MATHEMATICS 152 98-2 Solutions for Assignment 6

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2. If $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$ and $\theta = \sin^{-1}\frac{x}{2}$.

Note $\sin\theta = \frac{x}{2}$, so $\cos\theta = \frac{\sqrt{4-x^2}}{2}$ and $\tan\theta = \frac{x}{\sqrt{4-x^2}}$.

$$\int_0^2 x^3\sqrt{4-x^2}\,dx = \int_0^{\pi/2}(2\sin\theta)^3(2\cos\theta)(2\cos\theta\,d\theta) = \int_0^{\pi/2}32\sin^3\theta\cos^2\theta\,d\theta =$$

$$= \int_0^{\pi/2}32\sin\theta(1-\cos^2\theta)\cos^2\theta\,d\theta = \int_0^{\pi/2}32(\cos^2\theta - \cos^4\theta)\sin\theta\,d\theta =$$

$$= 32\left(-\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta\right)\Big]_0^{\pi/2} = \frac{64}{15}.$$

Alternatively if $x = 2\tanh u$, $dx = 2\operatorname{sech}^2 u\,du$, and $u = \tanh^{-1}\frac{x}{2} = \frac{1}{2}\ln\frac{1+\frac{x}{2}}{1-\frac{x}{2}} =$

$= \frac{1}{2}\ln\frac{2+x}{2-x}$. Note $\operatorname{sech} u = \frac{\sqrt{4-x^2}}{2}$, $\cosh u = \frac{2}{\sqrt{4-x^2}}$, and $\sinh u = \frac{x}{\sqrt{4-x^2}}$.

When $x = 0$, $u = 0$. As $x \to 2^-$, $u \to +\infty$.

$$\int_0^2 x^3\sqrt{4-x^2}\,dx = \int_0^{+\infty}(2\tanh u)^3(2\operatorname{sech} u)(2\operatorname{sech}^2 u\,du) = \int_0^{+\infty}32\tanh^3 u\operatorname{sech}^3 u\,du =$$

$$= \int_0^{+\infty}32\tanh^2 u\operatorname{sech}^2 u(\operatorname{sech} u\tanh u\,du) = \int_0^{+\infty}32(1-\operatorname{sech}^2 u)(\operatorname{sech}^2 u)(\operatorname{sech} u\tanh u\,du) =$$

$$= \int_0^{+\infty}32(\operatorname{sech}^2 u - \operatorname{sech}^4 u)(\operatorname{sech} u\tanh u\,du) = -32\left(\frac{1}{3}\operatorname{sech}^3 u - \frac{1}{5}\operatorname{sech}^5 u\right)\Big]_0^{+\infty} = \frac{64}{15}.$$

8. If $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$ and $\theta = \tan^{-1}x$.

When $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \frac{\pi}{4}$.

Note $\cos\theta = \frac{1}{\sqrt{x^2+1}}$ and $\sin\theta = \frac{x}{\sqrt{x^2+1}}$, while $\tan\theta = x$.

$$\int_0^1\sqrt{x^2+1}\,dx = \int_0^{\pi/4}\sec\theta(\sec^2\theta\,d\theta) = \int_0^{\pi/4}\sec^3\theta\,d\theta =$$

$$= \frac{1}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)\Big]_0^{\pi/4} = \frac{1}{2}\left(\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right).$$

Alternatively if $x = \sinh u$, $dx = \cosh u\,du$ and $u = \sinh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right)$.

Note $\cosh u = \sqrt{x^2+1}$. When $x = 0$, $u = 0$ and $\cosh u = 1$.

When $x = 1$, $u = \sinh^{-1}1 = \ln\left(1 + \sqrt{1^2+1}\right) = \ln\left(\sqrt{2}+1\right)$ and $\cosh u = \sqrt{2}$.

$$\int_0^1\sqrt{x^2+1}\,dx = \int_0^{\sinh^{-1}1}\cosh u(\cosh u\,du) = \frac{1}{2}(\cosh u\sinh u + u)\Big]_0^{\sinh^{-1}1} = \frac{1}{2}\left(\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right).$$

12. If $x = \frac{3}{4}\sec\theta$, $dx = \frac{3}{4}\sec\theta\tan\theta\, d\theta$ and $\theta = \sec^{-1}\frac{4}{3}x$.

Note $\sec\theta = \frac{4x}{3}$, $\cos\theta = \frac{3}{4x}$, $\sin\theta = \frac{\sqrt{16x^2-9}}{4x}$, and thus $\tan\theta = \frac{\sqrt{16x^2-9}}{3}$.

$$\int \frac{dx}{x^2\sqrt{16x^2-9}} = \int \frac{\frac{3}{4}\sec\theta\tan\theta\, d\theta}{\left(\frac{3}{4}\sec\theta\right)^2 \cdot 3\tan\theta} = \int \frac{4}{9}\cos\theta\, d\theta = \frac{4}{9}\sin\theta + C = \frac{\sqrt{16x^2-9}}{9x} + C.$$

Alternatively if $x = \frac{3}{4}\cosh u$ then $dx = \frac{3}{4}\sinh u\, du$.

Note $\cosh u = \frac{4x}{3}$, so $u = \cosh^{-1}\frac{4x}{3} = \ln\left(\frac{4}{3}x + \sqrt{\left(\frac{4x}{3}\right)^2 - 1}\right) = \ln\frac{4x+\sqrt{16x^2-9}}{3}$.

Since $\sinh u = \frac{\sqrt{16x^2-9}}{3}$, $\tanh u = \frac{\sqrt{16x^2-9}}{4x}$.

$$\int \frac{dx}{x^2\sqrt{16x^2-9}} = \int \frac{\frac{3}{4}\sinh u\, du}{\left(\frac{3}{4}\cosh u\right)^2 \cdot 3\sinh u} = \int \frac{4}{9}\operatorname{sech}^2 u\, du = \frac{4}{9}\tanh u + C = \frac{\sqrt{16x^2-9}}{9x} + C.$$

16. If $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$ and $\theta = \tan^{-1}\frac{x}{2}$.

Note $\tan\theta = \frac{x}{2}$, $\sin\theta = \frac{x}{\sqrt{x^2+4}}$, and $\cos\theta = \frac{2}{\sqrt{x^2+4}}$.

$$\int \frac{x}{(x^2+4)^{5/2}}\, dx = \int \frac{(2\tan\theta)(2\sec^2\theta\, d\theta)}{32\sec^5\theta} = \frac{1}{8}\int \cos^2\theta\sin\theta\, d\theta = -\frac{1}{24}\cos^3\theta + C =$$

$$= -\frac{1}{24}\left(\frac{2}{\sqrt{x^2+4}}\right)^3 + C = -\frac{1}{3(x^2+4)^{3/2}} + C.$$

Alternatively if $x = 2\sinh u$, then $dx = 2\cosh u\, du$, $\sinh u = \frac{x}{2}$,

$u = \sinh^{-1}\frac{x}{2} = \ln\left(\frac{x}{2} + \sqrt{\left(\frac{x}{2}\right)^2 + 1}\right) = \ln\frac{x+\sqrt{x^2+4}}{2}$, and $\cosh u = \frac{\sqrt{x^2+4}}{2}$.

$$\int \frac{x}{(x^2+4)^{5/2}}\, dx = \int \frac{(2\sinh u)(2\cosh u\, du)}{(2\cosh u)^5} = \int \frac{1}{8}\frac{\sinh u}{(\cosh u)^4} = -\frac{1}{24}(\cosh u)^{-3} + C =$$

$$= -\frac{1}{3(x^2+4)^{3/2}} + C.$$

Alternatively just make the substitution $u = x^2 + 4$, $du = 2x\, dx$, and integrate directly.

26. Notice that $5 - 4x - x^2 = 9 - (x+2)^2$.

If $x + 2 = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$ and $\theta = \sin^{-1}\frac{x+2}{3}$.

Note $\sin\theta = \frac{x+2}{3}$ and $\cos\theta = \frac{\sqrt{9-(x+2)^2}}{3} = \frac{\sqrt{5-4x-x^2}}{3}$.

$$\int \frac{dx}{(5-4x-x^2)^{5/2}} = \int \frac{3\cos\theta\, d\theta}{(3\cos\theta)^5} = \int \frac{1}{81}\sec^4\theta\, d\theta = \int \frac{1}{81}(1+\tan^2\theta)\sec^2\theta\, d\theta =$$

$$= \frac{1}{81}\left(\tan\theta + \frac{1}{3}\tan^3\theta\right) + C = \frac{1}{81}\left(\frac{x+2}{\sqrt{5-4x-x^2}} + \frac{1}{3}\frac{(x+2)^3}{(5-4x-x^2)^{3/2}}\right) + C.$$

Alternatively if $x + 2 = 3\tanh u$, then $u = \tanh^{-1}\frac{x+2}{3} = \frac{1}{2}\ln\frac{1+\frac{x+2}{3}}{1-\frac{x+2}{3}} = \frac{1}{2}\ln\frac{5+x}{1-x}$ and

$dx = 3\,\text{sech}^2 u\, du$. Note $\tanh u = \frac{x+2}{3}$, $\text{sech}\, u = \sqrt{1-\left(\frac{x+2}{3}\right)^2} = \frac{\sqrt{5-4x-x^2}}{3}$,

$\cosh u = \frac{3}{\sqrt{5-4x-x^2}}$, and $\sinh u = \frac{x+2}{\sqrt{5-4x-x^2}}$.

$$\int \frac{dx}{(5-4x-x^2)^{5/2}} = \int \frac{3\,\text{sech}^2 u\, du}{(3\,\text{sech}\, u)^5} = \int \frac{1}{81}\cosh^3 u\, du = \int \frac{1}{81}(1+\sinh^2 u)\cosh u\, du =$$

$$= \frac{1}{81}\left(\sinh u + \frac{1}{3}\sinh^3 u\right) + C = \frac{1}{81}\left(\frac{x+2}{\sqrt{5-4x-x^2}} + \frac{1}{3}\frac{(x+2)^3}{(5-4x-x^2)^{3/2}}\right) + C.$$

30. (a) If $x = a\tan\theta$, $dx = a\sec^2\theta\, d\theta$ and $\theta = \tan^{-1}\frac{x}{a}$.

Note $\tan\theta = \frac{x}{a}$, $\sin\theta = \frac{x}{\sqrt{x^2+a^2}}$, and $\cos\theta = \frac{a}{\sqrt{x^2+a^2}}$.

$$\int \frac{x^2}{(x^2+a^2)^{3/2}}\, dx = \int \frac{(a\tan\theta)^2(a\sec^2\theta\, d\theta)}{(a\sec\theta)^3} = \int \frac{\sin^2\theta}{\cos\theta}\, d\theta = \int \frac{1-\cos^2\theta}{\cos\theta}\, d\theta =$$

$$= \int(\sec\theta - \cos\theta)\, d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta + C = \ln\frac{\sqrt{x^2+a^2}+x}{a} - \frac{x}{\sqrt{x^2+a^2}} + C.$$

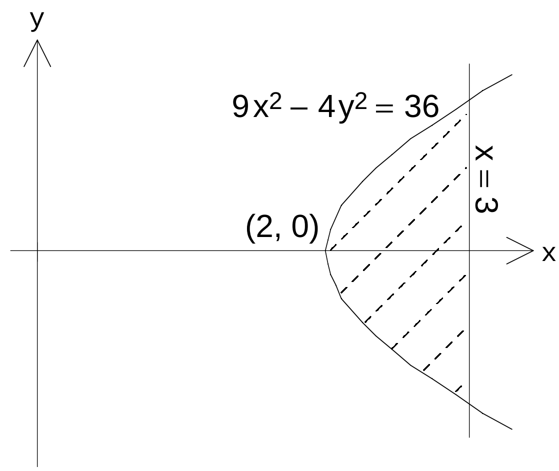
(b) If $x = a\sinh t$, $t = \sinh^{-1}\frac{x}{a} = \ln\left(\frac{x}{a} + \sqrt{1+\left(\frac{x}{a}\right)^2}\right) = \ln\frac{x+\sqrt{x^2+a^2}}{a}$, and

$dx = a\cosh t\, dt$. Note $\sinh t = \frac{x}{a}$ and $\cosh t = \frac{\sqrt{x^2+a^2}}{a}$ so $\tanh t = \frac{x}{\sqrt{x^2+a^2}}$.

$$\int \frac{x^2}{(x^2+a^2)^{3/2}}\, dx = \int \frac{a^2\sinh^2 t}{a^3\cosh^3 t}\, a\cosh t\, dt = \int \tanh^2 t\, dt = \int (1 - \text{sech}^2 t)\, dt =$$

$$= t - \tanh t + C = \ln\frac{x+\sqrt{x^2+a^2}}{a} - \frac{x}{\sqrt{x^2+a^2}} + C.$$

32. The upper half of the hyperbola $9x^2 - 4y^2 = 36$ is $y = \frac{3}{2}\sqrt{x^2-4}$. We can take the area of the region between the upper half of the hyperbola and the x-axis, between $x = 2$ and $x = 3$, and double it.



For Exercise 32

Area $= 2\int_2^3 \frac{3}{2}\sqrt{x^2-4}\,dx = 3\int_2^3 \sqrt{x^2-4}\,dx$.

If $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\,d\theta$ and $\theta = \sec^{-1}\frac{x}{2}$. Note $\sec\theta = \frac{x}{2}$, $\cos\theta = \frac{2}{x}$, $\sin\theta = \frac{\sqrt{x^2-4}}{x}$, and $\tan\theta = \frac{\sqrt{x^2-4}}{2}$.

Let $\alpha = \sec^{-1}\frac{3}{2} = \cos^{-1}\frac{2}{3}$.

Area $= 3\int_2^3 \sqrt{x^2-4}\,dx = 3\int_0^\alpha (2\tan\theta)(2\sec\theta\tan\theta\,d\theta) = 12\int_0^\alpha \tan^2\theta\sec\theta\,d\theta =$

$= 12\int_0^\alpha (\sec^2\theta - 1)\sec\theta\,d\theta = 12\int_0^\alpha (\sec^3\theta - \sec\theta)\,d\theta =$

$= 12\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| - \ln|\sec\theta+\tan\theta|\right)\Big]_0^\alpha =$

$= 6\left(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right)\Big]_0^\alpha = 6\left(\frac{3\sqrt{5}}{4} - \ln\frac{3+\sqrt{5}}{2}\right) = \frac{9}{2}\sqrt{5} - 6\ln\frac{3+\sqrt{5}}{2}$.

Alternatively if $x = 2\cosh u$, $dx = 2\sinh u\,du$ and $u = \cosh^{-1}\frac{x}{2} =$

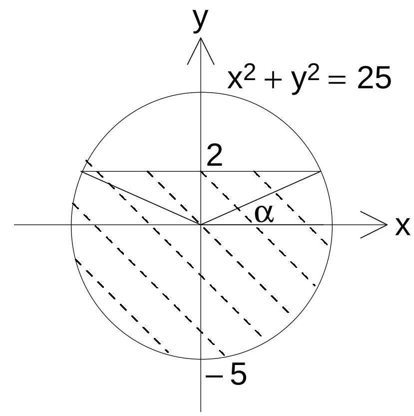
$= \ln\left(\frac{x}{2} + \sqrt{\left(\frac{x}{2}\right)^2 - 1}\right) = \ln\frac{x+\sqrt{x^2-4}}{2}$. Note $\cosh u = \frac{x}{2}$ and $\sinh u = \frac{\sqrt{x^2-4}}{2}$.

Let $\beta = \ln\frac{3+\sqrt{5}}{2} = \cosh^{-1}\frac{3}{2}$.

Area $= 3\int_2^3 \sqrt{x^2-4}\,dx = 3\int_0^\beta (2\sinh u)\,(2\sinh u\,du) = 12\int_0^\beta \sinh^2 u\,du =$

$= 12\left(\frac{1}{2}\sinh u\cosh u - \frac{1}{2}u\right)\Big]_0^\beta = 6\left(\frac{\sqrt{5}}{2}\frac{3}{2} - \ln\frac{3+\sqrt{5}}{2}\right) = \frac{9}{2}\sqrt{5} - 6\ln\frac{3+\sqrt{5}}{2}$.

36. We need to look at the circular disk forming a vertical cross-section of the cylindrical tank. If the disk is described by $x^2 + y^2 \le 25$ then we must find the ratio of the shaded area to that of the whole disk, 25π. It is easiest to integrate with respect to y and determine $A = 2\int_{-5}^{2} \sqrt{25 - y^2}\, dy$.



For Exercise 36

If $y = 5\sin\theta$, $dy = 5\cos\theta\, d\theta$, and $\theta = \sin^{-1}\frac{y}{5}$.

When $y = -5$, $\theta = -\frac{\pi}{2}$.

When $y = 2$, $\theta = \sin^{-1}\frac{2}{5} = \alpha$.

Notice that $\sin\alpha = 0.4$ and $\cos\alpha = \sqrt{0.84}$,

while $\sin(2\alpha) = 2\sin\alpha\cos\alpha = 0.8\sqrt{0.84}$.

$$A = 2\int_{-\pi/2}^{\alpha} (5\cos\theta)(5\cos\theta\, d\theta) = 25\int_{-\pi/2}^{\alpha} (1 + \cos(2\theta))\, d\theta = 25\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big]_{-\pi/2}^{\alpha} =$$

$$= 25\left(\alpha + \frac{\pi}{2} + \frac{0.8\sqrt{0.84}}{2}\right) = 25\left(\sin^{-1}0.4 + \frac{\pi}{2} + \frac{0.8\sqrt{0.84}}{2}\right).$$

The ratio of this area to the total disk area 25π is $\frac{\sin^{-1}0.4 + 0.4\sqrt{0.84}}{\pi} + \frac{1}{2} \approx 0.74768412$, or about 75%.

One can find the area A using geometry. It is easy to find both the area of the circular sector with central angle $2\alpha + \pi$ below the isosceles triangle and the area of the triangle itself. If you do this, compare the various terms involved to those in the answer obtained above.

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4. $$\frac{x^3 - x^2}{(x-6)(5x+2)^3} = \frac{A}{x-6} + \frac{B}{5x+3} + \frac{C}{(5x+3)^2} + \frac{D}{(5x+3)^3}.$$

8. $$\frac{x^4 + x^3 - x^2 - x + 1}{x^3 - x} = x + 1 + \frac{1}{x^3 - x} = x + 1 + \frac{1}{x(x+1)(x-1)} =$$
$$= x + 1 + \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}.$$

12. $$\frac{1 + 16x}{(2x+3)(x+5)^2(x^2+x+1)} = \frac{A}{2x-3} + \frac{B}{x+5} + \frac{C}{(x+5)^2} + \frac{Dx+E}{x^2+x+1}.$$

16. $$\frac{1}{x^6 - x^3} = \frac{1}{x^3(x^3-1)} = \frac{1}{x^3(x-1)(x^2+x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{Ex+F}{x^2+x+1}.$$

32. $\frac{18-2x-4x^2}{x^3+4x^2+x-6} = \frac{18-2x-4x^2}{(x+3)(x+2)(x-1)} = \frac{A}{x+3} + \frac{B}{x+2} + \frac{C}{x-1}$.

So $18-2x-4x^2 = A(x+2)(x-1) + B(x+3)(x-1) + C(x+3)(x+2)$.

Put $x=-3$ to see that $A=-3$.

Put $x=-2$ to see that $B=-2$.

Put $x=1$ to see that $C=1$.

$$\int \frac{18-2x-4x^2}{x^3+4x^2+x-6}\,dx = \int \left(\frac{-3}{x+3} + \frac{-2}{x+2} + \frac{1}{x-1}\right) dx =$$

$$= -3\ln|x+3| - 2\ln|x+2| + \ln|x-1| + K = \ln\frac{|x-1|}{|x+3|^3|x+2|^2} + K.$$

36. $\frac{x^3}{(x+1)^3} = 1 + \frac{-3x^2-3x-1}{(x+1)^3} = 1 + \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$.

So $-3x^2-3x-1 = A(x+1)^2 + B(x+1) + C$.

Put $x=-1$ to see that $C=-1$.

Differentiate to obtain $-6x-3 = 2A(x+1)+B$, and then put $x=-1$ to see that $B=3$.

Differentiate again to obtain $-6 = 2A$; thus $A=-3$.

$$\int \frac{x^3}{(x+1)^3}\,dx = \int\left(1 - \frac{3}{x+1} + \frac{3}{(x+1)^2} - \frac{1}{(x+1)^3}\right)dx = x - 3\ln|x+1| - \frac{3}{x+1} + \frac{1}{2(x+1)^2} + K.$$

For an easier way to get the partial fraction expansion write

$$\frac{x^3}{(x+1)^3} = \frac{[(x+1)-1]^3}{(x+1)^3} = \frac{(x+1)^3 - 3(x+1)^2 + 3(x+1) - 1}{(x+1)^3} = 1 - \frac{3}{x+1} + \frac{3}{(x+1)^2} - \frac{1}{(x+1)^3}.$$

46. $\frac{x^3}{x^3+1} = 1 - \frac{1}{x^3+1} = 1 - \frac{1}{(x+1)(x^2-x+1)} = 1 + \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$.

So $-1 = A(x^2-x+1) + (Bx+C)(x+1)$.

Put $x=-1$ to see that $A = -\frac{1}{3}$.

Replace A by $-\frac{1}{3}$, obtaining $\frac{1}{3}x^2 - \frac{1}{3}x - \frac{2}{3} = (Bx+C)(x+1) = Bx^2 + (B+C)x + C$.

Then it is clear that $B = \frac{1}{3}$ and $C = -\frac{2}{3}$.

But $\frac{\frac{1}{3}x - \frac{2}{3}}{x^2-x+1}$ is unpleasant to integrate. Replace it by

$$\frac{\frac{1}{6}(2x-1) - \frac{1}{2}}{x^2-x+1} = \frac{1}{6}\cdot\frac{2x-1}{x^2-x+1} - \frac{1}{2}\cdot\frac{1}{(x-0.5)^2+0.75} = \frac{1}{6}\cdot\frac{2x-1}{x^2-x+1} - \frac{2}{3}\cdot\frac{1}{\left((2x-1)/\sqrt{3}\right)^2+1}.$$

$$\int \frac{x^3}{x^3+1}\,dx = \int\left(1 - \frac{1}{3}\cdot\frac{1}{x+1} + \frac{1}{6}\cdot\frac{2x-1}{x^2-x+1} - \frac{2}{3}\cdot\frac{1}{\left((2x-1)/\sqrt{3}\right)^2+1}\right)dx =$$

$$= x - \frac{1}{3}\ln|x+1| + \frac{1}{6}\ln\left(x^2-x+1\right) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + K.$$

60. $\frac{2x+1}{4x^2+12x-7} = \frac{\frac{1}{4}(8x+12)-2}{4x^2+12x-7} = \frac{1}{4}\cdot\frac{8x+12}{4x^2+12x-7} - \frac{1}{2}\cdot\frac{1}{x^2+3x-\frac{7}{4}} =$

$= \frac{1}{4}\cdot\frac{8x+12}{4x^2+12x-7} - \frac{1}{2}\cdot\frac{1}{\left(x+\frac{3}{2}\right)^2-2^2}.$

$$\int\frac{2x+1}{4x^2+12x-7}dx = \frac{1}{4}\ln\left|4x^2+12x-7\right| - \frac{1}{8}\ln\left|\frac{x-\frac{1}{2}}{x+\frac{7}{2}}\right| + K = \frac{1}{4}\ln\left|(2x+3)^2-4^2\right| - \frac{1}{8}\ln\left|\frac{2x-1}{2x+7}\right| + K.$$

If $|2x+3|<4$, this can be rewritten as $\frac{1}{4}\ln\left(4^2-(2x+3)^2\right)+\frac{1}{4}\tanh^{-1}\frac{x+\frac{3}{2}}{2}+K=$

$=\frac{1}{4}\ln\left(4^2-(2x+3)^2\right)+\frac{1}{4}\tanh^{-1}\frac{2x+3}{4}+K.$

If $|2x+3|>4$, this can be rewritten as $\frac{1}{4}\ln\left((2x+3)^2-4^2\right)+\frac{1}{4}\coth^{-1}\frac{x+\frac{3}{2}}{2}+K=$

$=\frac{1}{4}\ln\left((2x+3)^2-4^2\right)+\frac{1}{4}\coth^{-1}\frac{2x+3}{4}+K.$

64. Using the method of disks, if the region under the curve $y=\frac{1}{x^2+3x+2}$ from $x=0$ to $x=1$ is rotated about the x-axis the resulting volume is

$$V=\pi\int_0^1\left(\frac{1}{x^2+3x+2}\right)^2dx=\pi\int_0^1\frac{1}{(x+1)^2(x+2)^2}\,dx.$$

$$\frac{1}{(x+1)^2(x+2)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+2}+\frac{D}{(x+2)^2}.$$

So $A(x+1)(x+2)^2+B(x+2)^2+C(x+1)^2(x+2)+D(x+1)^2=1.$

Putting $x=-1$, $B=1$; putting $x=-2$, $D=1$.

Differentiating, $A(x+2)(3x+4)+2B(x+2)+C(x+1)(3x+5)+2D(x+1)=0.$

Putting $x=-1$, $A+2B=0$ so $A=-2B=-2$; putting $x=-2$, $C-2D=0$ so $C=2D=2$.

Thus $\frac{1}{(x+1)^2(x+2)^2}=\frac{-2}{x+1}+\frac{1}{(x+1)^2}+\frac{2}{x+2}+\frac{1}{(x+2)^2}.$

Integrating, the volume is

$$V=\pi\left(-2\ln(x+1)-\frac{1}{x+1}+2\ln(x+2)-\frac{1}{x+2}\right)\Big]_0^1=\pi\left(\ln\frac{9}{16}+\frac{2}{3}\right).$$

The method of cylindrical shells could also be used, but one would have to treat the subintervals $0\le y\le\frac{1}{6}$ and $\frac{1}{6}\le y\le\frac{1}{2}$ separately, and for the latter region expressing x as a function of y would involve a square root with radicand containing $\frac{1}{y}$, an unpleasant prospect.

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2. If $u = \sqrt[3]{x}$, $x = u^3$ and $dx = 3u^2\,du$.

$$\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx = \int_0^1 \frac{1}{1+u}\cdot 3u^2\,du = \int_0^1 \left(3u - 3 + \frac{3}{1+u}\right)du = \left(\frac{3}{2}u^2 - 3u + 3\ln(1+u)\right)\Big]_0^1 =$$
$$= -\frac{3}{2} + 3\ln 2.$$

6. If $u = \sqrt{x+2}$, $x = u^2 - 2$ and $dx = 2u\,du$.

$$\int \frac{1}{x - \sqrt{x+2}}\,dx = \int \frac{2u\,du}{u^2 - u - 2} = \int \left(\frac{4/3}{u-2} + \frac{2/3}{u+1}\right)du = \frac{4}{3}\ln|u-2| + \frac{2}{3}\ln|u+1| + C =$$
$$= \frac{4}{3}\ln\left|\sqrt{x+2} - 2\right| + \frac{2}{3}\ln\left|\sqrt{x+2} + 1\right| + C.$$

10. If $u = \sqrt{x}$, $x = u^2$ and $dx = 2u\,du$.

$$\int_{1/3}^{3} \frac{\sqrt{x}}{x^2 + x}\,dx = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{u}{u^4 + u^2}\,2u\,du = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{2\,du}{u^2+1} = 2\tan^{-1}u\Big]_{1/\sqrt{3}}^{\sqrt{3}} = \frac{\pi}{3}.$$

14. If $u = \sqrt[6]{x}$, $x = u^6$ and $dx = 6u^5\,du$.

$$\int \frac{\sqrt{x}}{\sqrt{x} - \sqrt[3]{x}}\,dx = \int \frac{u^3}{u^3 - u^2}\cdot 6u^5\,du = \int \frac{6u^6}{u-1}\,du = \int 6\left(u^5 + u^4 + u^3 + u^2 + u + 1 + \frac{1}{u-1}\right)du =$$
$$= u^6 + \frac{6}{5}u^5 + \frac{3}{2}u^4 + 2u^3 + 3u^2 + 6u + 6\ln|u-1| + C =$$
$$= x + \frac{6}{5}x^{5/6} + \frac{3}{2}x^{2/3} + 2x^{1/2} + 3x^{1/3} + 6x^{1/6} + 6\ln\left|\sqrt[6]{x} - 1\right| + C.$$

18. If $u = \sin x$, $x = \sin^{-1}u$ and $du = \cos x\,dx$.

$$\int \frac{\cos x}{\sin^2 x + \sin x}\,dx = \int \frac{du}{u^2 + u} = \int \left(\frac{1}{u} - \frac{1}{u+1}\right)du = \ln|u| - \ln|u+1| + C = \ln\left|\frac{\sin x}{\sin x + 1}\right| + C.$$

20. If $u = \sqrt{1+e^x}$, $x = \ln(u^2 - 1)$ and $dx = \dfrac{2u\,du}{u^2-1}$.

$$\int \frac{dx}{\sqrt{1+e^x}} = \int \frac{2u\,du}{(u^2-1)u} = \int \frac{2\,du}{u^2-1} = \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right)du = \ln|u-1| - \ln|u+1| + C = \ln\frac{\sqrt{1+e^x} - 1}{\sqrt{1+e^x} + 1} + C.$$

26. If $t = \tan\dfrac{x}{2}$, $x = 2\tan^{-1}t$ and $dx = \dfrac{2\,dt}{1+t^2}$.

Then $\sin x = \dfrac{2t}{1+t^2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$, and $\tan x = \dfrac{2t}{1-t^2}$.

$$\int \frac{dx}{\sin x + \tan x} = \int \left(\frac{2t}{1+t^2} + \frac{2t}{1-t^2}\right)^{-1} \frac{2\,dt}{1+t^2} = \int \frac{(1-t^2)\,dt}{2t} = \frac{1}{2}\int \left(\frac{1}{t} - t\right)dt =$$
$$= \frac{1}{2}\left(\ln|t| - \frac{1}{2}t^2\right) + C = \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| - \frac{1}{4}\tan^2\frac{x}{2} + C.$$