Fenchel conjugates and subdifferentials in Maple

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November 17, 1997

Abstract

The notions of the Fenchel conjugate and the subdifferential of a convex function are fundamental in optimization. The package `fenchel` allows the symbolic computation of these objects for numerous convex functions defined on the real line.
Key words. Convex function, Fenchel conjugate, Legendre-Fenchel transform, Legendre transform, subdifferential, subgradient.

1 Motivation

Definitions

Convex function

Suppose f is a function defined on $\mathbb{R}^n$ with values in $(-\infty, +\infty]$. Recall that f is *convex*, if

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2),$$

for every $x_1, x_2 \in \mathbb{R}^n$, and all $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$. The *(effective) domain of* f, written $\operatorname{dom} f$, is the set of all points where f is not $+\infty$. Convex functions lie at the heart of Convex, Functional and Real Analysis. Several excellent monographs on the subject are available: Rockafellar's classical [13], Roberts and Varberg's gentler [12], and Hiriart-Urruty and Lemaréchal's recent [5, 6]. In these references, the reader may find further information and proofs of the results stated below.

Subdifferential and subgradient

In calculus, we learn that a minimizer of f, say $\bar{x}$, is necessarily a critical point: $\nabla f(\bar{x}) = 0$. Since various of the most interesting convex functions are *not* everywhere differentiable, this

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technique is not immediately available. Instead, one defines the so-called *subdifferential of f at x*, denoted $\partial f(x)$, by

$$\{y \in \mathbb{R}^n : \langle y, x' - x\rangle \leq f(x') - f(x), \forall x' \in \mathbb{R}^n\}.$$

(Here and elsewhere in the paper, $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$.) Members of the subdifferential are called *subgradients*. This idea generalizes differentiability: indeed, f is differentiable at x if and only if $\partial f(x) = \{\nabla f(x)\}$. The importance of subgradients in optimization stems from the fact that

$$\bar{x} \text{ is a (global) minimizer of } f \Leftrightarrow 0 \in \partial f(\bar{x}).$$

Since f is convex, the left (f'_-) and right (f'_+) derivatives exist at every point in $\operatorname{dom} f$. Moreover, the subdifferential is completely described by these directional derivatives:

$$\partial f(x) = [f'_-(x), f'_+(x)], \quad \forall x \in \operatorname{dom} f.$$

Fenchel conjugate

The *Fenchel conjugate* of f, denoted f^*, is defined by

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \langle y, x\rangle - f(x), \quad \forall y \in \mathbb{R}^n;$$

it is always a convex and lower semi-continuous function on $\mathbb{R}^n$. (Recall that a function g is called lower semi-continuous at y, if $\underline{\lim}_{y' \to y} g(y') \geq g(y)$.) Assuming lower semi-continuity, the operation of performing Fenchel conjugation twice recovers the original function. In fact, more is true:

$$f \text{ is convex and lower semi-continuous} \Leftrightarrow f = f^{**}.$$

Immediate from the definition of the Fenchel conjugate is the famous *Fenchel/Young inequality:*

$$f(x) + f^*(y) \geq \langle x, y\rangle, \quad \forall x \in \mathbb{R}^n, \forall y \in \mathbb{R}^n.$$

The *Fenchel/Young equality* is $f(x) + f^*(y) = \langle x, y\rangle$. Sufficient and necessary conditions for this equality to hold are formulated in terms of subgradients: for $x, y \in \mathbb{R}^n$, we have

$$f(x) + f^*(y) = \langle x, y\rangle \Leftrightarrow y \in \partial f(x) \Leftrightarrow x \in \partial f^*(y).$$

Convex Optimization Duality

Convex optimization studies convex programs of the following type:

$$(\mathrm{P}) \qquad p := \inf_{x \in \mathbb{R}^n} [f(x) + g(Ax)],$$

where $A \in \mathbb{R}^{m \times n}$ and f (resp. g) is convex and lower semi-continuous on $\mathbb{R}^n$ (resp. $\mathbb{R}^m$). The program (P) is called the *primal problem*. The problem *dual* to (P) is defined by

$$(\mathrm{D}) \qquad d := -\inf_{z \in \mathbb{R}^m} [f^*(-A^*z) + g^*(z)],$$

where A^* is the transpose of A.

We collect the fundamental results on this primal-dual pair of optimization problems in the following.

Fact (Fenchel's Duality Theorem) Suppose $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$. Then:
Weak duality: $p \geq d$.
Karush/Kuhn&Tucker conditions: x solves (P), z solves (D), and $p = d$ if and only if

$$Ax \in \partial g^*(z) \text{ and } -A^*z \in \partial f(x).$$

Strong duality: If $A(\operatorname{dom} f) \cap \operatorname{int}(\operatorname{dom} g) \neq \emptyset$, then $p = d$ and the infimum d is attained.
Primal solutions: If z is a solution of (D), then the solutions of (P) are equal to the (possibly empty) set

$$A^{-1}\partial g^*(z) \cap \partial f^*(-A^*z).$$

Fenchel's Duality Theorem clearly shows the importance of subdifferentials and Fenchel conjugates.

As an aside, we mention that it is possible to recover the well-known *Linear Programming Duality* from (a variant of) Fenchel's Duality Theorem.

One-dimensional applications

A convex function f defined on $\mathbb{R}^n$ is called *separable*, if f can be written as

$$f(x) = \sum_{j=1}^{n} f_j(x_j), \quad \forall x \in \mathbb{R}^n,$$

where each f_j is a convex function on $\mathbb{R}$; for such a function f, we have

$$\partial f(x) = \partial f_1(x_1) \times \cdots \times \partial f_n(x_n), \quad \forall x \in \mathbb{R}^n$$

and

$$f^*(y) = \sum_{j=1}^{n} f_j^*(y_j), \quad \forall y \in \mathbb{R}^n.$$

Consequently, computing subdifferentials and conjugates of separable convex functions is (essentially) no harder than dealing with one-dimensional convex functions.

We give two areas of applications.

Separable convex optimization
Separable convex programs are optimization problems with separable objective functions. This includes the problem of *network optimization*; see [3, Section 5.4.2].

The study of *penalty* and *proximal point-like methods* for solving convex programs relies implicitly on Fenchel conjugates. Separable penalty functions are often used in practice; see for instance [3, Sections 5.4.5 and 5.4.6]) and [11].

Inequalities
An important source for *inequalities in classical analysis* is the Fenchel/Young inequality and its characterization of equality; see [10, Sections XIV.9 and XXVI.4].

Aim of this paper

As the preceding subsections show, it is quite worthwhile and interesting to be able to compute Fenchel conjugates and subdifferentials, even in the one-dimensional case.

Our objective in this paper is the presentation of the package `fenchel`*, which allows the symbolic computation of Fenchel conjugates and subdifferentials for many convex functions defined on the real line.*

Why `fenchel`?

It is tempting to try to compute Fenchel conjugates in Maple by using `maximize`, perhaps in conjunction with texttttpiecewise. Unfortunately, neither is particularly well suited for this task: On the one hand, the normal form of a `piecewise`-function taking the value $+\infty$ may contain terms of the form $\infty - \infty$, which would yield an error. (See [15] and [14] for more on `piecewise` and its underlying theory.) On the other hand, the command `maximize` is not able to handle many of the examples listed below. We thus were led to create `fenchel` to conjugate and subdifferentiate convex functions. As far as we know, it is the first package able to manipulate convex functions in this generality.

Good algorithms for computing Fenchel conjugates *numerically* are available, see Lucet's [9] for further information.

Obtaining `fenchel`

The Centre for Experimental and Constructive Mathematics (CECM) maintains a *Computational Convex Analysis* page at `cecm.sfu.ca/projects/CCA`; the reader can find a link to `fenchel` from there.

A good class of functions

Let $\mathcal{F}$ be the class of all functions f satisfying the conditions listed next:
(i) f is a function from $\mathbb{R}$ to $(-\infty, +\infty]$;
(ii) f is convex;
(iii) f is continuous on its effective domain;
(iv) there are finitely many points $x_0 = -\infty < x_1 < \cdots < x_{m-1} < x_m = +\infty$ such that f restricted to each open interval (x_{i-1}, x_i) is one of the following: identically equal to $+\infty$, or affine, or differentiable with strictly increasing derivative.

Note that — because we deal with functions on the real line — condition (iii) is equivalent to the lower semi-continuity of f.

If f_1, f_2 belong to $\mathcal{F}$ and $\alpha_1, \alpha_2 \geq 0$, then $\alpha_1 f_1 + \alpha_2 f_2$ and f_1^* are in $\mathcal{F}$ as well. (It is possible that $\max\{f_1, f_2\}$ lies outside $\mathcal{F}$, but we do not discuss the maximum operation here.)

The class $\mathcal{F}$ is thus very well-suited for our purpose.

What can `fenchel` do?

We classify `fenchel`'s procedures as follows.

New commands: `cf` and `sd` (entering convex functions and subdifferentials); `cfplot` and `sdplot` (plotting); `cfeval` and `sdeval` (evaluating at a point); `subdiff` and `invert` (computing and inverting the subdifferential of a convex function); `conj` (computing the Fenchel conjugate); `convpw` (converts from `cf` to `piecewise`).
Supported Maple commands The following commands work for `cf`-functions: `convert`, `evalf`, `expand`, `normal`, `print`, `simplify`, `type`. Most useful to us were `simplify` and `convert` (from `piecewise` to `cf`).

Rather than formally discussing `fenchel`'s commands, we prefer to demonstrate it by examples.

How does `fenchel` work?

A convex lower semi-continuous function on the real line is very well-behaved. For instance, it is subdifferentiable on the interior of its domain. This allows the computation of the Fenchel conjugate at such a point y in two steps: firstly, solve $y \in \partial f(x)$ for x — this is the key step — and let $\bar{x}$ be such a solution. Secondly, use the Fenchel/Young equality to obtain $f^*(y) = \langle \bar{x}, y \rangle - f(\bar{x})$. The value of the Fenchel conjugate at a boundary point is then determined by the continuity of the function on its domain.

2 Ten examples

1. The notorious absolute value

Perhaps the simplest nondifferentiable convex function is the absolute value: $f := |\cdot|$. Its derivative at 0 fails to exist, since $f'_-(0) = -1 < 1 = f'_+(0)$. Accordingly, the subdifferential of f at 0 is $\partial f(0) = [-1, 1]$. The conjugate of f, f^*, is the *indicator* function of the interval $[-1, 1]$: $f^*(y) = 0$, if $y \in [-1, 1]$; $f^*(y) = \infty$, otherwise. We use the commands `subdiff` and `conj` to rederive this. The `convert` command makes entering the absolute value very easy.

```
> f1 := convert(abs(x),cf);
```

$$f1 := \begin{cases} -x & (-\infty < x) \textbf{ and } (x < 0) \\ 0 & x = 0 \\ x & (0 < x) \textbf{ and } (x < \infty) \end{cases}$$

```
> subdiff(f1);
```

$$\begin{cases} \{-1\} & (-\infty < x) \textbf{ and } (x < 0) \\ [-1,\ 1] & x = 0 \\ \{1\} & (0 < x) \textbf{ and } (x < \infty) \end{cases}$$

```
> g1 := conj(f1,y);
```

$$g1 := \begin{cases} \infty & (-\infty < y) \textbf{ and } (y < -1) \\ 0 & y = -1 \\ 0 & (-1 < y) \textbf{ and } (y < 1) \\ 0 & y = 1 \\ \infty & (1 < y) \textbf{ and } (y < \infty) \end{cases}$$

2. The Boltzmann/Shannon entropy

The exponential function together with the Boltzmann/Shannon entropy is one of the most famous pairs of Fenchel conjugates. It shows that the Fenchel conjugate can take the value ∞, even if the original function is very well-behaved.

$$f(x) = \exp(x) \text{ and } f^*(y) = \begin{cases} +\infty, & \text{if } y < 0; \\ 0, & \text{if } y = 0; \\ y \ln y - y & \text{otherwise.} \end{cases}$$

In `fenchel`, we recapture this computation as follows:

```
> f2 := convert(exp(x),cf);
```

$$f2 := \{ e^x \quad (-\infty < x) \textbf{ and } (x < \infty)$$

```
> g2 := conj(f2,y);
```

$$g2 := \begin{cases} \infty & (-\infty < y) \textbf{ and } (y < 0) \\ 0 & y = 0 \\ y \ln(y) - y & (0 < y) \textbf{ and } (y < \infty) \end{cases}$$

```
> conj(g2,x);
```

$$\{ e^x \quad (-\infty < x) \textbf{ and } (x < \infty)$$

3. De Pierro and Iusem's example

This pair was suggested by De Pierro and Iusem on page 438 of [4]; it was quite useful in [1] and [2]. This time, we enter the function directly via `cf` (`convert` would have worked, too).

```
> f3 := cf([[-infinity,1/2*(x^2-4*x+3),1],
            [1,0,1],[1,-ln(x),infinity]],x);
```

$$f3 := \begin{cases} \frac{1}{2}x^2 - 2x + \frac{3}{2} & (-\infty < x) \textbf{ and } (x < 1) \\ 0 & x = 1 \\ -\ln(x) & (1 < x) \textbf{ and } (x < \infty) \end{cases}$$

```
> g3 := conj(f3,y);
```

$$g3 := \begin{cases} \frac{1}{2}y^2 + 2y + \frac{1}{2} & (-\infty < y) \textbf{ and } (y < -1) \\ -1 & y = -1 \\ -1 - \ln(-y) & (-1 < y) \textbf{ and } (y < 0) \\ \infty & y = 0 \\ \infty & (0 < y) \textbf{ and } (y < \infty) \end{cases}$$

4. Affine and quadratic functions

An *affine* function is of the form $x \mapsto bx + c$, where b, c are real constants. Conjugates of affine functions are everywhere ∞, except at one point.

```
> convert(b*x+c,cf,x);
```

$$\{ b\,x + c \quad (-\infty < x) \textbf{ and } (x < \infty)$$

```
> conj(",y);
```

$$\begin{cases} \infty & (-\infty < y)\ \mathbf{and}\ (y < b) \\ -c & y = b \\ \infty & (b < y)\ \mathbf{and}\ (y < \infty) \end{cases}$$

On the other hand, the class of quadratic convex functions is closed under Fenchel conjugation:

```
> assume(a>0); convert(a*x^2+b*x+c,cf,x);
```

$$\{\, a\tilde{}\, x^2 + b\, x + c \quad (-\infty < x)\ \mathbf{and}\ (x < \infty)$$

```
> conj(",y);
```

$$\left\{ \frac{1}{4}\, \frac{y^2 - 2\, y\, b + b^2 - 4\, c\, a\tilde{}}{a\tilde{}} \quad (-\infty < y)\ \mathbf{and}\ (y < \infty) \right.$$

The function $\frac{1}{2}x^2$ is special since it is the only quadratic that coincides with its Fenchel conjugate. In fact, much more is true: the only self-conjugate convex lower semi-continuous function on $\mathbb{R}^n$ is $x \mapsto \frac{1}{2}\|x\|^2$.

5. Lagarias and Weiss's function

One of the most famous open problems in mathematics is the so-called $3x + 1$ *problem*:

> Define a self map T of the positive integers by
>
> $$T(n) := \begin{cases} n/2, & \text{if } n \text{ is even;} \\ 3n+1, & \text{if } n \text{ is odd.} \end{cases}$$
>
> Does repeated application of T eventually lead to 1, for every positive integer n?

An excellent survey is Lagarias's [7]. In [8], Lagarias and Weiss study a *random walk model* for the $3x + 1$ problem. They are led to consider the function $f(x) := \frac{1}{2}(2^x + (\frac{2}{3})^x)$ and its conjugate $g := f^*$. For instance, they prove that $\operatorname{dom} g \subseteq (\ln(2/3), \ln(2))$, that $g(\frac{1}{2}\ln(\frac{4}{3})) = 0$, and that $g(y) = y$ has a unique solution $\bar{y} \approx 0.02399$. (The quantity $\bar{y}$ is very important in their analysis.) We can at least illustrate their findings. In fact, `conj` returns a closed form for g, which yields $\operatorname{dom} g = [\ln(2/3), \ln(2)]$ and g takes the value $\ln(2)$ at both endpoints of the interval. We first show that $g(\frac{1}{2}\ln(\frac{4}{3})) = 0$ by using `fenchel`'s evaluation command `cfeval`. We then confirm Lagarias and Weiss's numerical approximation of $\bar{y}$ using `op` and `fsolve`.

```
> f5 := convert(ln((1/2)*(2^x+(2/3)^x)),cf);
```

$$f5 := \left\{ \ln(\frac{1}{2}\, 2^x + \frac{1}{2}\, (\frac{2}{3})^x) \quad (-\infty < x)\ \mathbf{and}\ (x < \infty) \right.$$

```
> g5 := conj(",y):

> cfeval(g5,y=1/2*ln(4/3));
```

$$0$$

```
> temp := op([1,3,2],g5);
```

$$temp := (y\ln(-\frac{y-\ln(2)+\ln(3)}{y-\ln(2)})+\ln(2)\ln(3)$$
$$-\ln((-\frac{y-\ln(2)+\ln(3)}{y-\ln(2)})^{(\frac{\ln(2)}{\ln(3)})}+(-\frac{y-\ln(2)+\ln(3)}{y-\ln(2)})^{(\frac{\ln(2)-\ln(3)}{\ln(3)})})\ln(3))/\ln(3)$$

```
> ybar := fsolve(temp=y,y);
```

$$ybar := .02399367662$$

6. A barrier function

In [11], Polyak and Teboulle study algorithms for convex optimizations and they use (implicitly) convex functions and their Fenchel conjugates. (We should point out that the authors decided to consider *concave* functions and corresponding *concave conjugates*. However, this is mathematically equivalent to our situation.) Here we recover the last example in their paper; also, we use `factor` to simplify the Fenchel conjugate. (The use of `expand` and `normal` is similar.)

```
> f6 := cf([[-infinity,exp(-4*x-2),-1/2],[-1/2,1,-1/2],
            [-1/2,-x/(1+x),infinity]],x);
```

$$f6 := \begin{cases} e^{(-4x-2)} & (-\infty < x)\ \textbf{and}\ (x < \frac{-1}{2}) \\ 1 & x = \frac{-1}{2} \\ -\frac{x}{1+x} & (\frac{-1}{2} < x)\ \textbf{and}\ (x < \infty) \end{cases}$$

```
> factor(conj(f6,y));
```

$$\begin{cases} \frac{1}{4}y\,(-1+2\ln(2)-\ln(-y)) & (-\infty < y)\ \textbf{and}\ (y < -4) \\ 1 & y = -4 \\ -\frac{y\sqrt{-y}-2y-\sqrt{-y}}{\sqrt{-y}} & (-4 < y)\ \textbf{and}\ (y < 0) \\ 1 & y = 0 \\ \infty & (0 < y)\ \textbf{and}\ (y < \infty) \end{cases}$$

7. An example from Bertsekas's text

The following pair can be found on page 468 in Bertsekas's recent text [3]. In fact, `fenchel` is good enough to recover all conjugate pairs in this book!

```
> f7 := convert(piecewise(x>=0,a*x+x^2,0),cf,x);
```

$$f7 := \begin{cases} 0 & (-\infty < x)\ \textbf{and}\ (x < 0) \\ 0 & x = 0 \\ a\tilde{}\,x + x^2 & (0 < x)\ \textbf{and}\ (x < \infty) \end{cases}$$

```
> g7 := factor(conj(f7,y));
```

$$g7 := \begin{cases} \infty & (-\infty < y)\ \textbf{and}\ (y < 0) \\ 0 & y = 0 \\ 0 & (0 < y)\ \textbf{and}\ (y < a\tilde{}) \\ 0 & y = a\tilde{} \\ \frac{1}{4}(y - a\tilde{})^2 & (a\tilde{} < y)\ \textbf{and}\ (y < \infty) \end{cases}$$

8. An example from Rockafellar's text

The next function can be found on [13, page 229]. We use `cfplot` to plot the function (Figure 1). To find its subdifferential, we use `subdiff`. Next, we plot it using `sdplot` (Figure 2). We then find the conjugate and plot it, too (Figure 3). Finally, we convert the conjugate to a `piecewise` function.

```
> f8 := convert(piecewise(-3<=x and x<=1,
         abs(x)-2*sqrt(1-x),infinity),cf);
```

$$f8 := \begin{cases} \infty & (-\infty < x)\ \textbf{and}\ (x < -3) \\ -1 & x = -3 \\ -x - 2\sqrt{1-x} & (-3 < x)\ \textbf{and}\ (x < 0) \\ -2 & x = 0 \\ x - 2\sqrt{1-x} & (0 < x)\ \textbf{and}\ (x < 1) \\ 1 & x = 1 \\ \infty & (1 < x)\ \textbf{and}\ (x < \infty) \end{cases}$$

```
> cfplot(f8,x=-4..2,scaling=constrained,axes=framed);
```

```
> sdf8 := subdiff(f8);
```

$$sdf8 := \begin{cases} \{\} & (-\infty < x)\ \textbf{and}\ (x < -3) \\ [-\infty, \frac{-1}{2}] & x = -3 \\ \{-\frac{\sqrt{1-x}-1}{\sqrt{1-x}}\} & (-3 < x)\ \textbf{and}\ (x < 0) \\ [0, 2] & x = 0 \\ \{\frac{\sqrt{1-x}+1}{\sqrt{1-x}}\} & (0 < x)\ \textbf{and}\ (x < 1) \\ \{\} & x = 1 \\ \{\} & (1 < x)\ \textbf{and}\ (x < \infty) \end{cases}$$

```
> sdplot(sdf8,x=-3..1,view=[-3..1,-3..5],axes=none);
```

```
> g8 := conj(f8,y);
```

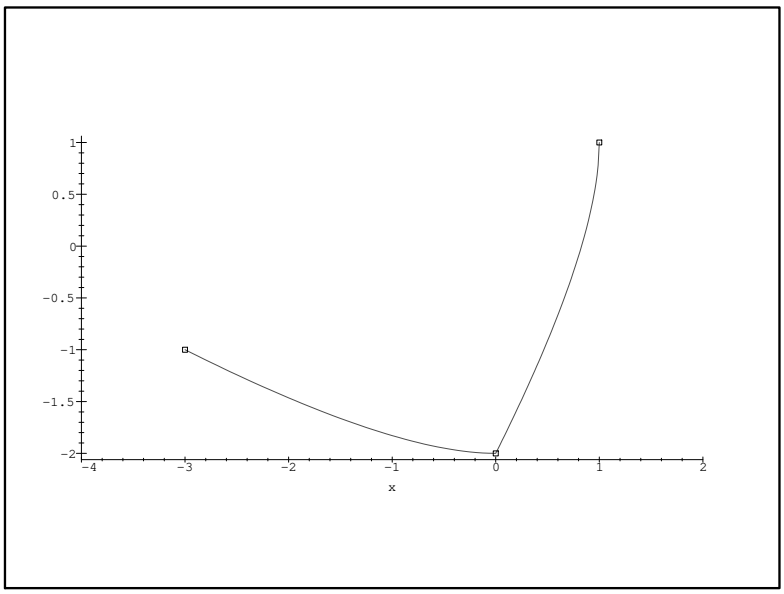

Figure 1: The graph of $f8$

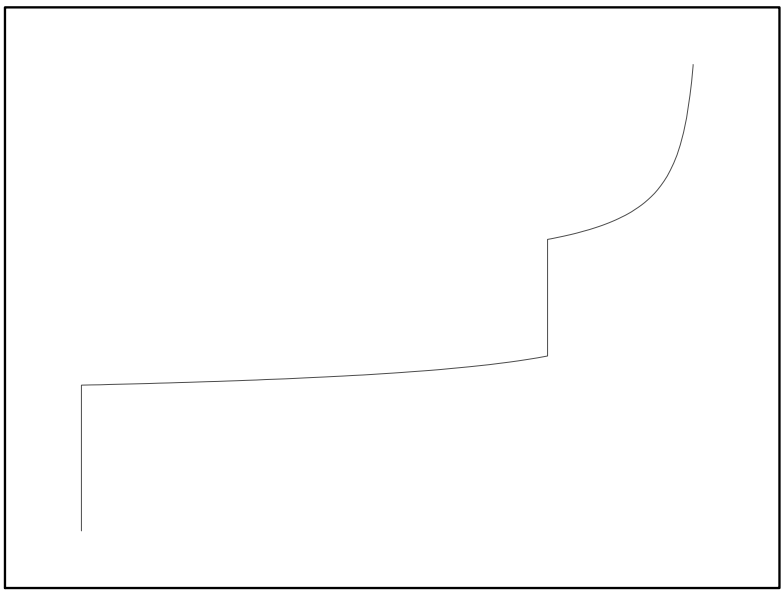

Figure 2: The subdifferential of $f8$

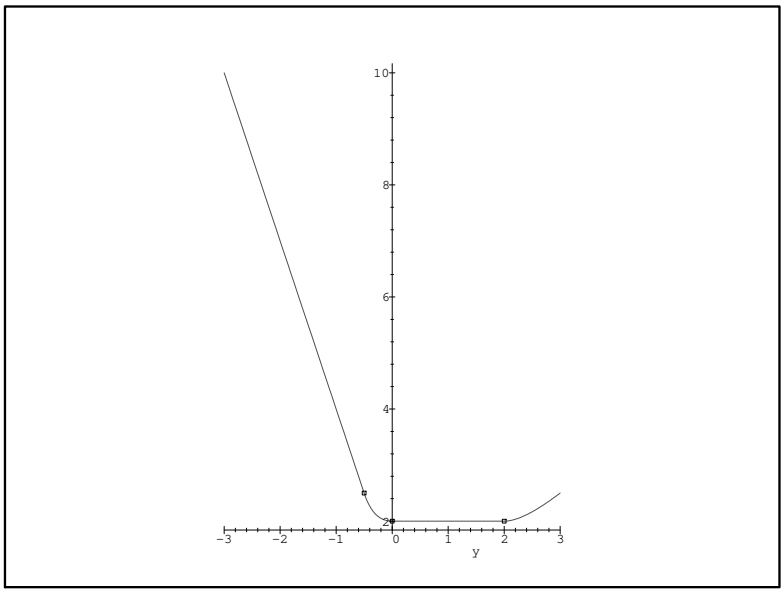

Figure 3: Graph of $g8 = (f8)^*$

$$g8 := \begin{cases} -3\,y+1 & (-\infty < y)\ \textbf{and}\ (y < \dfrac{-1}{2}) \\ \dfrac{5}{2} & y = \dfrac{-1}{2} \\ \dfrac{y^2+2\,y+2}{y+1} & (\dfrac{-1}{2} < y)\ \textbf{and}\ (y < 0) \\ 2 & y = 0 \\ 2 & (0 < y)\ \textbf{and}\ (y < 2) \\ 2 & y = 2 \\ \dfrac{y^2-2\,y+2}{y-1} & (2 < y)\ \textbf{and}\ (y < \infty) \end{cases}$$

```
> cfplot(g8,y=-3..3,scaling=constrained);
```

```
> simplify(convpw(g8));
```

$$\begin{cases} -3\,y+1 & y \le \dfrac{-1}{2} \\ \dfrac{y^2+2\,y+2}{y+1} & y \le 0 \\ 2 & y \le 2 \\ \dfrac{y^2-2\,y+2}{y-1} & 2 < y \end{cases}$$

9. An infimal convolution

Given two convex lower semi-continuous functions f and h, the function $(f^* + h^*)^*$ is called (the closure of) the *infimal convolution* of f and h. If one of the functions is differentiable, then so is their infimal convolution; hence this operation is a *regularization*. In the next example, we regularize the (nondifferentiable) absolute value function $|x|$ with $\frac{1}{2}x^2$. We also make use of `simplify`.

```
> f1; f9 := convert(1/2*x^2,cf);
```

$$\begin{cases} -x & (-\infty < x) \textbf{ and } (x < 0) \\ 0 & x = 0 \\ x & (0 < x) \textbf{ and } (x < \infty) \end{cases}$$

$$f9 := \begin{cases} \frac{1}{2}x^2 & (-\infty < x) \textbf{ and } (x < \infty) \end{cases}$$

```
> simplify(conj(f1,y) + conj(f9,y));
```

$$\begin{cases} \infty & (-\infty < y) \textbf{ and } (y < -1) \\ \frac{1}{2} & y = -1 \\ \frac{1}{2}y^2 & (-1 < y) \textbf{ and } (y < 1) \\ \frac{1}{2} & y = 1 \\ \infty & (1 < y) \textbf{ and } (y < \infty) \end{cases}$$

```
> conj(",x);
```

$$\begin{cases} -x - \frac{1}{2} & (-\infty < x) \textbf{ and } (x < -1) \\ \frac{1}{2} & x = -1 \\ \frac{1}{2}x^2 & (-1 < x) \textbf{ and } (x < 1) \\ \frac{1}{2} & x = 1 \\ x - \frac{1}{2} & (1 < x) \textbf{ and } (x < \infty) \end{cases}$$

10. A case of Young's inequality

Suppose $p \in (1, \infty)$ and let q be given by $\frac{1}{p} + \frac{1}{q} = 1$. The inequality

$$\tfrac{1}{p}a^p + \tfrac{1}{q}b^q \geq ab, \quad \forall a \geq 0, \forall b \geq 0,$$

is often referred to as *Young's inequality*. It can be used to give an easy proof of *Hölder's inequality*. In fact, since

$$\left(\tfrac{1}{p}|\cdot|^p\right)^* = \tfrac{1}{q}|\cdot|^q,$$

Young's is a special case of the Fenchel/Young inequality. In our last example, we derive the conjugate pair for $p = 3$.

```
> f10 := convert(1/3*abs(x)^3,cf);
```

$$f10 := \begin{cases} -\frac{1}{3}x^3 & (-\infty < x)\ \mathbf{and}\ (x < 0) \\ 0 & x = 0 \\ \frac{1}{3}x^3 & (0 < x)\ \mathbf{and}\ (x < \infty) \end{cases}$$

```
> g10 := conj(f10,y);
```

$$g10 := \begin{cases} -\frac{2}{3}y\sqrt{-y} & (-\infty < y)\ \mathbf{and}\ (y < 0) \\ 0 & y = 0 \\ \frac{2}{3}y^{3/2} & (0 < y)\ \mathbf{and}\ (y < \infty) \end{cases}$$

3 Limitations and known bugs

Because `limit` and `diff` are well implemented in Maple, finding subdifferentials with `subdiff` does not pose a problem. On the other hand, computing Fenchel conjugates is hard. Its key step is the inversion of the subdifferential, which relies on `solve`. Failure of `conj` can occur for two reasons.
(1) *Problems inverting the Fenchel/Young equality.* A typical example is $f(x) := \frac{1}{4}x^4$. Maple correctly finds three solutions of $\nabla f(x) = y$, *i.e.*, three cubic roots of y. But none of these expression is the real root for *all* values of y. In essence, this is why we cannot recover the p-powers discussed in Example 10.
(2) *Unrelated problems.* Illustrative is the function $f(x) := \cosh(x)$. `conj` indeed correctly finds $f^*(y) = y\mathrm{arcsinh}(y) - \sqrt{1+y^2}$. Now conjugating f^* should bring us back to f; instead, we obtain an error in `simplify`. It turns out that Maple V.4 does not like to
`simplify(arcsinh(sinh(x)));`

In summary, we expect that future releases of Maple and `fenchel` should result in fewer errors of both type (1) and (2).

4 Concluding remarks

In `fenchel`, we have implemented Fenchel conjugation and subdifferentials for convex functions on the real line. We have presented well-working examples and commented on limitations of the code. We hope that: • `fenchel` will be useful to both researchers and instructors in convex analysis and optimization; • `fenchel` will spark further activities in this area resulting perhaps in code that tackles nonseparable multi-dimensional convex functions; • last but not least, we have been able to transmit some of our enthusiasm for convex analysis to the reader.

Acknowledgment

We wish to thank Jon Borwein for discussions and comments that led to the creation of `fenchel` and to subsequent improvements.

A very special thanks goes to Greg Fee for several most helpful suggestions and comments.

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