Analytic ranks of Jacobians

Banff 8.2.2007

BSD

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w exterior form on A/K

$$\Omega = \prod_{\substack{v|\infty \\ \text{real}}} \int_{A(K_v)} |\omega| \prod_{\substack{v|\infty \\ \text{cplx}}} 2 \int_{A(K_v)} \omega \wedge \bar{\omega}$$

$$C = \prod_{v \nmid \infty} c_v \left| \frac{\omega}{\omega_v^{\text{Néron}}} \right|_v \qquad c_v = \left| \frac{A(K_v)}{A_0(K_v)} \right| \text{ Tamagawa number}$$

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- Weil restriction (Tate-Milne), so may work with AVs over $\mathbb{Q}$
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Non-trivial applications:
Examples of non-square $\text{Ш}[p]$ for $p < 25000$ (Stein)
Joint work with V.D. on the Parity Conjecture

Computing analytic ranks

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Want: compute $L(1)$ or $L^{(k)}(1)$ explicitly (assuming every conjecture) to predict the Mordell-Weil rank and $|\text{Ш}|$.

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$$\hat{L}(s) = \left(\frac{N}{\pi^g}\right)^{s/2} \Gamma\left(\frac{s}{2}\right)^g \Gamma\left(\frac{s+1}{2}\right)^g L(s)$$

Functional equation:

$$\hat{L}(s) = w\, \hat{L}(2-s), \qquad w = \pm 1$$

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with $G(s,x)$ a higher transcendental function depending *only* on g, exponentially decaying with x.
(E.g. for elliptic curves, $G(s,x) = \frac{\sqrt{4\pi}}{(2x)^s}\, \Gamma(s, 2x)$)

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- $N, F_p(T)$ for p bad. Theoretically hard:

$$g = 1: \quad F_p(T) = \begin{cases} 1 - T & \text{split mult. red.} \\ 1 + T & \text{non-split mult. red. (Tate's algorithm)} \\ 1 & \text{additive reduction} \end{cases}$$

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But possible in practice.

Question: An algorithm to do this, e.g. for hyperelliptic curves?

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Question: How to do this efficiently in practice?

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$y^2 = x^3 + 23231x + 1437209$	$g = 1$
$y^2 + y = x^5 + 101x$	$g = 2$
$y^2 + (x^9+x^8+x^4+x^3+x+1)y = -x^{17}+x^{15}-x^{14}-x^{11}$	$g = 8$

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with $E = 43A1$, $K = \mathbb{Q}(\sqrt[5]{2})$ $\qquad (N = 2^8 5^{10} 43^4)$.

Example: A genus 3 curve

$$C/\mathbb{Q}: \quad y^4 - y^3 - y^2 + y = x^4 - x$$

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Compute $L(C,s)|_{s=1}$

$$\begin{aligned} L(1) &= 0.00000 \\ L'(1) &= 0.00000 \\ \tfrac{1}{2!}L''(1) &= 0.00000 \\ \tfrac{1}{3!}L^{(3)}(1) &= 0.00000 \\ \tfrac{1}{4!}L^{(4)}(1) &= 0.00000 \\ \tfrac{1}{5!}L^{(5)}(1) &= 3.64636 \end{aligned}$$

So expect $J(C)$ have Mordell-Weil rank 5.

Example: Abelian twists of elliptic curves

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1214534039480.99999999999999998
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$(= 3^6 7^8 17^2)$

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- 3^∞-Selmer rank is even $\iff 11 \nmid m$ (Parity Conjecture)

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So $w = 1$ for $11 \nmid m$ and $w = -1$ for $11|m$,
thus expect even Mordell-Weil rank $\iff 11 \nmid m$. In fact,

- 3^∞-Selmer rank is even $\iff 11 \nmid m$ (Parity Conjecture)
- For $11 \nmid m$, 3-Selmer is non-trivial $\iff p|m$ with $\tilde{E}(\mathbb{F}_p)[p] \neq 0$.

Example: Non-abelian twists of elliptic curves

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Analytic computation:

m	2	3	5	6	7	11	$\cdots$	29	30	31	33	34	$\cdots$	71	$\cdots$
rk	0	0	0	0	0	1	$\cdots$	0	0	0	1	0	$\cdots$	2	$\cdots$
$\lvert Ш \rvert$	1	1	4	1	1	?	$\cdots$	9	1	1	?	25	$\cdots$	?	$\cdots$