MATHEMATICS 152 98-2 Solutions for Assignment 13

6. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$, so $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}$ for $|x| < 1$.

Multiplying by x^2, $\frac{x^2}{1-x^2} = \sum_{n=0}^{\infty} x^{2n+2} = \sum_{n=1}^{\infty} x^{2n}$ for $|x| < 1$.

Adding, $f(x) = \frac{1+x^2}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} + \sum_{n=1}^{\infty} x^{2n} = 1 + \sum_{n=1}^{\infty} 2x^{2n}$ for $|x| < 1$.

10. $f(x) = \frac{x}{x^2 - 3x + 2} = \frac{-1}{x-1} + \frac{2}{x-2} = \frac{1}{1-x} - \frac{1}{1-\frac{x}{2}} = \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=1}^{\infty} \left(1 - \frac{1}{2^n}\right) x^n$

for $|x| < 1$. (The $n = 0$ terms in the two series cancel.)

The series expansion for $\frac{1}{1-x}$ is valid for $|x| < 1$ and the series expansion for

$\frac{1}{1-\frac{x}{2}}$ is valid for $|x| < 2$. The common portion of these two intervals satisfies $|x| < 1$.

14. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$, so $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$ for $|x| < 1$.

Integrating, $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$ for $|x| < 1$.

Multiplying by x, $f(x) = x\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+2} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1} x^n$ for $|x| < 1$.

In fact, this series converges (conditionally) at $x = 1$, representing $\ln 2$ correctly.
It diverges however at $x = -1$, and of course there is no $\ln 0$ for it to try to represent!

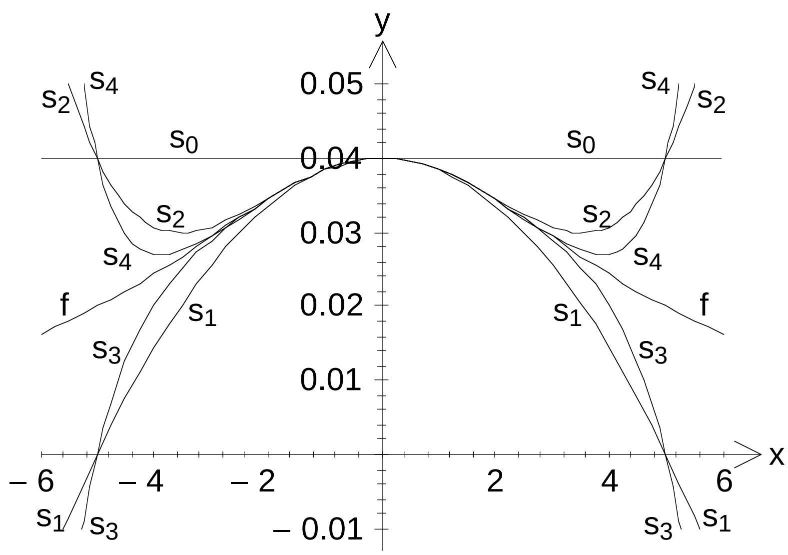
20. $f(x) = \frac{1}{x^2 + 25} = \frac{1}{25} \cdot \frac{1}{1 - \left(-\frac{x^2}{25}\right)} = \frac{1}{25} \sum_{n=0}^{\infty} \left(-\frac{x^2}{25}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{25^{n+1}} x^{2n}$ for $\left|-\frac{x^2}{25}\right| < 1$,

or in other words for $|x| < 5$. Notice that $s_0(x) = \frac{1}{25}$, $s_1(x) = \frac{1}{25} - \frac{1}{625}x^2$,

$s_2(x) = \frac{1}{25} - \frac{1}{625}x^2 + \frac{1}{15625}x^4$, $s_3(x) = \frac{1}{25} - \frac{1}{625}x^2 + \frac{1}{15625}x^4 - \frac{1}{390625}x^6$, and

$s_4(x) = \frac{1}{25} - \frac{1}{625}x^2 + \frac{1}{15625}x^4 - \frac{1}{390625}x^6 + \frac{1}{9765625}x^8$. Here $s_n(x) = T_{2n}(x)$.

On the next page is a *Maple* plot of $y = f(x)$, $y = s_0(x)$, $y = s_1(x)$, $y = s_2(x)$, $y = s_3(x)$, and $y = s_4(x)$, all on the interval $-6 \le x \le 6$, but with visible y values limited to $-0.01 \le y \le 0.04$. In the open interval $-5 < x < 5$, the higher-order Taylor polynomials do a better job of approximating the function $f(x)$, but at $x = \pm 5$ the error $R_k(\pm 5)$ is always exactly 0.02 in absolute value, and for $|x| > 5$ the higher-order Taylor polynomials do a worse job of approximating $f(x)$.



For Exercise 20

24. $\dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ for $|x|<1$.

Integrating, $\tan^{-1}(x) = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{2n+1}x^{2n+1}$ for $|x|<1$.

Replacing x by x^2, $\tan^{-1}(x^2) = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{2n+1}(x^2)^{2n+1} = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{2n+1}x^{4n+2}$ for $|x|<1$.

Integrating, $\int \tan^{-1}(x^2)\,dx = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)(4n+3)}x^{4n+3}$ for $|x|<1$.

As a matter of fact, the series given here for $\tan^{-1}(x)$, $\tan^{-1}(x^2)$, and $\int \tan^{-1}(x^2)\,dx$ all converge at the endpoints $x=\pm 1$ also, and correctly represent the functions there. In particular, the one for $\tan^{-1}(x)$ can be used to approximate $\pi/4 = \tan^{-1}1$, although the convergence is very slow.

26. By Exercise 24, $\int_0^{0.5}\tan^{-1}(x^2)\,dx = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)(4n+3)}\left(\dfrac{1}{2}\right)^{4n+3}$.

This is an alternating series. Successive partial sums are
$s_0(0.5) \approx 0.041666667$, $s_1(0.5) \approx 0.041294643$, $s_2(0.5) \approx 0.041303521$,
$s_3(0.5) \approx 0.041303230$, and $s_4(0.5) \approx 0.041303241$.

To six decimal places, $\int_0^{0.5}\tan^{-1}(x^2)\,dx \approx 0.041303$.

Given the command

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> evalf(int(arctan(x^2),x=0..0.5));
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Maple provides the response

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0.0413032408.
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2. $f(x) = \sin(2x)$ so $f^{(2n)}(x) = (-1)^n 2^{2n}\sin(2x)$ and $f^{(2n+1)}(x) = (-1)^n 2^{2n+1}\cos(2x)$.
$f^{(2n)}(0) = 0$, $f^{(2n+1)}(0) = (-1)^n 2^{2n+1}$, and the Maclaurin series for $f(x)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1}$.

If $u_n = \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1}$ then $\left|\frac{u_{n+1}}{u_n}\right| = \frac{4x^2}{(2n+2)(2n+3)} \to 0$ as $n \to \infty$ for all real x.

The radius of convergence is $R = \infty$.

4. $f(x) = \frac{x}{1-x} = -1 + \frac{1}{1-x}$ so $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$ and $f^{(n)}(0) = n!$ for $n \ge 1$, but $f^{(0)}(0) = 0$.

The Maclaurin series for $f(x)$ is thus $\sum_{n=1}^{\infty} x^n$.

If $u_n = x^n$ then $\left|\frac{u_{n+1}}{u_n}\right| = |x|$ so the series has radius of convergence $R = 1$.

8. $f(x) = \cos x$ and $a = -\pi/4$.
Since $f^{(4n)}(x) = \cos x$, $f^{(4n+1)}(x) = -\sin x$, $f^{(4n+2)}(x) = -\cos x$, and $f^{(4n+3)}(x) = \sin x$,
we have $f^{(4n)}(a) = \frac{1}{\sqrt{2}}$, $f^{(4n+1)}(a) = \frac{1}{\sqrt{2}}$, $f^{(4n+2)}(a) = -\frac{1}{\sqrt{2}}$, and $f^{(4n+3)}(a) = -\frac{1}{\sqrt{2}}$.

The Taylor series for $f(x)$ about $-\pi/4$ is $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2}} \left(\frac{\left(x+\frac{\pi}{4}\right)^{2n}}{(2n)!} + \frac{\left(x+\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} \right)$.

If $u_n = \frac{f^{(n)}(-\pi/4)}{n!}\left(x + \frac{\pi}{4}\right)^n$ then $\left|\frac{u_{n+1}}{u_n}\right| = \frac{\left|x+\frac{\pi}{4}\right|}{n+1} \to 0$ so the series has radius of convergence $R = \infty$.

20. $\cos x$ has Maclaurin series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$.

Hence $\cos(x^3)$ has Maclaurin series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x^3)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{6n}$.

Since the series for $\cos x$ converges absolutely for all real x, so does the one for $\cos(x^3)$, and its radius of convergence is $R = \infty$.

24. $\cos x$ has Maclaurin series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$.

Hence $\cos(2x)$ has Maclaurin series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (2x)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}$ and

$\cos^2 x = \frac{1}{2} [1 + \cos(2x)]$ has Maclaurin series $1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!} x^{2n}$.

All these series have $R = \infty$.

26. $\cos x$ has Maclaurin series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ so $1 - \cos x$ has Maclaurin series

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)!} x^{2n+2} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)!} x^{2n}$.

If $f(x) = \frac{1 - \cos x}{x^2}$ for $x \neq 0$ and $f(0) = \frac{1}{2}$ then $f(x)$ has Maclaurin series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)!} x^{2n}$.

32. $\sin x$ has Maclaurin series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$, an alternating series.

$3^\circ = \pi/60$ radians, so to have 5 decimal place accuracy we can calculate

$\sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{60}\right)^{2n+1}$ where $\frac{1}{(2N+3)!} \left(\frac{\pi}{60}\right)^{2n+3} < 0.000005$.

Since $\frac{1}{3!} \left(\frac{\pi}{60}\right)^3 > 0.00002$ and $\frac{1}{5!} \left(\frac{\pi}{60}\right)^5 < 0.000000004$, using $N = 1$ we conclude that $0.052335952 < \sin 3^\circ < 0.052335957$.

My TI–36 calculator gives $\sin 3^\circ \approx 0.052335956243$.
Maple gives $\sin 3^\circ \approx 0.0523359562429438327\ldots$.

36. e^x has Maclaurin series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ with radius of convergence $R = \infty$.

So e^{x^3} has Maclaurin series $\sum_{n=0}^{\infty} \frac{1}{n!} (x^3)^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^{3n}$ also with $R = \infty$.

Thus $\int e^{x^3} dx$ has Maclaurin series $\sum_{n=0}^{\infty} \frac{1}{n!(3n+1)} x^{3n+1} + C$, again with $R = \infty$.

42. $\cos x$ has Maclaurin series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$ so

$\sec x$ has Maclaurin series $1 + \frac{1}{2} x^2 + \frac{5}{24} x^4 + \cdots$, using long division.