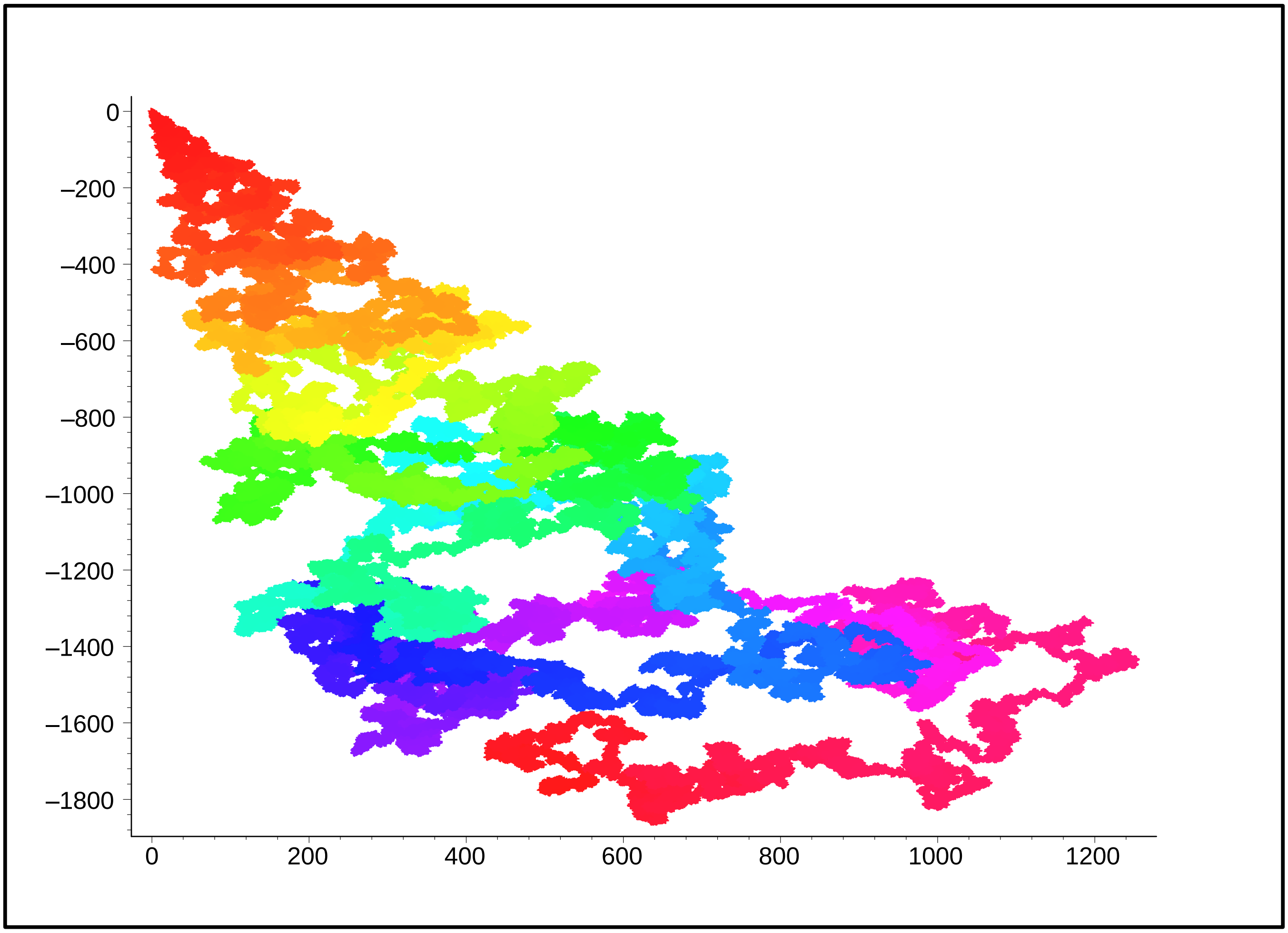
The Riemann Hypothesis

Peter Borwein



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RESEARCH INSTITUTE
UNANSWERED QUESTIONS
UNQUESTIONED ANSWERS
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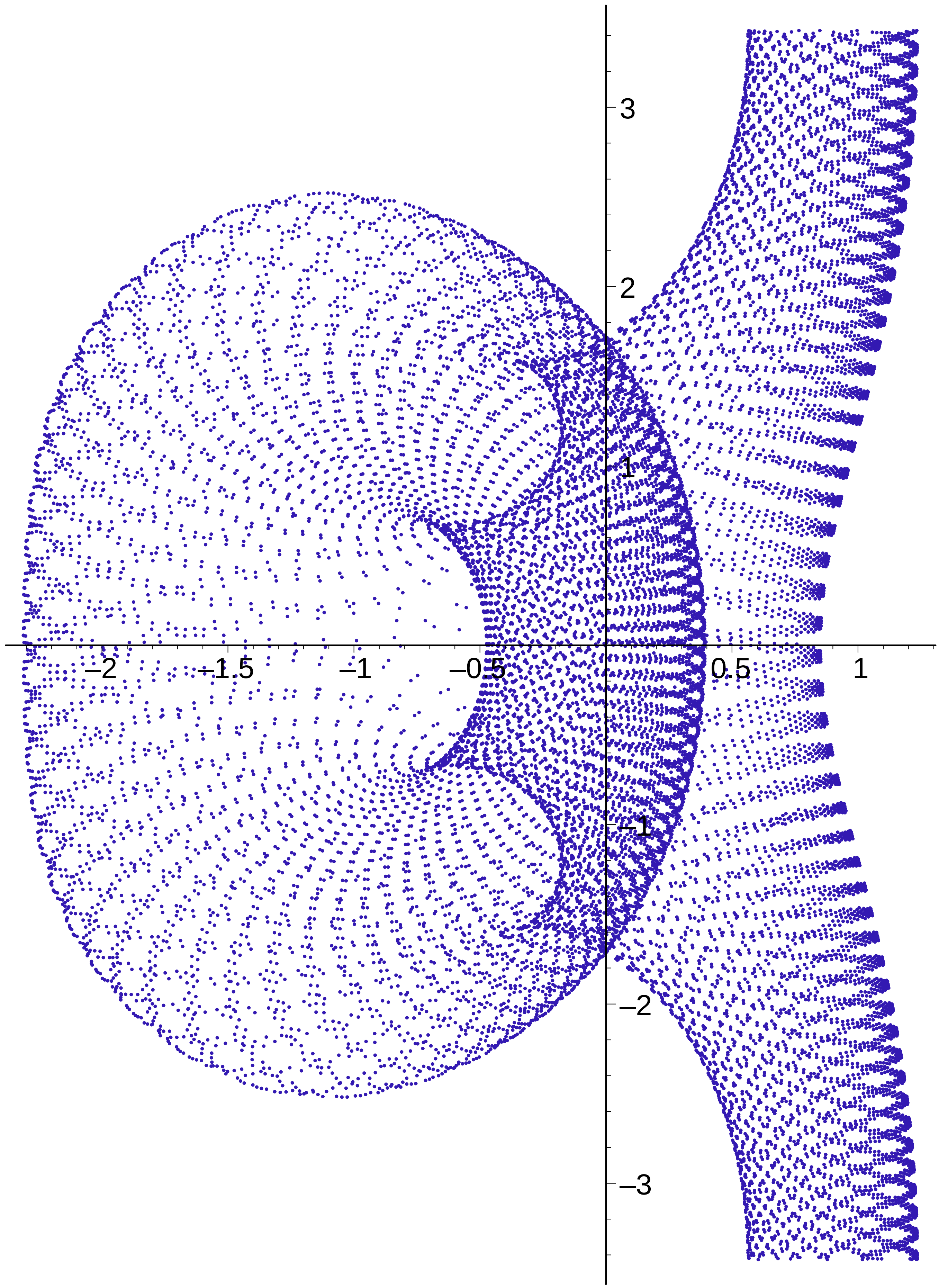

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Millennium Prize Problems

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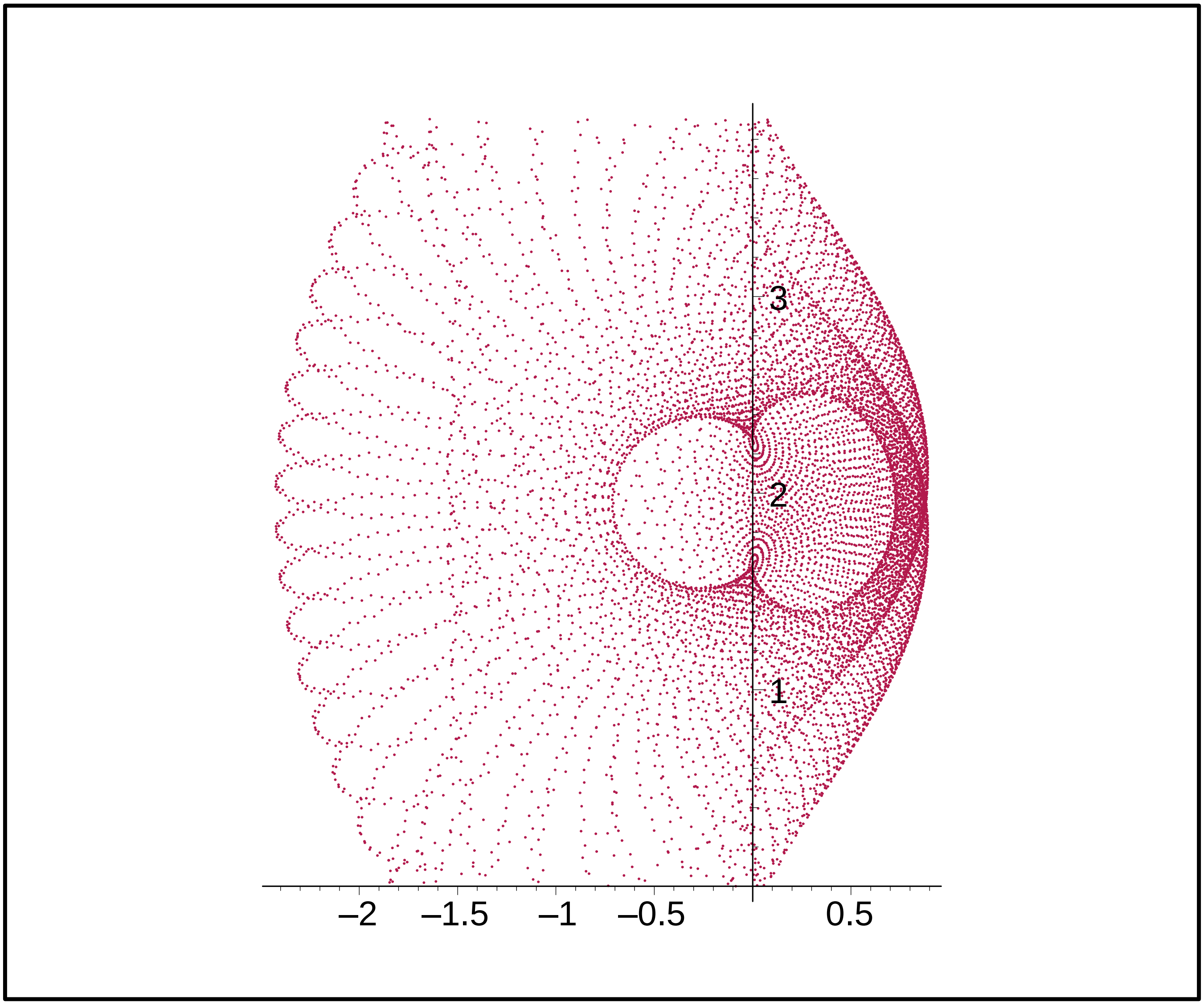
CMI attempts to further the beauty, the power, and the universality of mathematical thought. Toward this end, CMI currently pursues a series of programs."



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6. RIEMANN HYPOTHESIS

Some numbers have the special property that they cannot be expressed as the product of two smaller numbers, e.g., 2, 3, 5, 7, etc. Such numbers are called prime numbers, and they play an important role, both in pure mathematics and its applications.

The distribution of such prime numbers among all natural numbers does not follow any regular pattern, however the German mathematician G.F.B. Riemann (1826 - 1866) observed that the frequency of prime numbers is very closely related to the behavior of an elaborate function $\zeta(s)$ called the Riemann Zeta function.

The Riemann hypothesis asserts that all interesting solutions of the equation

$$\zeta(s) = 0$$

lie on a straight line. This has been checked for the first 1,500,000,000 solutions. A proof that it is true for every interesting solution would shed light on many of the mysteries surrounding the distribution of prime numbers.

More about the RIEMANN HYPOTHESIS

Ask any professional mathematician to name the most important unsolved problem of mathematics and the answer is virtually certain to be, “the Riemann Hypothesis.”

Keith Devlin – The Millennium Problems – 2002

On the Number of Prime Numbers less than a Given Quantity.

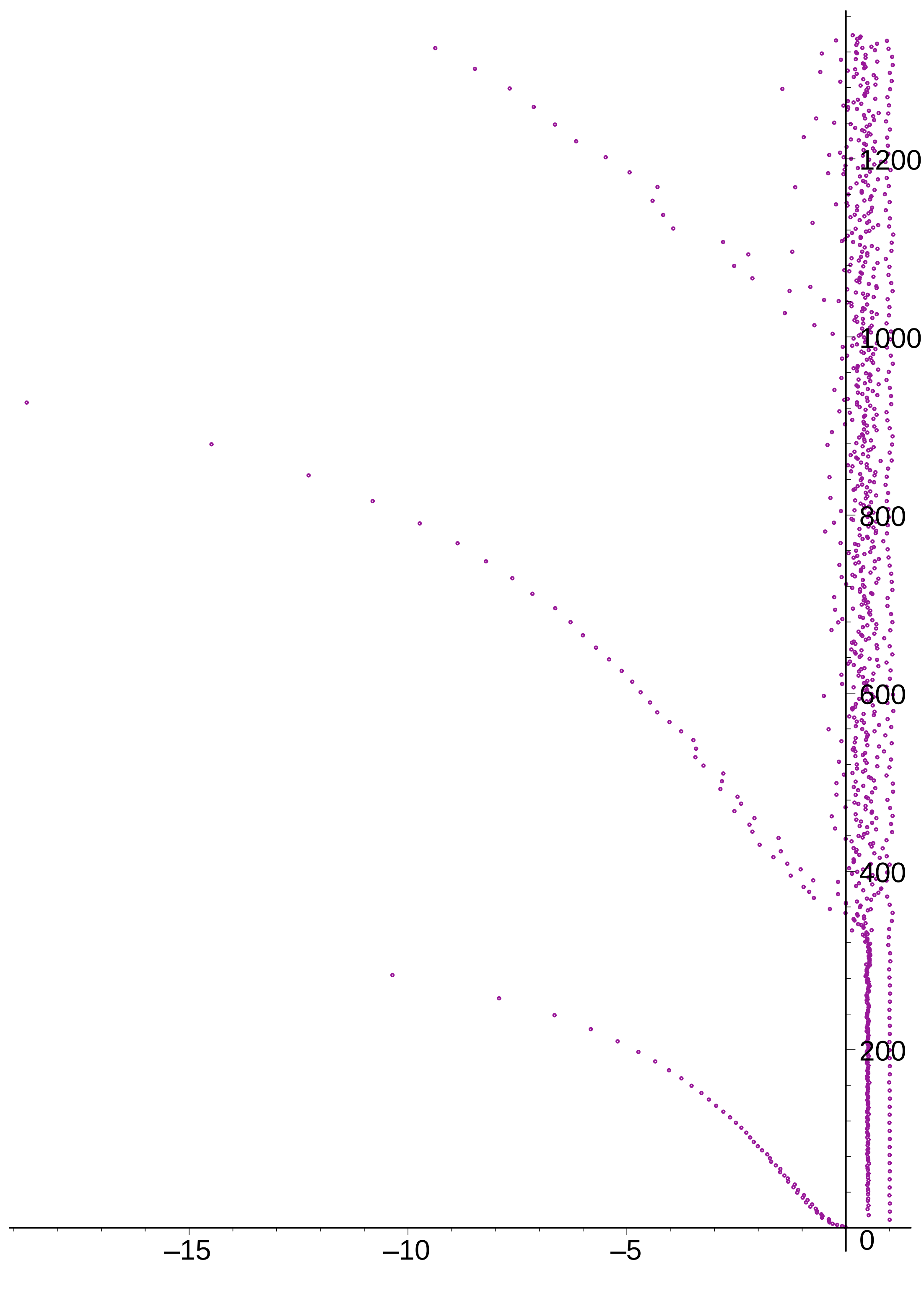
(Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.)

Bernhard Riemann

[Monatsberichte der Berliner Akademie, November 1859.]

Translated by David R. Wilkins

"One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation."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The Holy Grail

The Holy Grail in mathematics is the Riemann Hypothesis. The problem, formulated in 1859 by Bernard Riemann, one of the extraordinary mathematical talents of the 19th century, makes a very precise connection between two seemingly unrelated objects, and if solved, would tell us something profound about the nature of mathematics and, in particular, prime numbers.

Why is the Riemann Hypothesis so important? Why is it the problem that mathematicians would make a pact with the devil to solve?

There are a number of great old unsolved problems in mathematics but none of them have quite the stature of the Riemann Hypothesis – for a variety of reasons both mathematical and cultural.

In common with the other old great unsolved problems, the Riemann Hypothesis is clearly very hard. It has resisted solution for 150 years and has been attempted by many of the greatest minds in mathematics.

A feature of the mathematics related to the Riemann Hypothesis is that certain phenomena that appear likely true and that can be tested in part computationally are false but only false past computational range.

Accept for a moment that the Riemann Hypothesis is the greatest unsolved problem in mathematics and that the greatest achievement any young graduate student could aspire to is to solve it. Why isn't it better known? Why hasn't it permeated public consciousness? (The way black holes and unified field theory have, at least to some extent.)

Part of the reason for this is it is hard to state precisely. It requires most of an undergraduate degree in mathematics to be familiar with enough of the mathematical objects to even accurately state the Riemann Hypothesis. Our suspicion is that only a minority of professional mathematicians –perhaps a quarter – can state the Riemann Hypothesis if asked.

The Liouville λ function and RH

For $\Re s > 1$, the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

The Riemann Hypothesis is usually given as: the nontrivial zeros of the Riemann zeta function lie on the line $\Im s > 1$.

(There is already, of course, the problem that the above series doesn't converge on this line so one is already talking about an analytic continuation.)

Our immediate goal is to give as simple an (equivalent) statement of the Riemann Hypothesis as we can.

Loosely the statement is " the number of integers with an even number of prime factors is the same as the number of integers with an odd number of prime factors."

This is made precise in terms of the Liouville Function.

The Riemann Hypothesis is equivalent to the statement that an integer has equal probability of having an odd number or an even number of distinct prime factors, a statement with some intuitive appeal.

The Liouville Function gives the parity of the number of prime factors.

The **Liouville Function** is defined by

$$\lambda(n) = (-1)^{\omega(n)}$$

where $\omega(n)$ is the number of distinct prime factors in n with multiple factors counted multiply.

So

$$\lambda(1) = \lambda(2) = \lambda(5) = \lambda(7) = \lambda(8) = -1$$

and

$$\lambda(4) = \lambda(6) = \lambda(9) = \lambda(10) = 1.$$

(Alternatively one can define λ as the completely multiplicative function with $\lambda(p) = -1$ for any prime p.)

The connection between the Liouville function and the Riemann Hypothesis were explored by Landau in his doctoral thesis of 1899.

Theorem 1 *The Riemann Hypothesis is equivalent to*

$$\gamma(n) := \lambda(1) + \lambda(2) + \cdots + \lambda(n) \ll n^{1/2+\epsilon},$$

for every positive ϵ.

This is saying that the sequence

$$\{\lambda(i)\}_{i=1} := \{1, -1, -1, 1, -1, \ldots\}$$

behaves more-or-less like a random sequence of of plus and minus ones in that the difference between the number of plus one's and minus ones is not much larger that the square root of the number of terms.

The proof of the equivalence is relatively easy. We give the proof that the growth of $\gamma(n)$ implies the Riemann Hypothesis.

proof It is well known (Hardy and Wright p 255) and not very hard that

$$\frac{\zeta(2s)}{\zeta(s)} = \frac{1}{\prod_{p\,prime}(1+p^{-s})}$$

$$= 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s} - \frac{1}{5^s} - \cdots$$

$$= \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}.$$

and we have

$$\frac{\zeta(2s)}{\zeta(s)} := \int_0^{\infty} x^{-s} dW(x)$$

where W is the Stieltjes measure defined as follows. W is a step function, $W(0) = 0$ and W has a jump of $\lambda(n)$ at n. Also

$$\frac{W(n-\epsilon) + W(n+\epsilon)}{2} = \frac{1}{2}\lambda(n) + \sum_{j=1}^{n-1} \lambda(j).$$

Now, if, for some $\delta > 0$

$$|W(x)| \ll x^{\delta}$$

then the above integral actually converges for $\Re(s) > \delta$. So

$$\frac{\zeta(2s)}{\zeta(s)} = s\int_0^{\infty} W(x)x^{-s-1}dx$$

continues analytically for $\Re(s) > \delta$ and thus $\zeta(s)$ can't vanish here.

Landau in his doctoral thesis of 1899 also proved the following.

Theorem 2 (Landau)

$$\frac{\lambda(1) + \lambda(2) + \cdots + \lambda(n)}{n} \to 0$$

is equivalent to the Prime Number Theorem

This can be made the basis for an elementary (though not easy)proof of the Prime Number Theorem.

One also has

Theorem 3 (Landau)

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n}$$

converges is equivalent to the the Prime Number Theorem.

Turán's conjecture

Turán conjectured that that for all n

$$\sum_{i=1}^{n} \frac{\lambda(i)}{i} > 0.$$

This would imply the Riemann Hypothesis. However it is provably false.

Though no actual counterexample was known til now. It is true at least up to $n = 10^{12}$ and is a cautionary example on not trusting the numbers.

The first place where the sum hits zero is

$$n = 72196252762111.$$

Opinions and Quotations

One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation.

Bernard Riemann 1859.

If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?

David Hilbert.

Whoever can prove either the truth or the falsehood of this conjecture [The Riemann Hypothesis] will cover himself in glory.

C. de la Vallee Poussin, 1916.

There have probably been very few attempts at proving the Riemann hypothesis, because, simply, no one has ever had any really good idea for how to go about it!.

Atle Selberg.

I believe this to be false. There is no evidence whatever for it (unless one counts that it is always nice when any function has only real roots). One should not believe things for which there is no evidence. In the spirit of this anthology I should also record my feeling that there is no imaginable reason why it should be true. Titchmarsh devised a method, of considerable theoretical interest, for calculating the zeros. The method reveals that for a zero to be off the critical line a remarkable number of 'coincidences' have to happen. I have discussed the matter with several people who know the problem in relation to electronic calculation; they are all agreed that the chance of finding a zero off the line in a lifetime's calculation is millions to one against.

It looks then as if we may never know. It is true that the existence of an infinity of L-functions raising the same problems creates a remarkable situation.

Nonetheless life would be more comfortable if one could believe firmly that the hypothesis is false.

J. Littlewood, 1962.

It wouild be very discouraging if somewhere down the line you could ask a computer if the Riemann hypothesis is correct and it said, 'Yes, it is true, but you won't be able to understand the proof.'

Ronald Graham

The failure of the Riemann hypothesis would create havoc in the distribution of prime numbers. This fact alone singles out the Riemann hypothesis as the main open question of prime number theory.

E. Bombieri

Mother Nature has such beautiful harmonies, so you couldn't say that something like [the Riemann Hypothesis] is false.

H. Iwaniec

Right now, when we tackle problems without knowing the truth of the Riemann hypothesis, it's as if we have a screwdriver. But when we have it, it'll be more like a bulldozer.

P. Sarnak

You must know that Hardy had a running feud with God. In Hardy's view God had nothing more important to do than frustrate Hardy.

This led to a sort of insurance policy for Hardy one time when he was trying to get back to Cambridge after a visit to [Herald] Bohr in Denmark. The weather was bad and there was only a small boat available. Hardy thought there was a real possibility the boat would sink. So he sent a postcard to Bohr saying, "I proved the Riemann Hypothesis. G.H. Hardy." That way if the boat sank, everyone would think that Hardy had proved the Riemann Hypothesis. God could not allow so much glory for Hardy so he could not allow the boat to sink.

George Polya

Advice ...

If you want to climb the Matterhorn, you might first wish to go to Zermat, where those who have tried are buried

George Polya

...I don't believe or disbelieve the Riemann Hypothesis. I have a certain amount of data and a certain amount of facts. These facts tell me definitely that the thing has not been settled. Until it's been settled it's a hypothesis, that's all. I would like the Riemann Hypothesis to be true, like any decent mathematician, because it's a thing of beauty, a thing of elegance, a thing that would simplify many proofs and so forth, but that's all.

A. Ivic

Now, fifty years after the publication of Riemann's great paper "On the number of prime numbers less than a given quantity", we have only just begun to understand and absorb what Riemann's supremely creative imagination produced. Progress along the path that Riemann blazed so fearlessly has been hesitant and slow; and the justly famous hypothesis that lies at the kernel of that thesis has resisted all efforts at proof.

E. Landau, 1909

So for all practical purposes, the Riemann zeta function does not show its true colours in the range available by numerical investigations. You should go up to the height 10^{10000} then I would be much more convinced if things were still pointing strongly in the direction of the Riemann Hypothesis. So numerical calculations are certainly very impressive, and they are a triumph of computers and numerical analysis, but they are of limited capacity. The Riemann Hypothesis is a very delicate mechanism. It works as far as we know for all existing zeros, but we cannot, of course, verify numerically an infinity of zeros, so other theoretical ways of approach must be found, and for the time being they are insufficient to yield any positive conclusion.

A. Ivic

Timeline

1738,1742 Euler and Maclaurin invent the "Euler-Maclaurin summation".

1742 Goldbach proposes the Goldbach Conjecture to Euler in a letter.

* **1792** Gauss proposes the Prime Number Theorem.

1802 Haros discovers and proves results concerning the general properties of Farey Series.

1816 Farey re-discovers Farey series and is given credit for their invention. Hardy writes,
"*Farey is immortal because he failed to understand a theorem which Haros had proved perfectly fourteen years before* ."

1845 Bertrand postulates that for $a > 1$ there is always a prime that lies between a and $2a$.

* **1850** Chebyshev proves Bertrand's Postulate using elementary methods.

* **1859** Riemann publishes his *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* in which he proposes the RH.

"*One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation* "

Riemann's paper contains the functional equation,

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

$$= \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

and the formula,

$$\xi(t) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

for $s = \frac{1}{2} + it$.

His work also contains analysis of the Riemann zeta function.

Some of Riemann's analysis is overlooked until it is re-invigourated by Siegel (see 1932).

* **1885** Stieltjes claims to have proof of what would later be called the Mertens Conjecture. His proof is never published, nor found amongst his papers posthumously.

The Mertens Conjecture implies the Riemann Hypothesis. This is perhaps the first significant failed attempt at a proof.

1890-1920 Sometime in this range Lindelöf proposed the Lindelöf Hypothesis. His hypothesis is weaker than the actual Hypothesis, and concerns the distribution of the zeros of Riemann's zeta.

It is still unproven.

* **1896** Hadamard and de la Vallée Poussin independently prove the Prime Number Theorem. The proof relies on showing that $\zeta(s)$ has no zeros of the form $1 + it$ for $t \in \mathbb{R}$.

1897 Mertens publishes the Mertens Conjecture. (This conjecture is proved incorrect by Odlyzko and te Riele, see 1985).

1901 von Koch publishes that the Riemann Hypothesis is equivalent to the statement that,

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(\sqrt{x} \log x).$$

* **1903** Gram calculates the first 15 zeros of $\zeta(s)$ on the critical line.

* **1903-1912** Gram, Backlund and Hutchinson independently use Euler-Maclaurin summation to calculate $\zeta(s)$ and to verify the RH for $t \leq 300$ (where $s = \frac{1}{2} + it$).

1912 Littlewood proves that the Mertens Conjecture implies the RH.

1912 Littlewood proves that $\pi(n) < Li(n)$ fails for some n.

1912 Backlund develops a method of determining the number of zeros of $\zeta(s)$ in the critical strip $0 < \Re(s) < 1$ up to a given height. This method is used through 1932.

* **1914** Backlund calculates the first 79 zeros of $\zeta(s)$ on the critical line.

* **1914** Littlewood proves that $\pi(n) < Li(n)$ fails for infinitely many n.

1914 Bohr and Landau prove that if $N(\sigma, T)$ is the number of zeros of $\zeta(s)$ in the rectangle $0 \leq \Im(s) \leq T$, $\frac{1}{2} \leq \sigma \leq 1$ then $N(\sigma, T) = O(T)$ for any fixed σ.

* **1919** Pólya conjectures that the summatory Liouville function, $L(x)$, satisfies $L(x) \leq 0$ for $x \geq 2$. (This conjecture is proved incorrect by Haselgrove, see 1958).

1920 Carlson proves the density theorem.

$N(\sigma, T) = O(T^{4\sigma(1-\sigma)+\varepsilon})$ for any fixed $\varepsilon > 0$, $\frac{1}{2} \leq \sigma \leq 1$.

1922 Hardy and Littlewood show that the gRH implies Goldbach's Weak Conjecture.

* **1923** Hardy and Littlewood prove that if the gRH is true, then almost all even numbers are the sum of two primes.

  Specifically, if $E(N)$ denotes the number of even integers, $n < N$, that are not the sum of two primes, then $E(N) \ll N^{\frac{1}{2}+\varepsilon}$.

1924 Franel and Landau discover an equivalence to the RH involving Farey series. The details are not complicated, but are rather lengthy.

1925 Hutchinson calculates the first 138 zeros of $\zeta(s)$ on the critical line.

1928 Littlewood shows that assuming the gRH gives bounds on $L_D(1,\chi)$, where $L_D(1,\chi)$ is the Dirichlet L-series,

$$L_D(1,\chi) = \sum_{n=1}^{\infty} \frac{\chi_D(n)}{n},$$

and χ_D is a non-principal number theoretic character with modulus D. Littlewood bounds $|L_D(1,\chi)|$ as,

$$\frac{1}{\log\log D} \ll |L_D(1,\chi)| \ll \log\log D.$$

* **1932** Siegel analyzes Riemanns private (and public) papers. He finds (amongst other things) a formula for calculating values of $\zeta(s)$ that is more efficient than Euler-Maclaurin summation. The method is referred to as the Riemann-Siegel Formula and is used in some form up to the present.

  Siegel is credited with re-invigourating Riemann's most important results regarding $\zeta(s)$.

  In the words of Edwards,

  "It is indeed fortunate that Siegel's concept of scholarship derived from the older tradition of respect for the past rather than the contemporary style of novelty."

1934 Speiser publishes that the Riemann Hypothesis is equivalent to the non-vanishing of

$$\zeta'(s)$$

in $0 < \sigma < \frac{1}{2}$.

*** 1935** Titchmarsh calculates the first 1,041 zeros of $\zeta(s)$ on the critical line.

1937 Vinogradov proves the following result related to Goldbach's Conjecture without assuming any variant of the RH.

Every sufficiently large odd number, $N \geq N_0$, is the sum of 3 prime numbers.

1940 Ingham shows that

$$N(\sigma, T) = O(T^{3\frac{1-\sigma}{2-\sigma}} \log^5 T).$$

This is still the best known result for $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$.

1942 Ingham publishes a paper building on the conjectures of Mertens and Pólya. He proves that not only do both conjectures imply the truth of the Hypothesis, and the simplicity of the zeros, but they also imply a linear dependence between the imaginary parts of the zeros.

1943 Alan Turing publishes two important developments. The first is an algorithm for computing $\zeta(s)$ (made obsolete by better estimates to the error terms in the Riemann-Siegel Formula). The second is a method for calculating $N(T)$, and gives a powerful tool for verifying the RH up to a given height.

* **1943** Time magazine publishes a short article detailing a recent failed attempt at a proof of the RH. The proof was submitted for review and publication to Transactions of the American Mathematical Society by Hans Rademacher and subsequently withdrawn.

* **1948** Turán shows that if for all N sufficiently large, the N^{th} partial sum of $\zeta(s)$ does not vanish for $\sigma > 1$ then the Riemann Hypothesis follows.

* **1948** Weil proves that the Riemann Hypothesis is true for function fields.

* **1949–1950** Selberg and Erdös independently find an 'elementary' proof of the Prime Number Theorem. This adds to the heuristic evidence supporting the RH (as the RH implies the PNT).

* **1953** Turing calculates the first 1,104 zeros of $\zeta(s)$ on the critical line.

* **1955** Skewes bounds the first n such that $\pi(n) < Li(n)$ fails. This bound is improved in the future, but retains the name 'Skewes number'.

1955 A. Beurling finds the Nyman-Beurling equivalent form.

* **1956** Lehmer calculates the first 15,000 zeros of $\zeta(s)$ on the critical line.

* **1956** Lehmer calculates the first 25,000 zeros of $\zeta(s)$ on the critical line.

* **1958** Meller calculates the first 35,337 zeros of $\zeta(s)$ on the critical line.

1958 Haselgrove disproves Pólya's conjecture.

1966 Lehman improves Skewes bound.

* **1966** Lehman calculates the first 250,000 zeros of $\zeta(s)$ on the critical line.

1967 Hooley proves that Artin's Conjecture holds under the assumption of the eRH. Artin's Conjecture is,

Every $a \in \mathbb{Z}$, where a is not square and $a \neq -1$, is a primitive root modulo p for infinitely many primes p.

* **1968** Rosser, Yohe and Schoenfeld calculate the first 3,500,000 zeros of $\zeta(s)$ on the critical line.

1968 Louis de Branges makes the first of his several failed attempts to prove the RH.

1973 Montgomery conjectures that the correlation for the zeros of the zeta function is,

$$1 - \frac{\sin^2(\pi x)}{(\pi x)^2}.$$

* **1973** Chen proves that every sufficiently large even integer is the sum of a prime and the product of at most two primes.

1974 The probabilistic Solovay-Strassen algorithm for primality testing is published. It can be made deterministic under the gRH.

1975 The probabilistic Miller-Rabin algorithm for primality testing is published. It runs in polynomial time under the gRH.

1977 Redheffer shows that the Riemann Hypothesis is equivalent to the statement that,

$$\det(R_n) = \sum_{k=1}^{n} \mu(k)$$

where R_n is the $n \times n$ matrix with entries,

$$(i,j) = \begin{cases} 1 & \text{if } j = 1 \text{ or if } i|j \\ 0 & \text{otherwise.} \end{cases}$$

* **1977** Brent calculates the first 40,000,000 zeros of $\zeta(s)$ on the critical line.

* **1979** Brent calculates the first 81,000,001 zeros of $\zeta(s)$ on the critical line.

* **1982** Brent, van de Lune, te Riel and Winter calculate the first 200,000,001 zeros of $\zeta(s)$ on the critical line.

1983 van de Lune and te Riele calculate the first 300,000,001 zeros of $\zeta(s)$ on the critical line.

1983 H. Montgomery proves that the 1948 approach of Turán will not lead to a proof of the RH. This is because for any positive $c < \frac{4}{\pi} - 1$ the N^{th} partial sum of $\zeta(s)$ has zeros in the half-plane $\sigma > 1 + c\frac{\log\log N}{\log N}$.

1984 Ram Murty and Gupta prove that Artin's Conjecture holds for infinitely many a without assuming any variant of the Riemann Hypothesis.

* **1985** Odlyzko and te Riele prove that the Mertens conjecture is false. They speculate that, while not impossible, it is improbable that $M(n) = O(n^{\frac{1}{2}})$. The Riemann Hypothesis is in fact equivalent to the conjecture, $M(n) = O(n^{\frac{1}{2}+\varepsilon})$

* **1986** van de Lune, te Riele and Winter calculate the first 1,500,000,001 zeros of $\zeta(s)$ on the critical line.

1986 de Branges publishes another 'proof'.

1986 Heath-Brown proves that Artin's Conjecture fails for at most two primes.

1988 Odlyzko and Schönhage publish an algorithm for calculating values of $\zeta(s)$. The Odlyzko-Schönhage algorithm is currently the most efficient algorithm for determining values $t \in \mathbb{R}$ for which $\zeta(\frac{1}{2} + it) = 0$. It uses the fact that the most computationally complex part of the evaluation using the Riemann-Siegel formula are computations of the form,

$$g(t) = \sum_{k=1}^{M} k^{-it}.$$

The proposed algorithm makes use of the FFT to convert sums of this type into rational functions.

The algorithm presented computes the first n zeros of $\zeta(\frac{1}{2} + it)$ in $O(n^{1+\varepsilon})$ (as opposed to $O(n^{\frac{3}{2}})$ using previous methods).

1988 Barratt, Forcade and Pollington formulate a graph theoretic equivalent to the Riemann Hypothesis through Redheffer matrices.

* **1989** Odylzko computes 175 million consecutive zeros around $t = 10^{20}$.

* **1989** Conrey proves that more than 40% of the nontrivial zeros of $\zeta(s)$ lie on the critical line.

1992 de Branges publishes again.

1993 Julio Alcántara-Bode shows that the Riemann Hypothesis is true if and only if the operator A_ρ is injective. A_ρ is the Hilbert-Schmidt integral operator on $L^2(0,1)$ given by,

$$[A_\rho f](\theta) = \int_0^1 \rho\left(\frac{\theta}{x}\right) f(x)dx.$$

1994 de Branges publishes again.

1994 Verjovsky proves that the Riemannn Hypothesis is equivalent to a problem about the rate of convergence of certain discrete measures.

1995 Volchov proves that the statement,

$$\int_0^\infty (1-12t^2)(1+4t^2)^{-3} \int_{\frac{1}{2}}^\infty \ln|\zeta(\sigma+it)|$$

$$= \frac{\pi(3-\gamma)}{32}$$

is equivalent to the Riemann Hypothesis, where γ is Euler's constant.

1997 Hardy and Littlewood's 1922 result concerning Goldbach's Conjecture is improved by Deshouillers, Effinger, te Riele and Zinoviev. They prove,

Assuming the gRH, every odd number greater than 5 can be expressed as a sum of 3 prime numbers.

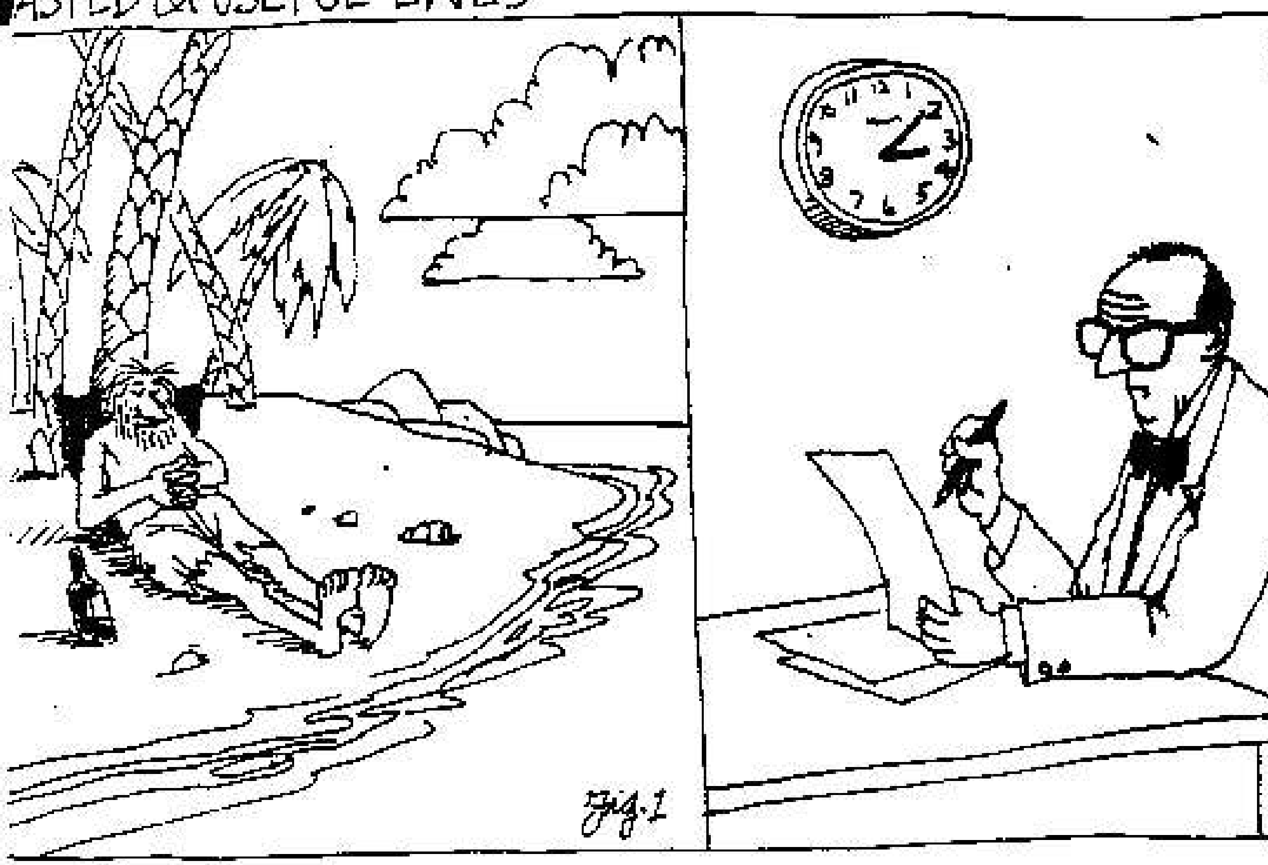
2000 Conrey and Li prove that the approach used by de Branges cannot lead to proof of the RH.

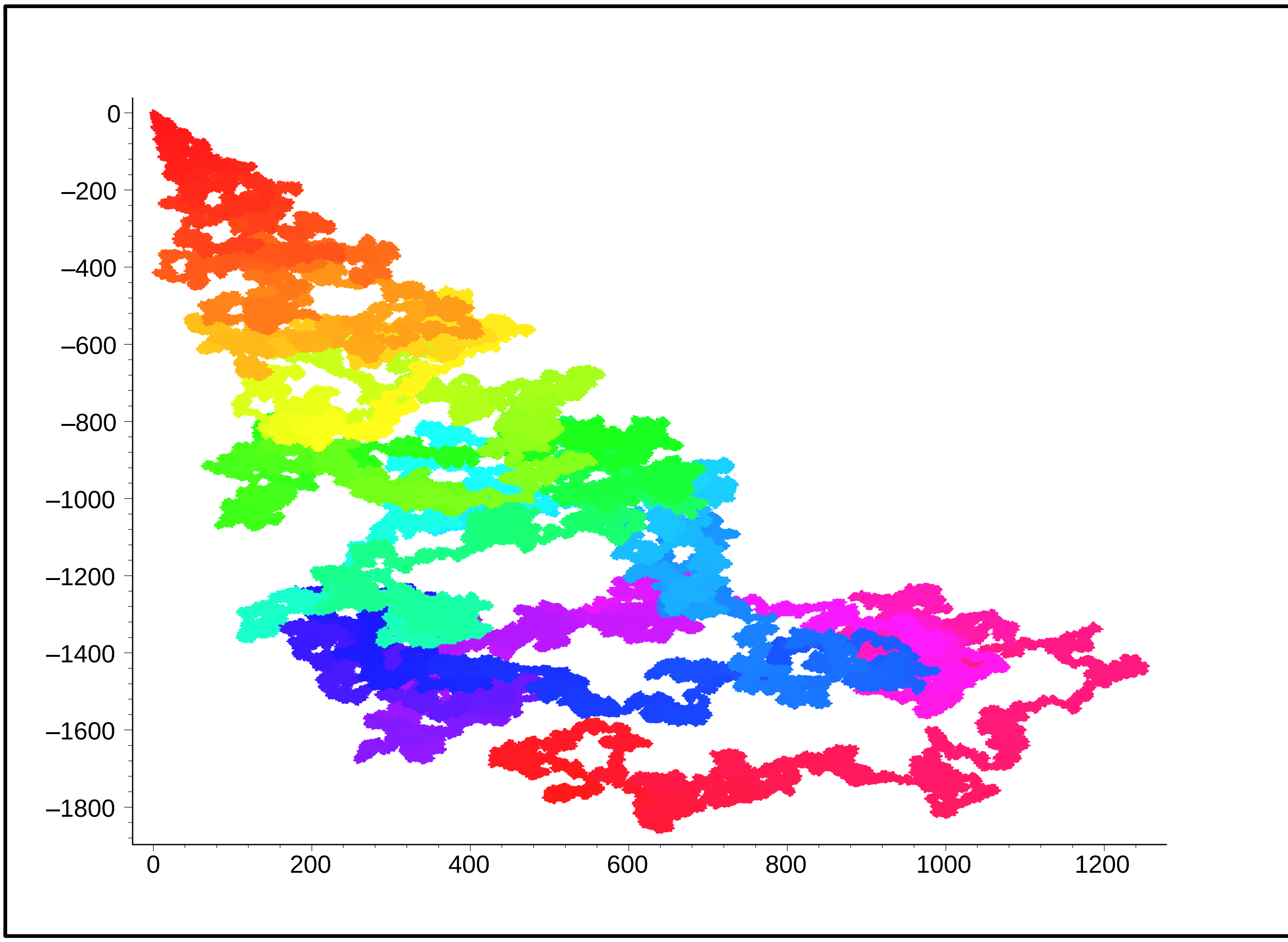
2000 Bays and Hudson lower Skewes number.

* **2001** van de Lune calculates the first 10,000,000,000 zeros of $\zeta(s)$ on the critical line.

* **2005** Wedeniwski calculates the first 1,000,000,000,000 zeros of $\zeta(s)$ on the critical line.

WASTED & USEFUL LIVES





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