Appendix A

BACKGROUND

A.1 Should I read this chapter?

This background chapter is not meant for the instructor but for the student. It is a mostly informal account of ideas that one needs to survive an elementary course in analysis. The chapters in the text itself are more formal and contain actual "mathematics". This chapter is *about* mathematics and should be an easier read.

You may skip around and select those topics that you feel you really need to read. The section on notation (Section A.2) may be looked through to be sure that you are familiar with the normal way of writing up many mathematical ideas such as sets and functions.

The sections on proofs (Sections A.4, A.5, A.6, A.7 and A.8) should be read if you have never taken any courses that required an ability to write up a proof. For many students this course on real analysis is the first exposure to these ideas and you may find these sections helpful.

A.2 Notation

If you are about to embark on a reading of the text without any further preliminaries then there is some notation that we should review.

A.2.1 Set Notation

Sets are just collections of objects. In the beginning we are mostly interested in sets of real numbers. If the word "set" becomes too often repeated you might find that words such as *collection*, *family*, or *class* are used. Thus a set of sets might become a family of sets. (One finds such variations in ordinary language, e.g., flock of sheep, gaggle of geese, pride of lions.)

The statement $x \in A$ means that x is one of those numbers belonging to A. The statement $x \notin A$ means that x is *not* one of those numbers belonging to A. (The stroke through the symbol $\in$ here is a familiar device, even on road signs or no smoking signs.)

Here are some familiar sets and notation. We use

(The empty set) $\emptyset$ to represent the set that contains no elements, the empty set.

(The natural numbers) $\mathbb{N}$ to represent the set of natural numbers (positive integers) 1, 2, 3, 4, etc..

(The Integers) $\mathbb{Z}$ to represent the set of integers (positive integers, negative integers and zero).

(The Rational Numbers) $\mathbb{Q}$ to represent the set of rational numbers, i.e., of all fractions m/n where m and n are integers (and $n \neq 0$).

(The Real Numbers) $\mathbb{R}$ to represent all the real numbers.

(Closed Intervals) $[a, b]$ to represent the set of all numbers between a and b including a and b. We assume that $a < b$. This is called the closed interval with endpoints a and b. (Some authors allow the possibility that $a = b$ in which case $[a, b]$ must be interpreted as the set containing just the one point a. This would then be referred to as a *degenerate* interval. We have avoided this usage.)

(Open Intervals) (a, b) to represent the set of all numbers between a and b excluding a and b. This is called the open interval with endpoints a and b.

(Infinite Intervals) (a, ∞) to represent the set of all numbers strictly greater than a. The symbol ∞ is not interpreted as a number. (It might have been better for most students if the notation had been $(a, \rightarrow)$ since that conveys the same meaning and the beginning student would not have presumed that there is some infinite number called "$\rightarrow$" at the extreme right hand "end" of the real line.)

The other infinite intervals are $(-\infty, a)$, $[a, \infty)$, $(-\infty, a]$ and $(-\infty, \infty) = \mathbb{R}$.

(Sets as a List) $\{1, -3, \sqrt{7}, 9\}$ to represent the set containing precisely the four real numbers 1, -3, $\sqrt{7}$, and 9. This is a useful way of describing a set (when possible): just list the elements that belong. Note that order does not matter in the world of sets so the list can be given in any order that one wishes.

(Set Builder Notation) $\{x : x^2 + x < 0\}$ to represent the set of all numbers x satisfying the inequality $x^2 + x < 0$. (It may take some time, see Exercise A:2.1, but if you are adept at inequalities and quadratic equations you can recognize that this set is exactly the open interval $(-1, 0)$.) This is another useful way of describing a set (when possible): just describe, by an equation or an inequality, the elements that belong. In general if $C(x)$ is some kind of assertion about an object x then $\{x : C(x)\}$ is the set of all objects x for which $C(x)$ happens to be true. Other formulations can be used. For example $\{x \in A : C(x)\}$ describes the set of elements x *that belong to the set* A and for which $C(x)$ is true. The example $\{1/n : n \in \mathbb{N}\}$ illustrates that a set can be obtained by performing computations on the members of another set.

described that consists of

Subsets, unions, intersection, differences The language of sets requires some special notation that is, doubtless, familiar. If you find you need some review take the time to learn this notation well as it will be used in all of your subsequent mathematics courses.

1. $A \subset B$ (A is a subset of B), if every element of A is also an element of B.

2. $A \cap B$ (the intersection of A and B), is the set consisting of elements of both sets.

3. $A \cup B$ (the union of the sets A and B), is the set consisting of elements of either set.

4. $A \setminus B$ (the difference[1] of the sets A and B) is the set consisting of elements belonging to A but not to B.

[1]Don't use $A - B$ for set difference since it suggests subtraction, which is something else.

In the text we will need also to form unions and intersections of large families of sets, not just of two sets. See the exercises for a development of such ideas.

De Morgan Laws Many manipulations of sets require two or more operations to be performed together. The simplest cases that should perhaps be memorized are

$$A \setminus (B_1 \cup B_2) = (A \setminus B_1) \cap (A \setminus B_2)$$

and a symmetrical version

$$A \setminus (B_1 \cap B_2) = (A \setminus B_1) \cup (A \setminus B_2).$$

If you sketch some pictures these two rules become quite evident.

There is nothing special that requires these "laws" to be restricted to two sets B_1 and B_2. Indeed any family of sets $\{B_i : i \in I\}$ taken over any indexing set I must obey the same laws:

$$A \setminus \left(\bigcup_{i \in I} B_i \right) = \bigcap_{i \in I} (A \setminus B_i)$$

and

$$A \setminus \left(\bigcap_{i \in I} B_i \right) = \bigcup_{i \in I} (A \setminus B_i) \,.$$

Here $\bigcup_{i \in I} B_i$ is just the set formed by combining all the elements of the sets B_i into one big set (i.e., forming a large union). Similarly $\bigcap_{i \in I} B_i$ is the set of points that are in all of the sets B_i, i.e., their common intersection.

Ordered Pairs Given two sets A and B often one needs to discuss pairs of objects (a, b) with $a \in A$ and $b \in B$. The first item of the pair is from the first set and the second item from the second. Since order matters here these are called *ordered pairs*. The set of all ordered pairs (a, b) with $a \in A$ and $b \in B$ is denoted

$$A \times B$$

and this set is called the *Cartesian product* of A and B.

Relations Often in mathematics we need to define a relation on a set S. Elements of S could be related by sharing some common feature, or could be related by a fact of one being "larger" than another. For example the statement $A \subset B$ is a relation on families of sets and $a < b$ a relation on a set of numbers. Fractions p/q and

a/b are related if they define the same number; thus we could define a relation on the collection of all fractions by $p/q \sim a/b$ if $pb = qa$.

A relation R on a set S then would be some way of deciding whether the statement xRy (read as x is related to y) is true. If we look closely at the form of this we see it is completely described by constructing the set

$$R = \{(x, y) : \ x \text{ is related to } y\}$$

of ordered pairs. Thus a relation on a set is not a new concept: it is merely a collection of ordered pairs. Let R be any set of ordered pairs of elements of S. Then $(x, y) \in R$ and xRy and "x is related to y" can be given the same meaning. This reduces relations to ordered pairs. In practice one usually views the relation from whatever perspective is most intuitive. (For example the order relation on the real line $x < y$ is technically the same as the set of ordered pairs $\{(x, y) : x < y\}$ but hardly anyone thinks about the relation this way.)

A.2.2 Function Notation

Analysis (indeed most of mathematics) is about functions. Do you recall that in elementary calculus courses you would often discuss some function such as $f(x) = x^2+x+1$ in the context of maxima and minima problems, or derivatives or integrals? The most important way of understanding a function in the calculus was by means of the graph: for this function the graph is the set of all pairs (x, x^2+x+1) for real numbers x and often this graph was sketched as a set of points in two dimensional space.

Definition of a Function What is a function really? Mathematicians noted long ago that the graph of a function carried all the information needed to describe the function. Indeed, since the graph is just a set of ordered pairs $(x, f(x))$, the concept of a function can be explained entirely within the language of sets without any need to invent a new concept. Thus the function *is* the graph and the graph is a set.

Definition A.1 Let A and B be nonempty sets. A set f of ordered pairs (a, b) with $a \in A$ and $b \in B$ is called a *function* from A to B, written symbolically as

$$f : A \to B,$$

provided that to every $a \in A$ there is precisely one pair (a, b) in f.

The notation $(a, b) \in f$ is often used in advanced mathematics but is awkward in expressing ideas in calculus and analysis. Instead we use the familiar expression $f(a) = b$. Also when we wish to think of a function as a graph we normally remind the reader by using the word "graph". Thus an analysis or calculus student would expect to see a question posed like this:

> Find a point on the graph of the function $f(x) = x^2+x+1$ where the tangent line is horizontal.

rather than the technically correct, but awkward looking

> Let f be the function
> $$f = \{(x, x^2 + x + 1) : x \in \mathbb{R}\}.$$
> Find a point in f where the tangent line is horizontal.

Domain of a Function The set of points A in the definition is called *the domain* of the function. It is an essential ingredient of the definition of any function. It should be considered incorrect to write

> Let the function f be defined by $f(x) = \sqrt{x}$.

Instead one should say

> Let the function f be defined with domain $[0, \infty)$ by $f(x) = \sqrt{x}$.

The first assertion is quite sloppy; it requires the reader to guess at the domain of the function. Calculus courses, however, often make this requirement leaving it to the student to figure out from a formula what domain should be assigned to the function. Often we, too, will require this of the reader.

Range of a Function The set of points B in the definition is sometimes called the *range* or *co-domain* of the function. Most writers do not like the term range for this, preferring to use the term range for the set

$$f(A) = \{f(x) : x \in A\} \subset B$$

that consists of the actual output values of the function f, not some larger set that merely contains all these values.

One-one and Onto Function If to each element b in the range of f there is precisely one element a in the domain so that $f(a) = b$ then f is said to be *one-one* or *injective.* We sometimes say, about the range $f(A)$ of a function, that f maps A *onto* $f(A)$. If $f : A \to B$ then f would be said to be *onto* B if B is the range of f, i.e., if for every $b \in B$ there is some $a \in A$ so that $f(a) = b$. A function that is onto is sometimes said to be *surjective.* A function that is both one-one and onto is sometimes said to be *bijective.*

Inverse of a Function Some functions allow an *inverse.* If $f : A \to B$ is a function there is, sometimes, a function $f^{-1} : B \to A$ that is the reverse of f in the sense that

$$f^{-1}(f(a)) = a \text{ for every } a \in A$$

and

$$f(f^{-1}(b)) = b \text{ for every } b \in B\ .$$

Thus f carries a to $f(a)$ and f^{-1} carries $f(a)$ back to a while f^{-1} carries b to $f^{-1}(b)$ and f carries $f^{-1}(b)$ back to b. This can happen only if f is one-one and onto B. See the exercises for some practice on these concepts.

Characteristic Function of a Set Let $E \subset \mathbb{R}$. Then a convenient function for discussing properties of the set E is the function χ_E defined to be 1 on E and to be 0 at every other point. This is called the *characteristic function* of E or, sometimes, *indicator function.*

Composition of Functions Suppose that f and g are two functions. For some values of x it is possible that the application of the two functions one after another

$$f(g(x))$$

has a meaning. If so this new value is denoted $f \circ g(x)$ and the function is called the *composition* of f and g. The domain of $f \circ g$ is the set of all values of x for which $g(x)$ has a meaning and for which then also $f(g(x))$ has a meaning, i.e., the domain of $f \circ g$ is

$$\{x : x \in \text{dom}(g) \text{ and } g(x) \in \text{dom}(f)\}.$$

Note that the order matters here so $f \circ g$ and $g \circ f$ have, usually, radically different meanings. This is likely one of the earliest appearances of an operation in elementary mathematics that is not commutative and which requires special attention by the reader.

Exercises

A:2.1 This exercise introduces the idea of set equality. The identity $X = Y$ for sets means that they have identical elements. To prove such an assertion assume first that $x \in X$ is any element. Now show $x \in Y$. Then assume that $y \in Y$ is any element. Now show $y \in X$.

(a) Show that $A \cup B = B$ if and only if $A \subset B$.

(b) Show that $A \cap B = A$ if and only if $A \subset B$.

(c) Show that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

(d) Show that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

(e) Show that $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.

(f) Show that $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$.

(g) Show that $\{x \in \mathbb{R} : x^2 + x < 0\} = (-1, 0)$.

A:2.2 This exercise introduces the notations $\bigcup_{n=1}^{N} A_i$ and $\bigcap_{n=1}^{N} A_i$ for the union and intersection of the sets A_1, A_2, ... A_N:

(a) Describe the sets

$$\bigcup_{n=1}^{N} (-1/n, 1/n) \text{ and } \bigcap_{n=1}^{N} (-1/n, 1/n).$$

(b) Describe the sets

$$\bigcup_{n=1}^{N} (-n, n) \text{ and } \bigcap_{n=1}^{N} (-n, n).$$

(c) Describe the sets

$$\bigcup_{n=1}^{N} [n, n+1] \text{ and } \bigcap_{n=1}^{N} [n, n+1].$$

A:2.3 This exercise introduces the notations $\bigcup_{n=1}^{\infty} A_i$ and $\bigcap_{n=1}^{\infty} A_i$ for the union and intersection of the sets A_1, A_2, ...

(a) Describe the sets

$$\bigcup_{n=1}^{\infty} (-1/n, 1/n) \text{ and } \bigcap_{n=1}^{\infty} (-1/n, 1/n).$$

(b) Describe the sets

$$\bigcup_{n=1}^{\infty} (-n, n) \text{ and } \bigcap_{n=1}^{\infty} (-n, n).$$

(c) Describe the sets

$$\bigcup_{n=1}^{\infty} [n, n+1] \text{ and } \bigcap_{n=1}^{\infty} [n, n+1].$$

A:2.4 Do you accept any of the following as an adequate definition of the function f? (The domain is not specified but it is assumed the reader will try to find a domain that might work.)

(a) $f(x) = 1/\sqrt{1-x}$.

(b) $f(x) = x$ if x is rational and $f(x) = -x$ if x is irrational.

(c) $f(x) = 1$ if x contains a 9 in its decimal expansion and $f(x) = 0$ if not.

(d) $f(x) = 1$ if x contains a 7 in its decimal expansion and $f(x) = 0$ if not.

(e) $f(x) = 1$ if x is a prime number and $f(x) = 0$ if it is not.

A:2.5 This exercise promotes the use of the terms *mapping* in the study of functions. If $f : X \to Y$ and $E \subset X$ then

$$f(E) = \{y : f(x) = y \text{ for some } x \in E\} \subset Y$$

is called the *image* of E under f and we say f *maps* E to the set $f(E)$.

(a) Let $f : \mathbb{R} \to \mathbb{R}$. Give an example of sets A, B so that $f(A \cap B) \neq f(A) \cap f(B)$.

(b) Would $f(A \cup B) = f(A) \cup f(B)$ be true in general?

(c) Find a function $f : \mathbb{R} \to \mathbb{R}$ so that $f([0,1]) = \{1,2\}$.

A:2.6 This exercise concerns the notion of one-one function (i.e., injective function):

(a) Show that $f : \mathbb{R} \to \mathbb{R}$ is one-one if and only if $f(A \cap B) = f(A) \cap f(B)$ for all sets A, B.

(b) Show that $f : \mathbb{R} \to \mathbb{R}$ is one-one if and only if $f(A \cap B) = \emptyset$ for all sets A, B with $A \cap B = \emptyset$.

A:2.7 This exercise concerns the notion of preimage. If $f : X \to Y$ and $E \subset Y$ then

$$f^{-1}(E) = \{x : f(x) = y \text{ for some } y \in E\} \subset X$$

is called the preimage of E under f. (There may or may not be an inverse function here; $f^{-1}(E)$ has a meaning even if there is no inverse function.)

(a) Show that $f(f^{-1}(E)) \subset E$ for every set $E \subset \mathbb{R}$.

(b) Show that $f^{-1}(f(E)) \supset E$ for every set $E \subset \mathbb{R}$.

(c) Can you simplify $f^{-1}(A \cup B)$ and $f^{-1}(A \cap B)$?

(d) Show that $f : \mathbb{R} \to \mathbb{R}$ is one-one if and only if $f^{-1}(\{b\})$ contains at most a single point for any $b \in \mathbb{R}$.

(e) Show that $f : \mathbb{R} \to \mathbb{R}$ is onto (i.e., the range of f is all of $\mathbb{R}$) if and only if $f(f^{-1}(E)) = E$ for every set $E \subset \mathbb{R}$.

A:2.8 This exercise concerns the notion of composition of functions:

(a) Give examples to show that $f \circ g$ and $g \circ f$ are distinct.

(b) Give an example in which $f \circ g$ and $g \circ f$ are not distinct.

(c) While composition is not commutative, is it associative, i.e., is it true that

$$(f \circ g) \circ h = f \circ (g \circ h)?$$

(d) Give several examples of functions f for which $f \circ f = f$.

A:2.9 This exercise concerns the notion of onto function (i.e., surjective function): Which of the following functions map $[0, 1]$ onto $[0, 1]$?

(a) $f(x) = x$.

(b) $f(x) = x^2$.

(c) $f(x) = x^3$.

(d) $f(x) = 2|x - \frac{1}{2}|$.

(e) $f(x) = \sin \pi x$.

(f) $f(x) = \sin x$.

A:2.10 This exercise concerns the notion of one-one and onto function (i.e., bijective function):

(a) Which of the functions of Exercise A:2.9 is a bijection of $[0, 1]$ to $[0, 1]$?

(b) Is the function $f(x) = x^2$ a bijection of $[-1, 1]$ to $[0, 1]$?

(c) Find a linear bijection of $[0, 1]$ onto the interval $[3, 6]$.

(d) Find a bijection of $[0, 1]$ onto the interval $[3, 6]$ that is not linear.

(e) Find a bijection of $\mathbb{N}$ onto $\mathbb{Z}$.

A:2.11 This exercise concerns the notion of inverse functions: for each of the functions of Exercise A:2.9 select an interval $[a, b]$ on which that function has an inverse and find an explicit formula for the inverse function. Be sure to state the domain of the inverse function.

A:2.12 This exercise concerns the notion of an equivalence relation. A relation $x \sim y$ on a set S is said to be an equivalence relation if

(a) $x \sim x$ for all $x \in S$.

(b) $x \sim y$ implies that $y \sim x$.

(c) $x \sim y$ and $y \sim z$ imply that $x \sim z$.

(a) Show that the relation $p/q \sim a/b$ if $pb = qa$ defined in the text on the collection of fractions is an equivalence relation.

(b) Define a relation on the collection of fractions that satisfies two of the requirements of an equivalence relation but is not an equivalence relation.

(c) Define nontrivial equivalence relations on the sets $\mathbb{N}$ and $\mathbb{Z}$.

A:2.13 Set builder notation can be used to "describe" some curious sets. For example

$$S_1 = \{S : S \text{ is a set}\}.$$

This has the peculiar property that $S_1 \in S_1$. (That is similar to joining a club where you find the club appearing on the membership list as a member of itself!) Worse yet is

$$S_2 = \{S : S \text{ is a set and } S \notin S\}.$$

This has the paradoxical property that if $S_2 \in S_2$ then $S_2 \notin S_2$ while if $S_2 \notin S_2$ then $S_2 \in S_2$. Any thoughts?

A.3 What is Analysis?

The term "analysis" now covers very large parts of mathematics. One pretty well needs to be a professional mathematician to understand what it might mean.

For a course at this level, though, "real analysis" mostly refers to the subject matter that you have already learned in your calculus courses: limits, continuity, derivatives, integrals, sequences, and series. The calculus as a subject can be thought of as an eighteenth century development, analysis as a nineteenth century creation. None of the ideas of the calculus rested on very firm foundation and the lack of foundations proved a barrier to further progress. There was much criticism by mathematicians and philosophers of the fundamental ideas of the calculus (limits especially) and often when new and controversial methods were proposed (such as Fourier series) the mathematicians of the time could not agree on whether they were valid.

In the first decades of the nineteenth century the foundations of the subject were reworked, most notably by Cauchy (whose name will appear very frequently in this text), and new and powerful methods developed. It is this that we are studying here.

We will look once again at notions of sequence limit, function limit, etc. that we have seen before in our calculus classes, but now from a more rigorous point of view. We want to know precisely what they mean and how to prove the validity of the techniques of the subject.

At first sight the student might wonder about this. Are we just reviewing our calculus but now we do not get to skip over the details of proofs? If, however, you persist you will see that we are entering instead a very new and different world. By looking closely at the details of why certain things work we gain a new insight. More than that we can do new things, things that could not have been imagined at a mere calculus level.

A.4 Why Proofs?

Can't we just do mathematics without proofs? Certainly there are many applications of mathematics carried on by people unable or unwilling to attempt proofs. But at the very heart and soul of mathematics is the proof, the careful argument that shows that a statement is true.

Compare this with the natural sciences. The advancement of knowledge in those subjects rests on the experiment. No scientist considers very seriously whether students can skip over experimental work and just learn the result. At the very core of all scientific discovery is the experimental method. It is too central to the discipline to be removed. It is the reason for the monumental success of the subject.

Mathematicians feel the same way about proofs. One can, with imagination and insight, make reasonable conjectures. But one can't be sure a conjecture is true until one proves it. The history of mathematics is filled with plausible (but false) statements made by mathematicians, even famous ones.

Proofs are an essential part of the subject. If you can master the art of reading and writing proofs you enter properly into the subject. If not you remain forever on the periphery looking in, a spectator able to learn some superficial facts about mathematics, but unable to *do* mathematics.

What is a Proof Mathematicians are always prepared to define exactly what everything in their subject means. Certainly it is possible to define exactly what constitutes a proof. But that is best left to a course in logic.

For a course in analysis just understand that a proof is a short or long sequence of arguments meant to convince us that some statement is true. You will understand what a proof is after you have read some proofs and find that you do in fact follow the argument.

A proof is always intended for a specific audience. Proofs in this text are intended for readers who have some experience in calculus and good reasoning skills, but little experience in analysis. Proofs in more advanced texts would be much shorter and have less motivation. Proofs in professional research journals, intended for other professional mathematicians, can be very terse and mysterious indeed.

Traditionally courses in analysis do not start with much of a discussion of proofs even though the students will be expected to produce proofs of their own, perhaps for the first time in their career. The best advice may be merely to jump in. Start studying the proofs in the text, the proofs given in lectures, the proofs attempted by your fellow students. Try to write them yourself. Read a proof, understand its main ideas and then attempt to write the argument up in your own words.

How to Read a Proof While a proof may look like a short story it is often much harder to read than one. Usually some of the computations will not seem clear and you will have to figure out how they were done. Some of the arguments (this is true and hence that is true) will not be immediate, but will require some thinking. Many of the steps will appear completely strange and it will seem that the proof is going off in a weird direction that is entirely mysterious.

Basically you must unravel the proof. Find out what the main ideas are and the various steps of the proof.

One important piece of advice while reading a proof: try to remember what it is that has to be proved. Before reading the proof decide what it is that must be proved exactly. Ask yourself "what would I have to show to prove that?"

How to Write a Proof Practice! One learns to write proofs by writing proofs. Start off by just copying nearly word for word a proof in a text that you find interesting. Vary the wording to use your own phrases. Write out the proof using more steps and more details than you found in the original. Try to find a different proof of the same statement and write out your new proof. Try to change the order of the argument if it is possible. If it is not possible you'll soon see why.

We all have learned the art of proof by imitation at first.

A.5 Indirect Proof

Many proofs in analysis are achieved as indirect proofs. This refers to a very specific method.

The method argues as follows. I wish to prove a statement **P** is true. Either **P** is true or else **P** is false, not both. If I suppose **P** is false perhaps I can prove that then something entirely unbelievable must be true. Since that unbelievable something is not true it follows that it cannot be the case that **P** is false. Therefore **P** is true.

The method appears in the classical subject of rhetoric under the label *reductio ab absurdam* (I reduce to the absurd).

> Ladies and gentlemen my worthy opponent claims **P** but I claim the opposite, namely **Q** . Suppose his claim were valid. Then ... and then ... and that would mean .... . But that's ridiculous so his claim is false and my claim must be true.

The pattern of all indirect proofs (also known as "proofs by contradiction") follows this structure. We wish to prove statement **P** is true. Suppose, in order to obtain a contradiction, that **P** is false. This would imply the following statements. [Statements follow.] But this is impossible. It follows that **P** is true as we were required to prove.

Here is a simple example. Suppose we wish to prove that

> For all positive numbers x, the fraction $1/x$ is also positive.

An indirect proof would go like this.

Proof. Suppose the statement is false. Then there is a positive number x and yet $1/x$ is not positive. This means

$$\frac{1}{x} \leq 0.$$

Since x is positive we can multiply both sides of the inequality by x and the inequality sign is preserved (this is a property of inequalities that we learned in elementary school and so we need not explain it). Thus

$$x \times \frac{1}{x} \leq x \times 0$$

or

$$1 \leq 0.$$

This is impossible. From this contradiction it follows that the statement must be true. ■

Indirect proofs are wonderfully useful and will be found throughout analysis. In some ways, however, they can be unsatisfying. After the statement "suppose not" the proof enters a fantasy world where all manipulations work towards producing a contradiction. None of the statements that you make along the way to this contradiction is necessarily of much interest because it is based on a false premise. In a direct proof, on the other hand, every statement you make is true and may be interesting on its own, not just as a tool to prove the theorem you are working on.

Also indirect proofs reside inside a logical system where any statement not true is false and any statement not false is true. Some people have argued that we might wish to live in a mathematical world where, even though you have proved that something is not false, you have still not succeeded in proving that it is true.

Exercises

A:5.1 Show that $\sqrt{2}$ is irrational by giving an indirect proof.

A:5.2 Show that there are infinitely many prime numbers.

A.6 Contraposition

The most common mathematical assertions that we wish to prove can be written symbolically as

$$\mathbf{P} \Rightarrow \mathbf{Q}$$

which we read aloud as "statement **P** implies statement **Q** ". The real meaning attached to this is simply that if statement **P** is true then statement **Q** is true.

A moments reflection about the meaning shows that the two versions

If **P** is true, then **Q** must be true.

and

If **Q** is false, then **P** must be false.

are identical in meaning. These are called contrapositives of each other. Any statement

$$\mathbf{P} \Rightarrow \mathbf{Q}$$

has a contrapositive

$$\text{not } \mathbf{Q} \Rightarrow \text{not } \mathbf{P} .$$

To prove a statement it is sometimes better to prove the contrapositive.

Here is a simple example. Suppose, as calculus students we were required to prove that

> Suppose that $\int_0^1 f(x)\,dx \neq 0$. Then there must be a point $\xi \in [0,1]$ such that $f(\xi) \neq 0$.

At first sight it might seem hard to think of how we are going to find that point $\xi \in [0,1]$ from such little information. But let us instead prove the contrapositive. The contrapositive would say that if there is no point $\xi \in [0,1]$ such that $f(\xi) \neq 0$ then it would not be true that $\int_0^1 f(x)\,dx \neq 0$. Let's get rid of the double negatives. Restating this, now, we see that the contrapositive says that if $f(\xi) = 0$ for every $\xi \in [0,1]$ then $\int_0^1 f(x)\,dx = 0$. Even the C- students (none of whom are reading this book) would have now been able to proceed.

Exercises

A:6.1 Prove the following assertion by contraposition: If x is irrational then $x + r$ is irrational for all rational numbers r.

A.7 Counterexamples

The polynomial

$$p(x) = x^2 + x + 17$$

has an interesting feature: it generates prime numbers for some time. For example $p(1) = 19$, $p(2) = 23$, $p(3) = 29$, $p(4) = 37$ are all prime. More examples can be checked. After many more computations we would be tempted to make the claim

> For every integer $n = 1, 2, 3, \ldots$ the value $n^2 + n + 17$ is prime.

To prove that this is true (if indeed it is true) we would be required to show for any n, no matter what, that the value $n^2 + n + 17$

is prime. What would it take to disprove the statement, i.e., to show that it is false?

All it would take is one instance where the statement fails. Only one! In fact there are many instances. It is enough to give one of them. Take $n = 17$ and observe that

$$17^2 + 17 + 17 = 17(17 + 1 + 1) = 17 \cdot 19$$

which is certainly not prime. This one example is enough to prove that the statement is false. We refer to this as a *proof by counterexample.*

The Converse In analysis we shall often need to invent counterexamples. One frequent situation that occurs is the following. Suppose that we have just completed, successfully, the proof of a theorem expressed symbolically as

$$\mathbf{P} \Rightarrow \mathbf{Q}.$$

A natural question is whether the converse is also true. The *converse* is the opposite implication

$$\mathbf{Q} \Rightarrow \mathbf{P}.$$

Indeed once we have proved any theorem it is nearly routine to ask if the converse is true. Many converses are false and a proof usually consists in looking for a counterexample.

For example in calculus courses (and here too in analysis courses) it is shown that every differentiable function is continuous. Expressed as an implication it looks like this:

$$f \text{ is differentiable } \Rightarrow f \text{ is continuous}$$

and, hence, the converse statement is

$$f \text{ is continuous } \Rightarrow f \text{ is differentiable.}$$

Is the converse true? If it is then it, too, should be proved. If it is false then a counterexample must be found. To prove it false we need supply just one function that is continuous and yet not differentiable. The reader may remember that the function $f(x) = |x|$ is continuous and yet not differentiable since at the point 0 there is no derivative.

Exercises

A:7.1 Disprove this statement: For any natural number n the equation $4x^2 + x - n = 0$ has no rational root.

A:7.2 Every prime greater than two is odd. Is the converse true?

A:7.3 State both the converse and the contrapositive of the assertion "every differentiable function is continuous". Is there a difference between them? Are they both true?

A.8 Induction

There is a convenient formula for the sum of the first n natural numbers:

$$1 + 2 + 3 + \ldots (n-1) + n = \frac{n(n+1)}{2}.$$

An easy direct proof of this would go as follows. Let S be the sum so that

$$S = 1 + 2 + 3 + \ldots (n-1) + n$$

or, expressed in the other order

$$S = n + (n-1) + (n-3) + \cdots + 2 + 1.$$

Adding these two equations gives

$$2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)$$

and hence

$$2S = n(n+1)$$

or

$$S = \frac{n(n+1)}{2}$$

which is the formula we require.

Suppose instead that we had been unable to construct this proof. Lacking any better ideas we could just test it out for $n = 1$, $n = 2$, $n = 3$, ... for as long as we had the patience. Eventually we might run into a counterexample (proving the theorem is false) or have an inspiration as to why it is true. Well indeed we find

$$1 = \frac{1(1+1)}{2}$$

$$1 + 2 = \frac{2(2+1)}{2}$$

$$1 + 2 + 3 = \frac{3(3+1)}{2}$$

and we could go on for quite some time. On a computer we could very rapidly check for several million values, each time finding that the formula is valid.

Is this a proof? If a formula works this well for untold millions of values of n how can we conceive that it is false? We would certainly have strong emotional reasons for believing the formula if we have checked it for this many different values but this would not be a mathematical proof.

Instead there is a proof that, at first sight, seems to be just a matter of checking many times. Suppose that the formula does fail for some value of n. Then there must be a first occurrence of the failure, say for some integer N. We know $N \neq 1$ (since we already checked that) and so the previous integer $N-1$ does allow a valid formula. It is the next one N that fails. But if we can show that this never happens, i.e., there is never a situation with $N-1$ valid and N invalid, then we will have proved our formula.

For example here, if the formula

$$1+2+3+\ldots M = \frac{M(M+1)}{2}$$

is valid, then

$$1+2+3+\ldots M+(M+1) = \frac{M(M+1}{2} + (M+1)$$
$$= \frac{M(M+1)+2(M+1)}{2} = \frac{(M+1)(M+2)}{2}$$

which is indeed the correct formula for $n = M+1$. Thus there never can be a situation in which the formula is correct at some stage and fails at the very next stage. It follows that the formula is always true. This is a proof by induction.

This may be used to try to prove any statement about an integer n. Here are the steps:

Step 1 Verify the statement for $n = 1$.

Step 2 (The induction step) Show that whenever the statement is true for any positive integer m it is necessarily also true for the next integer $m+1$.

Step 3 Claim that the formula holds for all n by the principle of induction.

In the exercises you are asked for induction proofs of various statements. You might try too to give direct (noninductive) proofs. Which method do you prefer?

Exercises

A:8.1 Prove by induction that for every $n = 1, 2, 3, \ldots$,

$$1^2+2^2+3^2+\ldots n^2 = \frac{n(n+1)(2n+1)}{6}.$$

A:8.2 Compute for $n = 1, 2, 3, 4$ and 5 the value of

$$1 + 3 + 5 + \cdots + (2n - 1).$$

This should be enough values to suggest a correct formula. Verify it by induction.

A:8.3 Prove by induction for every $n = 1, 2, 3, \ldots$, that the number

$$7^n - 4^n$$

is divisible by 3.

A:8.4 Prove by induction that for every $n = 1, 2, 3, \ldots$,

$$(1 + x)^n \geq 1 + nx$$

for any $x > 0$.

A:8.5 Prove by induction that for every $n = 1, 2, 3, \ldots$,

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

for any real number $r \neq 1$.

A:8.6 Prove by induction for every $n = 1, 2, 3, \ldots$, that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$

A:8.7 Prove by induction that for every $n = 1, 2, 3, \ldots$,

$$\frac{d^n}{dx^n} e^{2x} = e^{2x + n \log 2}.$$

A:8.8 Show that the following two principles are equivalent, i.e., assuming the validity of either one of the them prove the other.

> **(Principle of Induction)** Let $S \subset \mathbb{N}$ such that $1 \in S$ and for all integers n if $n \in S$ then so also is $n + 1$. Then $S = \mathbb{N}$.

and

> **(Well Ordering of $\mathbb{N}$)** If $S \subset \mathbb{N}$ and $S \neq \emptyset$ then S has a first element (i.e., a minimal element).

A:8.9 Criticize the following "proof".

(Birds of a feather flock together) Any collection of n birds must be all of the same species.
Proof: This is certainly true if $n = 1$. Suppose it is true for some value n. Take a collection of $n + 1$ birds. Remove one bird and keep him in your hand. The remaining birds are all of the same species. What about the one in your hand? Take a different one out and replace the one in your hand. Since he now is in a collection of n birds he must be the same species too. Thus all birds in the collection of $n + 1$ birds are of the same species. The statement is now proved by induction.

A.9 Quantifiers

In all of mathematics and certainly in all of analysis the student will encounter two phrases used repeatedly:

For all ... it is true that ...

and

There exists a ... so that it is true that ...

For example the formula $(x+1)^2 = x^2 + 2x + 1$ is true *for all* real numbers x. *There is a* real number x such that $x^2 + 2x + 1 = 0$ (indeed $x = -1$).

It is extremely useful to have a symbolic way of writing this. It is universal for mathematicians of all languages to use the symbol $\forall$ to indicate "for all" or "for every" and to use

$\exists$ to indicate "there exists". Originally these were chosen since it was easy enough for typesetters to turn the characters "A" and "E" around or upside down. These are called by the logicians *quantifiers* since they answer (vaguely) the question "how many?". For how many x is it true that $(x+1)^2 = x^2 + 2x + 1$? For all real x. In symbols

$$\forall\, x \in \mathbb{R},\ (x+1)^2 = x^2 + 2x + 1.$$

For how many x is it true that $x^2 + 2x + 1 = 0$? Not many, but there do exist numbers x for which this is true. In symbols

$$\exists\, x \in \mathbb{R},\ x^2 + 2x + 1 = 0.$$

It is important to become familiar with statements involving one or more quantifiers whether symbolically expressed using $\forall$ and $\exists$ or merely using the phrases "for all" and "there exists". The exercises give some practice. You will certainly gain more familiarity by the time you are deeply into an analysis course in any case.

Negations of quantified statements Here is a tip that helps in forming negatives of assertions involving quantifiers. The two quantifiers $\forall$ and $\exists$ are complementary in a certain sense. The negation of the statement "all birds fly" would be (in conventional language) "some bird does not fly". More formally the negation of

For all birds b, b flies.

would be

There exists a bird b, b does not fly.

In symbols let B be the set of all birds. Then the form here is

$\forall b \in B$ "statement about b" is true

and the negation of this is

$\exists b \in B$ "statement about b" is NOT true

This allows a simple device for forming negatives. The negation of a statement with $\forall$ is a statement with $\exists$ replacing it, and the negation of a statement with $\exists$ is a statement with $\forall$ replacing it. For a complicated example what is the negation of the statement

$\exists a \in A,\ \forall b \in B,\ \forall c \in C$
"statement about a, b and c" is true

even without assigning any meaning? It would be

$\forall a \in A,\ \exists b \in B,\ \exists c \in C,$
"statement about a, b and c" is NOT true.

Exercises

A:9.1 Let $\mathbb{R}$ be as usual the set of all real numbers. Express in words what these statements mean and determine whether they are true or not. Do not give proofs just decide on the meaning and whether you think they are valid or not.

(a) $\forall x \in \mathbb{R}, x \geq 0$.
(b) $\exists x \in \mathbb{R}, x \geq 0$.
(c) $\forall x \in \mathbb{R}, x^2 \geq 0$.
(d) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = 1$.
(e) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 1$.
(f) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = 1$.
(g) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 1$.
(h) etc

A:9.2 Form the negations of each of the statements in the preceding exercise. If you decided that a statement was true (false) before, you should naturally now agree that the negative is false (true).

A:9.3 Explain what must be done in order to prove an assertion of the following form:

(a) $\forall s \in S$ "statement about s" is true

(b) $\exists s \in S$ "statement about s" is true

Now explain what must be done in order to disprove such assertions.

A:9.4 In the preceding exercise suppose that $S = \emptyset$. Could either statement be true? Must either statement be true?