3:5.3; We will show that $\sum a_n b_n$ is absolutely convergent. To do this note that $\sum_{n=1}^{m} |a_n b_n| \leq \sum_{n=1}^{m} |a_n| . \sum_{n=1}^{m} b_n$ but $\sum |a_n|$ and $\sum b_n$ are both absolutely convergent then $\sum |a_n b_n|$ is bounded and so it is convergent.

3:5.4; Example:
Let $a_n = b_n = \frac{(-1)^{n+1}}{\sqrt{n}}$ it is easy to show that $\sum |a_n| = \sum b_n$ are convergent but they are not absolutely convergent because $\sum |a_n| = \sum \frac{1}{\sqrt{n}} > \sum \frac{1}{n}$. But $\sum a_n b_n = \sum \frac{1}{n}$ and we know that it is divergent.

3:5.5; Let $\sum a_n$ be the new series which is obtained from $\sum \frac{1}{k}$ by deleting a specified digit and let $P_m = \sum \frac{1}{k}$ where sum is over all m digit numbers which do not have that specified digit. Clearly $\sum a_n = \sum P_m$. Note that for a specified m each term in the sum $P_m = \sum \frac{1}{k}$ is at most $\frac{1}{10^{m-1}}$, also note that there is at most 9^m terms in this sum then $P_m \leq \frac{9^m}{10^{m-1}}$ so $\sum P_m \leq 9 \sum_{m=1}^{\infty} (\frac{9}{10}^{m-1}) = 9 \times \frac{1}{1-\frac{9}{10}} = 90$. Hence $\sum P_m$ is bounded then $\sum a_n$ is bounded and it is convergent.

3:5.12; For one of the best known proofs consider the binomial

$$\begin{aligned} F(x) &= (a_1 x - b_1)^2 + (a_2 x - b_2)^2 + \ldots + (a_n x - b_n)^2 \\ &= (\sum a_i^2) x^2 - 2(\sum a_i b_i) x + \sum b_i^2 \end{aligned}$$

$F(x)$ is always nonnegative so $\Delta' \leq 0$ and equivalently we have

$$(\sum a_i b_i)^2 \leq \sum a_i^2 . \sum b_i^2$$

.

3:6.2;

(a) It is divergent because

$$\lim_{n\to\infty} \frac{n(n+1)}{(n+2)^2} = 1$$

(b) It is divergent because

$$\sum \frac{3n(n+1)(n+2)}{n^3\sqrt{n}} > \sum \frac{3n^3}{n^3\sqrt{n}} = \sum \frac{3}{\sqrt{n}}$$

.

(c) For $s > 1$ it is convergent since $\frac{1}{n^s \log n} < \frac{1}{n^s}$,

For $s = 1$ it is divergent by Cauchy's Condensation test,

For $s < 1$ from inequality $\frac{1}{n^s \log n} > \frac{1}{n^s}$ it follows that $\sum \frac{1}{n^s \log n}$

is divergent too.

(e) This series is divergent for all nonnegatives a except $a = 1$ to prove this consider the case $a > 1$ and let $a = 1 + \epsilon$ from Bernoli's inequality we know that $(1+\epsilon)^{1/n} > 1 + \frac{1}{n}\epsilon$, so $\sum a^{1/n} - 1 > \sum \frac{1}{n}\epsilon = \epsilon \sum \frac{1}{n}$.

For the case $a < 1$ use the second part of Bernoli's inequality, $(1 - x)^{1/n} < 1 - \frac{1}{n}\epsilon$.

(f) By Cauchy's Condensation test it is easy to show that it is convergent if $t > 1$ and it is divergent for $t \leq 1$.

(g) The case $s = 1$ has been answered in the previous one, for the cases $s < 1$ or $s > 1$ one can compare this series with $\sum \frac{1}{n^s}$ to obtain that the series is convergent for $s > 1$ and it is divergent for $s < 1$.

(h) Note that $\lim_{n\to\infty}(1 - \frac{1}{n})^n = e^{-1}$, then

$$\lim_{n\to\infty} \frac{(1-\frac{1}{n})^{n^2}}{e^n} = 1$$

but $\sum \frac{1}{e^n}$ converges and we obtain from Comparison Test that

$\sum(1 - \frac{1}{n})^{n^2}$ converges too.

3:6.2;

(a) Note that $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=x$, then this series converges for $|x|<1$ and diverges for $|x|>1$. For the case $x=1$ one can compare it with $\sum\frac{1}{n}$ to obtain that the series is convergent and for the case $a=-1$ the Alternating series Test can be used to obtain that it is convergent.

(b) Again by Ratio Test it is easy to see that the series converges for $|x|<1$ and it diverges for $|x|>1$. For the cases $|x|=1$ it is divergent too because of trivial test.

(c) Since $\lim_{n\to\infty}\sqrt[n]{a_n}=e^{-x}$ then by Root test we obtain that this series is convergent for $x>0$ and divergent for $x<0$. It is trivial to see that it is divergent for the case $x=0$ too.

(d) By Ratio Test:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}(1+\frac{1}{n})^n.x=e.x$$

So it is convergent for $|x|<1/e$ and it is divergent for $|x|>1/e$.

3:6.6; Answer is negative, for an example consider the series $\sum\frac{1}{n^2}$.

3:6.18; The sequence $\frac{(-1)^{k-1}}{k^p}$ decreases monotonically to zero for all positive value of p and then by Alternating Series Test it is convergent for all $p>0$. Also the series abtained from its absolute values is well known series which is convergent only for $0<p<1$. Then this series is nonabsolutely convergent for $0<p<1$ and it is absolutely convergent for $p\geq 1$.

3:6.22; By the methods have been used in the proof of Integral Test we know that limit existes and $0<\gamma<1-\int_n^{n+1}\frac{1}{x}dx<1$. It remaines to prove that $\gamma>1/2$ and to do this note that $1-\int_1^2\frac{1}{x}dx>1/2$ and use the above inequality again to show that the remining terms i.e. $\sum_{k=2}^{n}-\int_2 n+1\frac{1}{x}dx$ is positive.