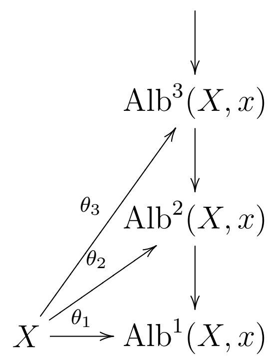
HIGHER ALBANESE MANIFOLDS

RICHARD HAIN

Goal: Suppose X is a smooth variety over $\mathbb{C}$ and $x \in X$. We want to make a tower



where $\mathrm{Alb}^1(X,x)$ is the classical Albanese variety of X.[1]

1. Unipotent completion

Let Γ be a discrete group. Let k be a field of characteristic 0. A *unipotent group* is a closed subgroup of the subgroup of $\mathrm{GL}_n(k)$ consisting of upper triangular matrices with 1s on the diagonal. Unipotent groups correspond to nilpotent Lie algebras via the logarithm and exponential maps, which are polynomial bijections.

Define the pro-unipotent group

$$\Gamma^{\mathrm{un}}_{/k} := \varprojlim_{\substack{\rho\colon \Gamma\to U(k)\\ \text{Zariski dense}\\ U \text{ unipotent}}} U.$$

It is also π_1 of the Tannakian category of unipotent representations of Γ over k.

Define the pro-nilpotent Lie algebra

$$\mathrm{Lie}(\Gamma^{\mathrm{un}}_{/k}) := \varprojlim \mathrm{Lie}(U).$$

A homomorphism $\Gamma \to U(k)$ from Γ to the k-points of a unipotent k-group U induces a homomorphism $\Gamma^{\mathrm{un}}_{/k} \to U$. The original representation factors $\Gamma \to U(k) \to \Gamma^{\mathrm{un}}_{/k} \to \Gamma^{\mathrm{un}}(k)$.

Let J be the kernel of the augmentation map $k\Gamma \xrightarrow{\epsilon} k$ sending each $\gamma \in \Gamma$ to 1. Define

$$(k\Gamma)^\wedge := \varprojlim_r (k\Gamma/J^r).$$

Then

$$\Gamma^{\mathrm{un}}_{/k} = \{x \in (k\Gamma)^\wedge : \Delta x = x \otimes x\} - \{0\}.$$

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[1]The original reference is *Unipotent variations of mixed Hodge structure*: Hain-Zucker, *Invent. Math.* 88 (1987).

Base change: if K/k is any field extension, then $\Gamma_K^{\mathrm{un}} = \Gamma_k^{\mathrm{un}} \otimes_k K$. In particular, $\Gamma_k^{\mathrm{un}} = \Gamma_{\mathbb{Q}}^{\mathrm{un}} \otimes_{\mathbb{Q}} k$.

Example 1.1. If Γ is the free group $\langle x_1, \ldots, x_n \rangle$, then $x_j \mapsto \exp(X_j)$ defines a map Γ to $\mathbb{Q}\langle\langle X_1, \ldots, X_n \rangle\rangle$, which contains the completed free Lie algebra $\mathbb{L}(X_1, \ldots, X_n)^\wedge$. Then $\operatorname{Lie} \Gamma_{/\mathbb{Q}}^{\mathrm{un}} = \mathbb{L}(X_1, \ldots, X_n)^\wedge$.

Example 1.2.

$$\operatorname{Lie}(\pi_1^{\mathrm{un}}(\text{genus-}g \text{ curve}, *)) = \mathbb{L}(A_1, \ldots, A_g, B_1, \ldots, B_g)^\wedge / \left(\sum_{j=1}^{g} [A_j, B_j] \right)$$

2. Profinite case

Let Γ be profinite. Let ℓ be a prime number. Then

$$\Gamma_{/\mathbb{Q}_\ell}^{\mathrm{cts,un}} := \varprojlim_{\substack{\rho\colon \Gamma \to U(\mathbb{Q}_\ell) \\ \text{cts, Zariski dense} \\ U \text{ unipotent}}} U.$$

Fact: If Γ is discrete, then

$$\hat{\Gamma}_{/\mathbb{Q}_\ell}^{\mathrm{cts,un}} = \Gamma_{/\mathbb{Q}_\ell}^{\mathrm{un}}.$$

3. De Rham version

Chen's iterated integral:[2] Let M be a smooth manifold. Let $\omega_1, \ldots, \omega_r \in E^1(M)$ (smooth 1-forms). Let $\gamma\colon [0,1] \to M$ be a piecewise smooth path. Chen defined

$$\int_\gamma \omega_1 \cdots \omega_r = \int \cdots \int_{0 \le t_1 \le \cdots \le t_r \le 1} f_1(t_1) \cdots f_r(t_r)\, dt_1 \cdots dt_r$$

where $\gamma^* \omega_j = f_j(t)\, dt$.

Remark 3.1. This works equally well for $\omega_j \in E^1(M) \otimes_{\mathbb{R}} A$ for any associative algebra A.

Example 3.2. Let $M = \mathbb{C} - \{0, 1\}$. Then

$$\begin{aligned}
\int_0^x \frac{dz}{1-z} \frac{dz}{z} &= \int_0^x \left(\sum_{n=0}^{\infty} z^n \, dz \right) \frac{dz}{z} \\
&= \int_0^x \left(\sum_{n=1}^{\infty} \frac{z^n}{n} \right) \frac{dz}{z} \\
&= \sum_{n=1}^{\infty} \frac{z^n}{n^2} \Bigg|_0^x \\
&= \sum_{n=1}^{\infty} \frac{x^n}{n^2} \\
&= \ln_2 x,
\end{aligned}$$

the dilogarithm function.

[2] Basic reference: K.-T. Chen, Bull. AMS, 1977. See R. Hain, *The geometry of the MHS on the fundamental group*, Proc. Symp. Pure Math., 46 (1987) for an introduction.

Proposition 3.3. *For loops γ and μ starting at the same point,*

$$\int_{\gamma\mu\gamma^{-1}\mu^{-1}} \omega_1\omega_2 = \begin{vmatrix} \int_\gamma \omega_1 & \int_\gamma \omega_2 \\ \int_\mu \omega_1 & \int_\mu \omega_2 \end{vmatrix}$$

For fixed $\omega_1, \dots, \omega_r$, we may view

$$\int \omega_1 \cdots \omega_r$$

as a function $PM \to \mathbb{R}$ on the path space or $P_{x,x}M \to \mathbb{R}$ on the loop space.

Definition 3.4. Let $\mathrm{Ch}(P_{x,x}M)$ be the $\mathbb{R}$-span of $\{\int \omega_1 \cdots \omega_r \colon P_{x,x}M \to \mathbb{R}\}$.

The space $\mathrm{Ch}(P_{x,x}M)$ is a Hopf algebra, with operations inspired by the following identities:

Product:

$$\int_\alpha \omega_1 \cdots \omega_r \int_\alpha \omega_{r+1} \cdots \omega_{r+s} = \sum_{\sigma \in \mathrm{Sh}(r,s)} \int_\alpha \omega_{\sigma(1)} \cdots \omega_{\sigma(r+s)}$$

where $\mathrm{Sh}(r,s)$ is the set of permutations σ of $\{1, \dots, r+s\}$ such that $\sigma^{-1}(1) < \cdots < \sigma^{-1}(r)$ and $\sigma^{-1}(r+1) < \cdots < \sigma^{-1}(r+s)$ (with the convention that $\int_\gamma \phi_1 \cdots \phi_s = 1$ if $s = 0$).

Coproduct:

$$\int_{\alpha\beta} \omega_1 \cdots \omega_r = \sum_{j=0}^{r} \int_\alpha \omega_1 \cdots \omega_j \int_\beta \omega_{j+1} \cdots \omega_r.$$

Antipode:

$$\int_{\gamma^{-1}} \omega_1 \cdots \omega_r = (-1)^r \int_\gamma \omega_r \cdots \omega_1.$$

Definition 3.5. $F \colon P_{x,x}M \to A$ is a *homotopy functional* if all pairs of homotopic paths γ_0 and γ_1 (in which the homotopy does not move the endpoints) satisfy $F(\gamma_0) = F(\gamma_1)$.

Let $H^0(\mathrm{Ch}(P_{x,x}M))$ be the subspace of homotopy functionals. This is a Hopf subalgebra, so $\mathrm{Spec}\, H^0(\mathrm{Ch}(P_{x,x}M))$ is a group scheme over $\mathbb{R}$. Let $L_r H^0(\mathrm{Ch}(P_{x,x}M)$ be the span of the elements of $H^0(\mathrm{Ch}(P_{x,x}M))$ of length $\leq r$. Since the diagonal preserves the length filtration

$$\Delta : L_r H^0(\mathrm{Ch}(P_{x,x}M) \to \sum s + t = r L_s H^0(\mathrm{Ch}(P_{x,x}M) \otimes L_t H^0(\mathrm{Ch}(P_{x,x}M).$$

This implies that $L_r H^0(\mathrm{Ch}(P_{x,x}M$ is a pro-unipotent group.

Let $\pi_1^{\mathrm{un}}(M,x)_{/\mathbb{R}}$ be $\Gamma_{\mathbb{R}}^{\mathrm{un}}$ where $\Gamma := \pi_1(M,x)$. By the product identity given above, there is an "integration" map

$$\pi_1(M,x) \to \mathrm{Hom}_{\mathbb{R}\text{-algebras}}(H^0(\mathrm{Ch}(P_{x,x}M)), \mathbb{R}) = \left(\mathrm{Spec}\, H^0(\mathrm{Ch}(P_{x,x}M))\right)(\mathbb{R}).$$

With the group structure on the right hand side induced by the comultiplication and antipode, this map is a group homomorphism. It induces a homomorphism of pro-algebraic groups

$$\pi_1^{\mathrm{un}}(M,x)_{/\mathbb{R}} \to \mathrm{Spec}\, H^0(\mathrm{Ch}(P_{x,x}M)).$$

Theorem 3.6 (Chen)**.** *The following three equivalent statements hold:*

(1) $\pi_1^{\mathrm{un}}(M,x)_{/\mathbb{R}} = \mathrm{Spec}\, H^0(\mathrm{Ch}(P_{x,x}M))$ *(i.e., the homomorphism just constructed is an isomorphism).*

(2) $\mathcal{O}(\pi_1^{\mathrm{un}}(M,x)_{/\mathbb{R}}) = H^0(\mathrm{Ch}(P_{x,x}M))$ *as Hopf algebras.*
(3) *Let* $L_r H^0(\mathrm{Ch}(P_{x,x}M))$ *be the space of iterated integrals of length* $\leq r$*. Integration gives an isomorphism*

$$L_r H^0(\mathrm{Ch}(P_{x,x}M)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}\text{-vector spaces}}\left(\frac{\mathbb{R}\pi_1(M,x)}{J^{r+1}}, \mathbb{R}\right).$$

Now suppose that X is a smooth projective variety over $\mathbb{C}$. Let $E^\bullet(X) = \bigoplus E^{p,q}(X)$ be the $\mathbb{C}$-valued smooth forms on X. Let $F^p E^\bullet(X) = \bigoplus_{s\geq p} E^{s,\bullet}(X)$. This extends to define a Hodge filtration of $\mathrm{Ch}(P_{x,x}X)$: namely, $F^p\,\mathrm{Ch}(P_{x,x}X)$ is defined as the span of $\int \omega_1\cdots\omega_r$ with $\omega_j \in F^{p_j}$ and $\sum p_j \geq p$.

When X is the complement of a normal crossings divisor D in a smooth projective variety Y, then one has the C^∞ log complex $\oplus E^{p,q}(Y \log D)$, where

$$E^{p,q}(Y \log D) := H^0(Y, \Omega_Y^p(\log D) \otimes_{\mathcal{O}_Y} \mathcal{E}_Y^{0,p}).$$

It is a fact that every element of $H^0(\mathrm{Ch}(P_{x,x}X))$ can be represented by iterated integrals of elements of $E^\bullet(Y \log D)$. The Hodge filtration of $H^0(\mathrm{Ch}(P_{x,x}X))$ is defined using the Hodge filtration of $E^\bullet(Y\log D)$ as in the projective case.

Example 3.7. We have $\int \frac{dz}{1-z}\frac{dz}{z} \in F^2$ and $\int d\bar{z}\,dz \in F^1$.

This restricts to define a Hodge filtration of $H^0(\mathrm{Ch}(P_{x,x}X))$ compatible with product, coproduct, and antipode.

Theorem 3.8. *This is part of the natural mixed Hodge structure on* $\pi_1^{\mathrm{un}}(X,x)$*. (The weight filtration is* $L^\bullet$ *when* X *is smooth and projective.)*

The Lie algebra of $\pi_1^{\mathrm{un}}(X,x)$ is the dual of $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}(\pi_1^{\mathrm{un}}(X,x)) = H^0(\mathrm{Ch}(P_{x,x}X)$ corresponding to evaluation at the trivial loop. The bracket is dual to the "cobracket" $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \otimes \mathfrak{m}/\mathfrak{m}^2$, which is the linear map induced by

$$\Delta - \tau\circ\Delta : H^0(\mathrm{Ch}(P_{x,x}X)) \to H^0(\mathrm{Ch}(P_{x,x}X)) \otimes H^0(\mathrm{Ch}(P_{x,x}X)),$$

where $\tau(f\otimes g) = g\otimes f$. This leads to a Hodge filtration on $\mathrm{Lie}\,\pi_1(X,x)$ compatible with $[\ ,\ ]$ (this means that $[F^p, F^q] \subseteq F^{p+q}$). On $\mathrm{Lie}\,\pi_1(X,x)$

$$\cdots \supseteq F^{-3} \supseteq F^{-2} \supseteq F^{-1} \supseteq F^0 \supseteq F^1 = 0.$$

Let

$$G = \pi_1^{\mathrm{un}}(X,x)_{/\mathbb{C}}$$
$$\mathfrak{g} = \mathrm{Lie}\,G.$$

Then $F^0\mathfrak{g}$ is a Lie subalgebra. So we have a subgroup F^0G of G.

4. Higher Albanese Manifolds

We have $\pi_1(X,x) \xrightarrow{\rho} G \supseteq F^0G$. Let Γ be the image of ρ.

Definition 4.1.

$$\mathrm{Alb}(X,x) = \Gamma\backslash G/F^0G.$$

Let $L^r\Gamma$ be the r-th term of the lower central series of Γ; i.e., $\Gamma = L^1\Gamma \supseteq L^2\Gamma \supseteq \cdots$ where $L^{i+1}\Gamma = [L^i\Gamma, \Gamma]$. For each r, define

$$G_r = G/L^{r+1}G,$$

define Γ_r as the image of $\pi_1(X,x)$ in G_r, and define

$$\mathrm{Alb}^r(X,x) = \Gamma_r \backslash G_r / F^0 G_r.$$

In general, these are not algebraic except when $r=1$. $\mathrm{Alb}(X,x)$ should be considered as the inverse limit of the $\mathrm{Alb}^r(X,x)$.

Example 4.2. If $r=1$, then $G_1 = H_1(X;\mathbb{C}) = H^{-1,0} \oplus H^{0,-1}$. Then $F^{-1} = H^{-1,0} \oplus H^{0,-1}$ and $F^0 = H^{0,-1}$. We have

$$G_1/F^0 = H^{-1,0} = H^0(\Omega^1_X)^*$$

and

$$\Gamma_1 = H_1(X,\mathbb{Z})/\mathrm{torsion}$$

so

$$\mathrm{Alb}^1 = H_1(X,\mathbb{Z}) \backslash H^0(\Omega^1_X)^*.$$

Example 4.3. Let A be a principally polarized abelian variety. Let Θ be the θ-divisor. Assume that Θ is irreducible and $0 \notin \Theta$. Let $X = A - \Theta$. Let $\mathcal{L}$ be the line bundle corresponding to the line sheaf $\mathcal{O}_A(\Theta)$. Let $\mathcal{L}^*$ be $\mathcal{L}$ minus the zero section. Then

$$\begin{array}{ccc} & & \mathcal{L}^* = \mathrm{Alb}^2(X,0) \\ & \nearrow & \downarrow \\ X & \longrightarrow & A = \mathrm{Alb}^1(X,0) \end{array}$$

5. Higher Albanese Mappings

Denote $\mathrm{Lie}(\pi_1^{\mathrm{un}}(X,x)(\mathbb{C}))$ by $\mathfrak{g}$. This is a quotient of the completion of the free complete Lie algebra $\mathrm{Lie}(H_1(X))^\wedge$ generated by $H_1(X;\mathbb{C})$. For simplicity, suppose that X is projective. Then

$$H_1(X;\mathbb{C}) = H^{-1,0}(X) \oplus H^{0,-1}(X).$$

Let $\{W_j'\}$ be a basis of $H^{-1,0}(X)$ and $\{W_j''\}$ be the complex conjugate basis of $H^{0,1}$. Let $\{w_j\}$ be the dual basis of $H^0(\Omega^1_X)$.

Proposition 5.1. *There is a* $\mathfrak{g}$*-valued* 1*-form*

$$\omega \in F^0\big(E^1(X) \mathbin{\widehat{\otimes}} \mathfrak{g}\big)$$

and which is congruent to

$$\sum_j w_j W_j' + \overline{w}_j W_j'' \bmod [\mathfrak{g},\mathfrak{g}]$$

and satisfies the integrability condition

$$d\omega + \frac{1}{2}[\omega,\omega] = 0.$$

When X is not compact, then one has $\omega \in W_{-1}F^0\big(E^1(X) \mathbin{\widehat{\otimes}} \mathfrak{g}\big)$. These statements are proved in *Higher Albanese Manifolds*, R. Hain, LNM 1246, 1987.

Define

$$T = 1 + \int \omega + \int \omega\omega + \int \omega\omega\omega + \cdots$$

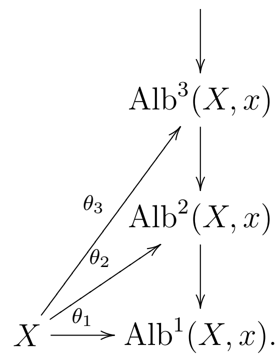
This is a $\widehat{U}\mathfrak{g}$-valued iterated integral. The integrability condition implies that T is a homotopy functional on PX. For all $\gamma \in PX$, $T(\gamma) \in \exp \mathfrak{g} = G$.

The Albanese mapping $\theta : X \to \mathrm{Alb}(X, x)$ is given by

$$\theta(y) = T(\gamma) \in \Gamma \backslash G / F^0.$$

It is clearly well defined as T is a homotopy functional and we have taken the quotient by Γ. On the space of paths that begin at x we have $dT = T\omega$. This and the fact that $\omega \in \mathbb{F}^0(E^1(X)\widehat{\otimes}\mathfrak{g})$, θ imply that θ is holomorphic. One can show that θ is independent of the choice of ω.

Truncating this construction by L_r gives the tower:



6. Algebraic Approach

The constructions above use smooth forms. There is also a version of Chen's π_1-de Rham Theorem that uses only iterated integrals of algebraic 1-forms. This is sketched in *Iterated Integrals and Algebraic Cycles: Examples and Prospects*, Nankai Tracts in Mathematics, vol. 5, World Scientific, 2002. The Hodge filtration should correspond to a "pole filtration," but this has yet to be worked out.

A different version of the algebraic de Rham theorem given in the same paper allows one to prove that if X is defined over F and $F \subseteq \mathbb{C}$, and $x \in X(F)$, then the Hopf algebra $H^0(\mathrm{Ch}\, P_{x,x}X)$ has a natural F-structure, and the Hodge filtration is defined over F.