THE 40 TRILLIONTH BINARY DIGIT OF PI IS 0

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Typeset by $\mathcal{AMS}$-TEX

"I AM ASHAMED TO TELL YOU TO HOW MANY FIGURES I CARRIED THESE COMPUTATIONS, HAVING NO OTHER BUSINESS AT THE TIME."

Isaac Newton 1665–1666

$$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{12} - \frac{1}{5\cdot 2^5} - \frac{1}{28\cdot 2^7} - - \cdots\right)$$

$$= \frac{3\sqrt{3}}{4} + 24\int_0^{\frac{1}{4}} \sqrt{x - x^2}\, dx$$

Prehistory

"Also he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about."

Old Testament, 1 Kings 7:23

Various Babylonian and Egyptian estimates include

$$\pi \doteq 3, 3\ \frac{1}{7},\ 3\frac{1}{8},\ 3\frac{13}{81}, \ldots$$

The First Age of Pi (Archimedes)

$$a_{n+1} := \frac{2a_n b_n}{a_n + b_n} \qquad a_0 := 2\sqrt{3}$$

$$b_{n+1} := \sqrt{a_{n+1} \cdot b_n} \qquad b_0 := 3$$

This computes the length of circumscribed and inscribed regular $6 \cdot 2^n$-gons.

- Archimedes, 3–6 Digits

- Ludolph van Ceulen, 34 Digits, (1540–1610)

The Second Age of Pi (Calculus)

$\arctan(x) := x - x^3/3 + x^5/5 - \cdots$

$\pi = 16 \arctan(1/5) - 4 \arctan(1/239)$

With various other Machin-like formulae

- 100 Digits – Machin (1706)
- 205 Digits – Dase (1844)
- 707 Digits (527 correct) – Shanks (1853)
- 2037 Digits – ENIAC (1949) – (von Neuman et al)
- 1-million Digits – CDC 7600 (1973) – (Guilloud and Bouyer)

The 3rd Age (higher order methods)

- 17-million Digits,
Symbolics 3670 (1985), (Gosper)

- 29-million Digits,
Cray-2 (1986), (Bailey)

- 1-4 Billion Digits,
Home-brew, (Chudnovskys)

- 1-200 Billion Digits,
various Hitachi, (Kanada)

The Current State of our Knowledge

We know that ...

- π is irrational. Lambert (1761)
- π is transcendental. Lindemann (1882)
- π is not Liouville. Mahler (1953)
- $\pi+\sqrt{2}\log 2$ transcendental. Baker(1966)
- Many billions of digits behave regularly.
- 123456789 appears at the 523,551,502th digit.
- The billionth digit of π is 9.

The Current State of Our Ignorance

We do not know ...

- The decimal expansion of π contains infinitely many 2's (or anything else concerning normality).

- The continued fraction expansion is unbounded.

- $\pi + e$ is transcendental (or even irrational).

- $\pi / \log \pi$ is transcendental (or even irrational).

- whether π is computable as fast as multiplication.

Bill No. 246, 1897. State of Indiana.

"Be it enacted by the General Assembly of the State of Indiana: It has been found that the circular area is to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side."

"In further proof of the value of the author's (E.J. Goodman, M.D.) proposed contribution to education, and offered as a gift to the State of Indiana, is the fact of his solutions of the trisection of the angle, duplication of the cube and quadrature of the circle having been already accepted as contributions to science by the American Mathematical Monthly, the leading exponent of mathematical thought in this country."

Passes three readings, Indiana House, 1897. (Via House Committee on Swamp Lands.)

Passes first reading, Indiana Senate, 1897. (Via Senate Committee on Temperance.)

"The case is perfectly simple. If we pass this bill which establishes a new and correct value of π, the author offers our state without cost the use of this discovery and its free publication in our school textbooks, while everyone else must pay him a royalty."

Professor C.A. Waldo of Purdue intercedes and the bill is tabled.

François Viète (1593)

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots}}$$

John Wallis (1655)

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdots}$$

William Brouncker (1620–1684)

$$\pi = \frac{4}{1+\frac{1}{2+\frac{9}{2+\frac{25}{2+\cdots}}}}$$

James Gregory (1671), Leibnitz (1674)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

How Big is 100 Billion

- 100 billion (12pt) digits stretch from Halifax to Vancouver ten times.

- 100 billion (12pt) digits fill 120 football fields.

- 100 billion digits read off at 3 digits/second take a millenium

- 100 billion is roughly the U.S. national debt in thousand dollar bills.

- 100 billion fill roughly 24000 Bibles.

High Order Iterative Algorithms

The arithmetic-geometric mean iteration:

$$a_0 := 1, \quad b_0 := \sqrt{2}$$

and

$$a_{n+1} := \frac{a_n + b_n}{2}, \quad b_{n+1} := \sqrt{a_n b_n}$$

Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n =$$

$$\frac{\pi/2}{\int_0^1 \frac{dt}{\sqrt{1-t^4}}}$$

In 1799, Gauss observed this purely numerically and wrote that this result

“will surely open a whole new field of analysis.”

Equivalent Modular Parameterization

This is equivalent to the identities

$$\theta_3(q^2) = \frac{\theta_3(q) + \theta_4(q)}{2}$$

and

$$\theta_4(q^2) = \sqrt{\theta_3(q)\theta_4(q)}$$

where

$$\theta_3(q) = \sum_{-\infty}^{\infty} q^{n^2}$$

$$\theta_4(q) = \sum_{-\infty}^{\infty} (-q)^{n^2}$$

These are modular forms. So for example

$$\sqrt{s}\,\theta_3\left(e^{-\pi s}\right) = \theta_3\left(e^{-\pi/s}\right)$$

Quartic Algorithm. Let $\alpha_0 := 6-4\sqrt{2}$, $y_0 := \sqrt{2} - 1$,

$$y_{n+1} := \frac{1-(1-y_n^4)^{1/4}}{1+(1-y_n^4)^{1/4}}$$

$$\alpha_{n+1} :=$$

$$(1+y_{n+1})^4\alpha_n - 2^{2n+3}y_{n+1}(1+y_{n+1}+y_{n+1}^2)$$

Then $\alpha_n \to \frac{1}{\pi}$ quartically.

• One hundred billion digits requires a couple of hundred **full precision** additions, multiplications, multiplications, divisions, and root extractions. Eighteen iterations give at least 100-billion digits and has been used in a number of the recent records.

- Multiplication is the key. Division and root extraction are Newton's method.

Ramanujan Type Series

Ramanujan's remarkable series for $1/\pi$ include

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{[1103 + 26390n]}{(4 * 99)^{4n}}.$$

This series adds roughly eight digits per term.

Gosper in 1985 computed 17 million terms of the continued fraction for π using this.

Such series exist because various modular invariants are rational (which is more-or-less equivalent to identifying those imaginary quadratic fields of class number 1).

Chudnovskys' series with $d = -163$ is

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!}{(n!)^3 (3n)!} \frac{13591409 + n545140134}{(640320^3)^{n+1/2}}.$$

Quadratic versions correspond to class number two imaginary quadratic fields. The largest example has $d = -427$ and

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!}{(n!)^3 (3n)!} \frac{(A + nB)}{C^{n+1/2}}$$

where

$$A := 212175710912\sqrt{61} + 1657145277365$$

$$B := 13773980892672\sqrt{61} + 107578229802750$$

$$C := [5280(236674 + 30303\sqrt{61})]^3.$$

This series adds roughly 25 digits per term.

Frauds

Gregory's series for π, truncated at $500,000$ terms gives to forty places

$$4 \sum_{k=1}^{500,000} \frac{(-1)^{k-1}}{2k-1}$$

$$= 3.14159\underline{0}6535897932\underline{40}4626433832\underline{6}9502884197.$$

Only the underlined digits are wrong.

Excessive Fraud

Sum (correct to over 42 billion digits)

$$\left(\frac{1}{10^5} \sum_{n=-\infty}^{\infty} e^{-\frac{n^2}{10^{10}}} \right)^2 \doteq \pi.$$

Conjecture No one will ever know the 10^{1000}th digit of π (or the 10^{51}th).

But there is more to this story...

Individual Digits of Pi

• We give algorithms for the computation of the d-th digit of certain transcendental numbers in various bases.

• These algorithms can be easily implemented (multiple precision arithmetic is not needed), require virtually no memory, and feature run times that scale nearly linearly with the order of the digit desired.

• They make it feasible to compute, for example, the billionth binary digit of log (2) or π on a modest work station in a few hours run time.

These calculations rest on three observations.

• First, the d-th digit of $1/n$ is "easy" to compute.

• Secondly, this allows for the computation of certain polylogarithm and arctangent series.

• Thirdly, very special polylogarithmic ladders exist for certain numbers like π, π^2, $\log(2)$ and $\log^2(2)$.

For example the critical identity for π:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right).$$

• It is widely believed that computing just the d-th digit of a number like π is really no easier than computing all of the first d digits.

• From a bit complexity point of view this may well be true, although it is probably very hard to prove.

• What we will show is that it is possible to compute just the d-th digit of many transcendentals in (essentially) linear time and logarithmic space.

• We are interested in computing in SC, polynomially logarithmic space and polynomial time.

Actually we are most interested in the space we will denote by SC* of polynomially logarithmic space and (almost) linear time (here we want the time $= O(d \log^{O(1)}(d))$).

• It is not known whether division is possible in SC, similarly it is not known whether base change is possible in SC.

• The situation is even worse in SC*, where it is not even known whether multiplication is possible. If two numbers are in SC* (in the same base) then their product computes in time $= O(d^2 \log^{O(1)}(d))$ and is in SC but not obviously in SC*.

• We will show that the class of numbers we can compute in SC* in base b includes all numbers of the form

$$\sum_{k=1}^{\infty} \frac{1}{p(k)b^{ck}}$$

where p is a polynomial with integer coefficients and c is a positive integer.

• Since addition is possible in SC*, integer linear combinations of such numbers are also feasible (provided the base is fixed).

• The algorithm for the binary digits of π, which also shows that π is in SC* in base 2, rests on the following remarkable identity:

Theorem.

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right).$$

Proof. This is equivalent to:

$$\pi = \int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1 - x^8} dx.$$

which on substituting $y := \sqrt{2}x$ becomes

$$\pi = \int_0^1 \frac{16\, y - 16}{y^4 - 2\, y^3 + 4\, y - 4} dy\, .$$

The equivalence follows from the identity

$$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8}\, dx = \int_0^{1/\sqrt{2}} \sum_{i=0}^{\infty} x^{k-1+8i}\, dx$$

$$= \frac{1}{\sqrt{2}^k} \sum_{i=0}^{\infty} \frac{1}{16^i(8i+k)}$$

□

Identities. As usual, we define the m-th polylogarithm L_m by

$$L_m(z) := \sum_{i=1}^{\infty} \frac{z^i}{i^m}, \qquad |z| < 1\,.$$

The most basic identity is

$$-\log(1 - 2^{-n}) = L_1(1/2^n)$$

which shows that $\log(1 - 2^{-n})$ is in SC* base 2 for integer n.

- Less obvious are the identities

$$\pi^2 =$$

$$36L_2(1/2) - 36L_2(1/4) - 12L_2(1/8) + 6L_2(1/64)$$

$$\log^2(2) =$$

$$4L_2(1/2) - 6L_2(1/4) - 2L_2(1/8) + L_2(1/64)\,.$$

These rewrite as

$$\frac{\pi^2}{36} = \sum_{i=1}^{\infty} \frac{a_i}{2^i i^2}, \qquad [a_i] = [\overline{1, -3, -2, -3, 1, 0}]$$

$$\frac{\log^2(2)}{2} = \sum_{i=1}^{\infty} \frac{b_i}{2^i i^2}, \quad [b_i] = [\overline{2, -10, -7, -10, 2, -1}]\,.$$

• Thus we see that π^2 and $\log^2(2)$ are in SC* in base 2.

• These are polylogarithmic ladders in the base 1/2 in the sense of Lewin.

We found them by searching for identities of this type using an integer relation algorithm. We have not found them directly in print. However they follow from known results pretty easily.

- There are several ladder identities involving L_3:

$$35/2\zeta(3) - \pi^2 \log(2) =$$

$$36L_3(1/2) - 18L_3(1/4) - 4L_3(1/8) + L_3(1/64)\,,$$

$$2\log^3(2) - 7\zeta(3) =$$

$$-24L_3(1/2) + 18L_3(1/4) + 4L_3(1/8) - L_3(1/64)\,,$$

$$10\log^3(2) - 2\pi^2 \log(2) =$$

$$-48L_3(1/2) + 54L_3(1/4) + 12L_3(1/8) - 3L_3(1/64)\,.$$

• The favored algorithms for π of the last centuries involved some variant of Machin's 1706 formula:

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239} .$$

There are many related formula but to be useful to us all the arguments of the arctans have to be a power of a common base, and we have not discovered any such formula for π .

• One can however write

$$\frac{\pi}{2} = 2 \arctan \frac{1}{\sqrt{2}} + \arctan \frac{1}{\sqrt{8}}$$

This rewrites as

$$\sqrt{2}\pi = 4f(1/2) + f(1/8) : \quad f(x) := \sum_{i=1}^{\infty} \frac{(-1)^i x^i}{2i+1}$$

and allows for the calculation of $\sqrt{2}\pi$ in SC*.

• Another two identities involving Catalan's constant G, π and log(2) are:

$$G - \frac{\pi \log(2)}{8} = \sum_{i=1}^{\infty} \frac{c_i}{2^{\lfloor \frac{i+1}{2} \rfloor} i^2},$$

where

$$[c_i] = [\overline{1, 1, 1, 0, -1, -1, -1, 0}]$$

and

$$\frac{5}{96}\pi^2 - \frac{\log^2(2)}{8} = \sum_{i=1}^{\infty} \frac{d_i}{2^{\lfloor \frac{i+1}{2} \rfloor} i^2},$$

where

$$[d_i] = [\overline{1, 0, -1, -1, -1, 0, 1, 1}]$$

• Thus $8G - \pi \log(2)$ is also in SC* in base 2.

That G is itself in SC* in base 2 is a recent result of Broadhurst.

The Algorithm.

•We wish to evaluate the n-th base b digit of

$$\sum_{k=1}^{\infty} \frac{1}{p(k)b^{ck}}$$

by evaluating the fractional part of

$$(*)\,. \qquad \sum_{k=1}^{\infty} \frac{b^n}{p(k)b^{ck}}$$

Here p is a simple polynomial and c is a fixed integer.

• Evaluating the fractional part of the second sum will evaluate the first sum to as many base b digits after the n-th place as the precision of the calculation.

• The keys are that the fractional part of (*) is the same as the fractional part of

$$(**) \qquad \sum_{k=1}^{\infty} \frac{b^{n-ck} \bmod p(k)}{p(k)}$$

and that $b^{n-ck} \bmod p(k)$ can be evaluated quickly.

• Fast evaluation of $b^{n-ck} \bmod p(k)$ is well understood; it rests on the simple fact that if

$$b^m \equiv r \bmod k$$

then

$$(b^m)^2 \equiv r^2 \bmod k.$$

The allows for fast exponentiation mod k by the so called binary method.

• (According to Knuth this trick goes back at least to 200 B.C.)

• One evaluates x^n rapidly by successive squaring and multiplication. This reduces the number of multiplications to less than $2\log_2(n)$.

• Note that this is entirely performed with positive integers that do not exceed c^2 in size. Further, it is not subject to round-off error, provided adequate numeric precision is used.

• The key observation is that the the fractional part of b^m/k can be computed quickly.

• For example, in base 10. If we solve

$$10^n \equiv \alpha \bmod k$$

then

$$\frac{10^n}{k} - \frac{\alpha}{k} \in \mathcal{Z}$$

and so $10^n/k$ and a/k have the same fractional parts.

• In particular α/k gives the digits of $1/k$ staring after the n-th place.

• This allows for the calculation of the n-th digit of $10^{-j}/k$ from the computation of

$$10^{n-j} \equiv \alpha \bmod \; k\,.$$

This explains (**) above.

• We are now in a position to evaluate the n-th "digit" (base b) of any series of the type

$$S = \sum_{k=0}^{\infty} \frac{1}{b^{ck} p(k)}$$

where p is a polynomial with integer co-efficients. We seek the fractional part of $b^n S$ and so write

$$b^n S \mod 1 = \sum_{k=0}^{\infty} \frac{b^{n-ck}}{p(k)} \mod 1 =$$

$$\sum_{k=0}^{\lfloor n/c \rfloor} \frac{b^{n-ck}}{p(k)} \mod 1 + \sum_{k=\lfloor n/c \rfloor + 1}^{\infty} \frac{b^{n-ck}}{p(k)} \mod 1$$

• For each term of the first summation, the binary exponentiation scheme is used to evaluate the numerator mod $p(k)$.

- The second summation, where powers of b are negative, may be evaluated as written using floating-point arithmetic. It is only necessary to compute a few terms of this summation.

- This is then converted to the desired base b.

Computations.

• Each of our computations employed quad precision floating-point arithmetic for division and sum mod 1 operations.

• Quad precision is supported on the Sun Sparc/20, the IBM RS6000/590, and the SGI Power Challenge (R8000), which were employed in these computations. Quad precision was also used for the exponentiation algorithm on the Sun system.

• On the IBM and the SGI systems, however, we were able to avoid the usage of explicit quad precision, at least in the exponentiation scheme, by exploiting a hardware feature common to these two systems, namely the 106-bit internal registers in the multiply-add operation. This saved considerable time, because quad precision operations are significantly more expensive than 64-bit operations.

• Our results are given below. The first entry, for example, gives the 10^6-th through $10^6 + 13$-th hexadecimal digits of π after the "decimal" point. We believe that all the digits shown below are correct.

• We did the calculations twice. The second calculation, performed for verification purposes, was similar to the first but shifted back one position (this changes all the arithmetic performed).

Constant:	Base:	Position:	Digits:
π	16	10^6	26C65E52CB4593
		10^8	ECB840E21926EC
		10^{10}	921C73C6838FB2
$\log(2)$	16	10^6	418489A9406EC9
		10^9	B1EEF1252297EC
π^2	16	10^6	685554E1228505
		10^9	437A2BA4A13591
$\log^2(2)$	16	10^6	2EC7EDB82B2DF7
		10^9	8BA7C885CEFCE8
$\log(9/10)$	10	10^6	80174212190900
		10^9	44066397959215
		10^{10}	82528693381274

The Forty Trillionth bit of Pi is 0

Between April 19, 1998, and February 9, 1998, one hundred and twenty-six computers from eighteen different countries set a new record for calculating specific bits of Pi. This is due to Colin Percival.

The calculation took a total of about 84,500 cpu hours, and was done using 'idle' time slices (time slices which no other program wants to make use of) under Windows 95 and Windows NT.

The 'average' computer participating was a 200MHz Pentium-based system.

The answer, starting at the 39,999,999,999,997th bit of Pi:

1010 0000 1111 1001 1111 1111 0011 0111 0001 1101

For more information see the PiHex webpage at

http://www.cecm.sfu.ca/projects/pihex/

Logs in base 2. It is easy to compute, in base 2, the d-th binary digit of

$$\log(1 - 2^{-n}) = L_1(1/2^n)\,.$$

So it is easy to compute $\log m$ for any integer m that can be written as

$$m := \frac{(2^{a_1} - 1)(2^{a_2} - 1)\cdots(2^{a_h} - 1)}{(2^{b_1} - 1)(2^{b_2} - 1)\cdots(2^{b_j} - 1)}.$$

• In particular the n-th cyclotomic polynomial evaluated at 2 is so computable. The beginning of this list is:

$$\{2, 3, 5, 7, 11, 13, 17, 31, 43, 57\}\,.$$

• Since

$$2^{18} - 1 = 7 \cdot 9 \cdot 19 \cdot 73,$$

and since 7, $\sqrt{9}$ and 73 are all on the above list we can compute log(19) in SC* from

$$\log(19) = \log(2^{18} - 1) - \log(7) - \log(9) - \log(73).$$

• Note that $2^{11} - 1 = 23 \cdot 89$ so either both log(23) and log(89) are in SC* or neither is.

One can show that no formula of the type on the previous page exists for log(23).

Questions.

• The hardest part of our method is finding an appropriate base b expansion.

• We cannot, at present, compute decimal digits of π by our methods because we know of no identity like Theorem 1 in base 10. But it seems unlikely that this is inherently impossible.

• This raises the following obvious problem.

1] Find an algorithm for the n-th decimal digit of π in SC*.

• It is not even so clear that π is in SC in base 10. This is a result of Plouffe.

• Numbers that are not given by special values of polylogarithms aren't susceptible to our methods. Is this necessarily the case?

2] Are e and $\sqrt{2}$ in SC (SC*) in any base?

• Similarly the treatment of log is incomplete.

3] Is log(2) in SC* in base 10?

4] Is log(23) in SC* in base 2?

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