SYMBOLIC COMPUTATION OF THE LEGENDRE–FENCHEL CONJUGATE AND BICONJUGATE OF A QUARTIC UNIVARIATE POLYNOMIAL

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ABSTRACT. Given a quartic univariate polynomial P, we compute symbolically its Legendre–Fenchel biconjugate and conjugate.

The Legendre–Fenchel conjugate of a univariate function P is defined as

$$P^*(s) := \sup_{x \in \mathbb{R}} [sx - P(x)]. \tag{1}$$

We want to compute *explicitly* the point $\hat{X}(s)$ where this maximum is attained when P is a quartic univariate polynomial.

The key idea is to note that the biconjugate or convex hull P^{**} of P is obtained by adding a bitangent to Epi$P$ (see Figure 1). In other words,

$$P^{**}(x) = \begin{cases} P(x) & \text{if } x \notin [x_1, x_2], \\ P'(x)(x - x_1) + P(x_1) & \text{otherwise;} \end{cases} \tag{2}$$

with $x_1 \neq x_2$ satisfying $P'(x_1) = P'(x_2)$ and $x_1 P'(x_1) - P(x_1) = x_2 P'(x_2) - P(x_2)$ (see Figure 1).

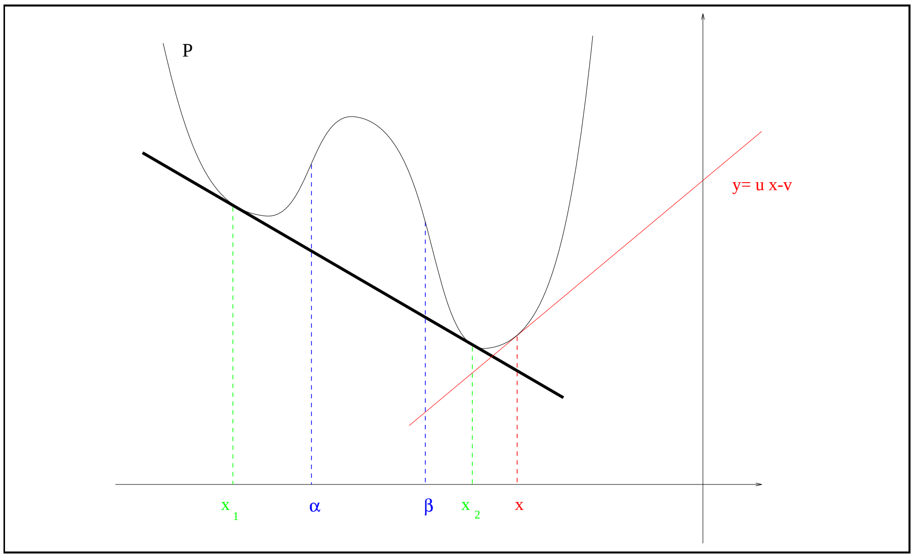

FIGURE 1. Computing the biconjugate of P amounts to adding bitangents to Graph$P$

Date: September 30, 1998.

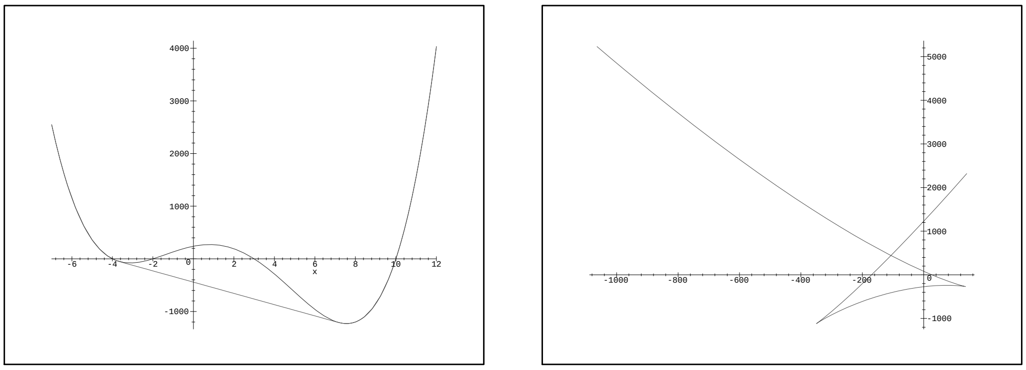

FIGURE 2. The biconjugate and the Legendre–Transform of a quartic polynomial: $P(x) = x^4 - 7\,x^3 - 40\,x^2 + 76\,x + 240$

In addition, we can parametrized GraphP^* by

$$\begin{cases} s & = P'(x) \\ P^*(s) & = sx - P(x) \end{cases} \quad x \in (-\infty, x_1] \cup [x_2, +\infty). \tag{3}$$

So computing P^{**} amounts to finding x_1, x_2, and $\overline{u} := P'(x_1)$.

If we let the parameter x describe the whole real line in (3), we obtain a contact transformation (see Figure 2). To obtain P^*, we must remove a part of the graph of the function $F : x \mapsto (P'(x), xP'(x) - P(x))$. We note that if $\alpha \leq \beta$ are the real roots of P'' (if there is no real root, P is convex and we are done), the "singular" points of Graph$F$ are $F(\alpha)$ and $F(\beta)$.

All in all, computing the convex hull amounts to computing x_1 and x_2. Next, solving a 3rd degree polynomial equation gives the usual nonparametric representation of the conjugate.

1. Computing the convex hull

We define

$$\begin{aligned} P_1(x) &:= u - P'(x), \\ P_2(x) &:= v - xP'(x) + P(x). \end{aligned}$$

The line $y = ux - v$ is tangent to Graph$P$ at x (see Figure 1). We are looking for $x_1 \neq x_2$ such that $F(x_1) = F(x_2)$. In other words, x_1 and x_2 are solutions of the polynomial system

$$\begin{cases} P_1(x) = 0, \\ P_2(x) = 0. \end{cases}$$

We deduce that the resultant $Q(u, v)$ of P_1 and P_2 is null (the function $Q(u, v)$ is null if and only if P_1 and P_2 have a common root). It allows us to eliminate x between both equations to obtain an implicit representation of Graph$F$: $Q(u, v) = 0$.

The next step is to note that if $\nabla Q(\tilde{u}, \tilde{v}) \neq 0$, there is no multiple point in a neighborhood of $(\tilde{u}, \tilde{v})$. Indeed, from the Implicit Function Theorem $Q(u, v) = 0$

can be locally written as $v = q(u)$ with q a smooth function. So there can be no double point in a neighborhood of $(\tilde{u}, \tilde{v})$.

Lemma. *All multiple point of GraphF lie in*
$S := \{(u, v) \in \mathbb{R} \times \mathbb{R} \,:\, \nabla Q(u, v) = 0$ *and* $Q(u, v) = 0\}$.

Finally S contains $F(\alpha)$ and $F(\beta)$ and can have at most 3 points. Hence the remaining one is $(\overline{u}, \overline{v})$ defining the bitangent.

Eventually

- First, compute $Q(u, v)$ the resolvant of P_1 and P_2.
- Next, solve $\nabla Q(u, v) = 0$ and deduce $(\overline{u}, \overline{v})$.
- Finally, solve $P'(x) = \overline{u}$ to obtain x_1 and x_2.

If we apply our method to a quartic polynomial, we deduce the following result.

Proposition (Convex hull computation)**.** *Let* $P(x) := x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. *If* $3a_3^2 - 8a_2 \leq 0$, P *is convex:* $P^{**} = P$.
Otherwise $3a_3^2 - 8a_2 > 0$ *and*

$$\begin{aligned}
\overline{u} &= -\frac{1}{2}\,a_3\,a_2 + \frac{1}{8}\,{a_3}^3 + a_1 \\
\overline{v} &= \frac{1}{64}\,{a_3}^4 - \frac{1}{8}\,a_2\,{a_3}^2 + \frac{1}{4}\,{a_2}^2 - a_0 \\
x_1 &= -\frac{1}{4}\,a_3 - \frac{1}{4}\,\sqrt{3\,{a_3}^2 - 8\,a_2} \\
x_2 &= -\frac{1}{4}\,a_3 + \frac{1}{4}\,\sqrt{3\,{a_3}^2 - 8\,a_2}
\end{aligned}$$

Proof. Only the last part is non-trivial. To proove it, check directly that equalities $P'(x_1) = P'(x_2)$ and $x_1P'(x_1) - P(x_1) = x_2P'(x_2) - P(x_2)$ hold with $x_1 \neq x_2$ (in fact, $x_1 < \alpha < \beta < x_2$, where $\alpha < \beta$ are the real roots of $P''(x) = 0$). □

2. Computing the conjugate

As for now, we only know a parametric representation of P^*, namely (3). Obtaining an explicit writting implies solving a 3rd degree polynomial equation.

We summarize a well-known method to do it. Using $X := x + a_3/4$, we transform the equation $P'(x) = s$ into

$$X^3 + pX + q - \frac{s}{4} = 0, \tag{4}$$

with $p := \dfrac{1}{4}(2a_2 - \dfrac{3}{4}a_3^2)$ and $q := \dfrac{1}{4}(\dfrac{1}{8}a_3^3 - \dfrac{1}{2}a_2a_3 + a_1)$. Now, denoting $j := \exp(2i\pi/3)$ the (usual) cubic root of 1, we have

$$(X - A - B)(X - jA - j^2B)(X - j^2A - jB) = X^3 - 3ABX - (A^3 + B^3).$$

Hence we are looking for $X := A + B$ with $AB = -p/3$ and $A^3 + B^3 = s/4 - q$. Thus A^3 and B^3 are roots of the resolvant equation: $X^2 + (q - s/4)X - p^3/27 = 0$.

Let $\Delta := (q - s/4)^2 + 4p^3/27$ and consider the following three cases.

Case $\Delta > 0$ There are two distincts roots of the resolvant equation. The unique real root of (4) is given by Cardan's Formula:

$$\hat{X}(s) := \left[\frac{1}{2}\left(\frac{s}{4} - q + \sqrt{\left(q - \frac{s}{4}\right)^2 + \frac{4}{27}p^3}\right)\right]^{1/3} + \left[\frac{1}{2}\left(\frac{s}{4} - q - \sqrt{\left(q - \frac{s}{4}\right)^2 + \frac{4}{27}p^3}\right)\right]^{1/3} \tag{5}$$

Case $\Delta = 0$ There is a double root of the resolvant equation. Equation (4) has two distinct real roots. The unique *single* real root is

$$\hat{X}(s) := 2\left(\frac{1}{2}(\frac{s}{4} - q)\right)^{1/3}. \tag{6}$$

A continuity argument implies that the single root is the one we are looking for.

Case $\Delta < 0$ Define $r := 1/2(s/4 - q + i\sqrt{-\Delta})$ with i the usual square root of -1. Name γ a cubic root of r. Equation (4) has three distinct real roots: $\gamma + \overline{\gamma}$, $j\gamma + \overline{j\gamma}$, and $j^2\gamma + \overline{j^2\gamma}$ (where overline denotes complex conjugacy). In fact, Galloi's theory tells us that one cannot write these real roots with only *real* radicals[1].

The root we are looking for is given by the Maple function surd:

$$\hat{X}(s) := \operatorname{surd}\left(\frac{1}{2}\left[\frac{s}{4} - q + i\sqrt{-\left(q - \frac{s}{4}\right)^2 - 4\frac{p^3}{27}}\right], 3\right) + \overline{\operatorname{surd}\left(\frac{1}{2}\left[\frac{s}{4} - q + i\sqrt{-\left(q - \frac{s}{4}\right)^2 - 4\frac{p^3}{27}}\right], 3\right)}. \tag{7}$$

Indeed, the same continuity argument as above implies that we want the cubic root of r *closest* to the real axis. This is precisely the root the function surd computes.

Hence Formulas (5)–(7) allow us to compute the conjugate of P.

Theorem (Conjugate computation). *Let $P(x) := x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. If $3a_3^2 - 8a_2 \le 0$, P is convex, $\Delta \ge 0$, and the point where the maximum is attained in (1) is given by Formulas (5) and (6).*

Otherwise $3a_3^2 - 8a_2 > 0$, the convex hull of P is given by Formula (2) with x_1 and x_2 as in the previous proposition. The point where the maximum is attained in (1) is given by Formulas (5)–(7).

Proof. Only the convex case needs to be proved. Since $p = 1/16(8a_2 - 3a_3^2)$ is non-negative, $\Delta = (q - s/4)^2 + 4p^3/27$ is non-negative too, and the theorem follows. $\square$

Example 1. *Take $P(x) := (x^2 - 1)^2 = x^4 - 2x^2 + 1$. We easily compute $\bar{u} = \bar{v} = 0$, $x_1 = -1$, $x_2 = 1$, $p = -1$, and $q = 0$. The resolvant equation is: $X^2 - sX/4 + 1/27 = 0$ and the discriminant $\Delta = -4/27 + 1/16s^2$.*

[1] To compute the real and imaginary part of r, one has to solve a 3rd degree equation, thus coming back to the same problem. For that reason that case was refered to as "casus irreducibilis".

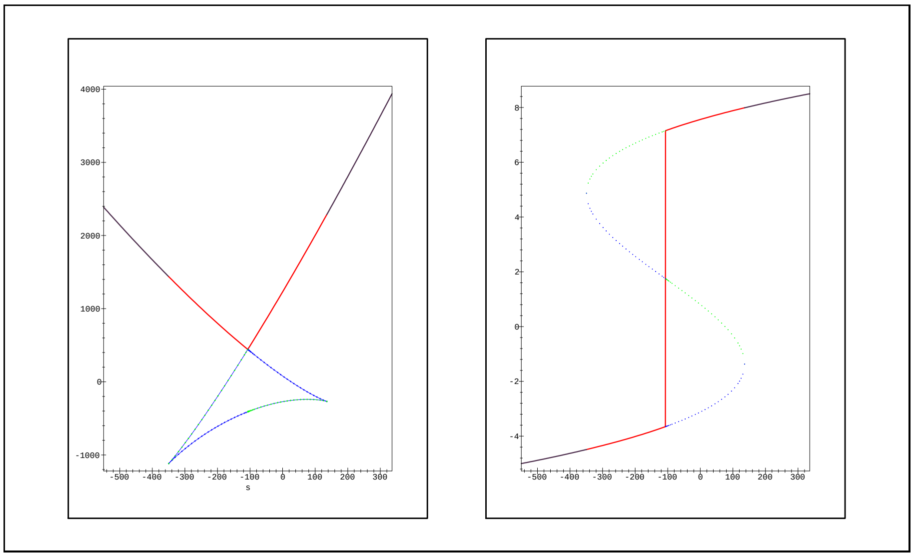

FIGURE 3. Removing two of the three branches of $P'(x) = s$ gives the conjugate

Consequently, if Δ is non-negative the solution is

$$\hat{X}(s) \;:= \left(\frac{1}{8}\,s + \frac{1}{72}\,\sqrt{81\,s^2 - 192}\right)^{1/3} + \left(\frac{1}{8}\,s - \frac{1}{72}\,\sqrt{81\,s^2 - 192}\right)^{1/3},$$

otherwise Δ is negative and the solution is

$$\hat{X}(s) \;:= 2\,Re(\mathrm{surd}(\frac{1}{8}\,s + \frac{1}{72}\,I\,\sqrt{192 - 81\,s^2},\,3));$$

where Re denotes the real part of a complex number.

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