \text{and} \qquad v:=v(q):=R(q^2)$$

solve (2.10) parametrically. From (2.12) it is natural to examine the cubic modular equation for R. This leads to the following result.

Theorem 2. *Let*

$$L(q):=\sum_{-\infty}^{\infty} q^{m^2+mn+n^2}$$

and

$$M(q):=\frac{3L(q^3)-L(q)}{2}.$$

Then, L and M parameterize the mean iteration of (1.1) and (1.2) in the sense that if $a:=L(q)$ and $b:=M(q)$, then

$$L(q^3):=\frac{a+2b}{3}$$

and

$$M(q^3)=\sqrt[3]{\frac{b(a^2+ba+b^2)}{3}}$$

and the limit function F (of Theorem 1) satisfies

$$F\left(1,\frac{M(q)}{L(q)}\right)=\frac{1}{L(q)}\,.$$

The derivation of this, which requires some modular function theory, will be discussed elsewhere [3].

References

[1] B.C. Berndt, *Ramanujan's Notebooks Part II*, Springer-Verlag, New York, 1989.

[2] J.M. Borwein and P.B. Borwein, *Pi and the AGM — A Study in Analytic Number Theory and Computational Complexity*, Wiley N.Y., New York, 1987.

[3] J.M. Borwein and P.B. Borwein, *A Cubic Counterpart of Jacobi's Identity and the AGM*, Trans. A.M.S., to appear.

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