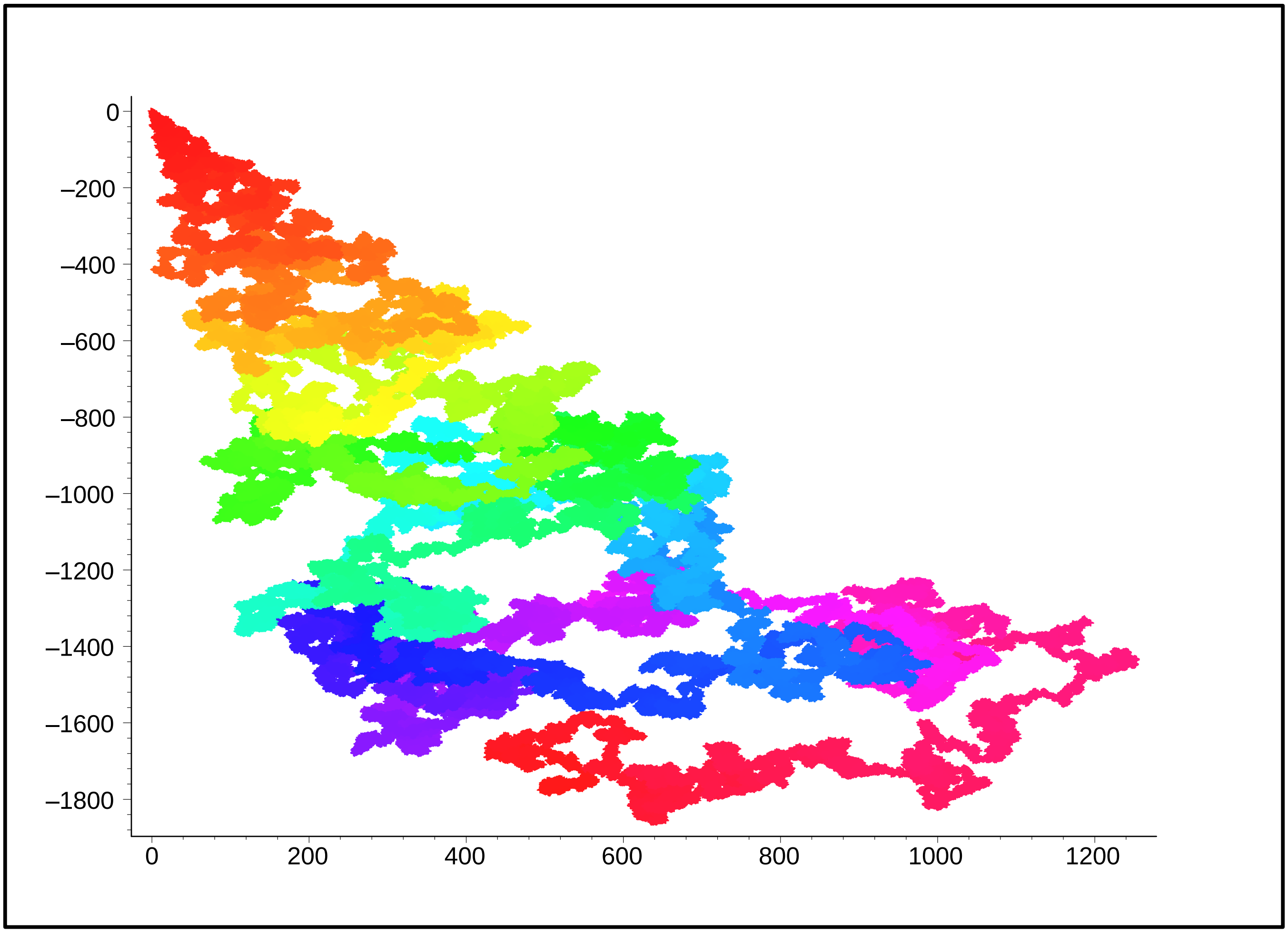
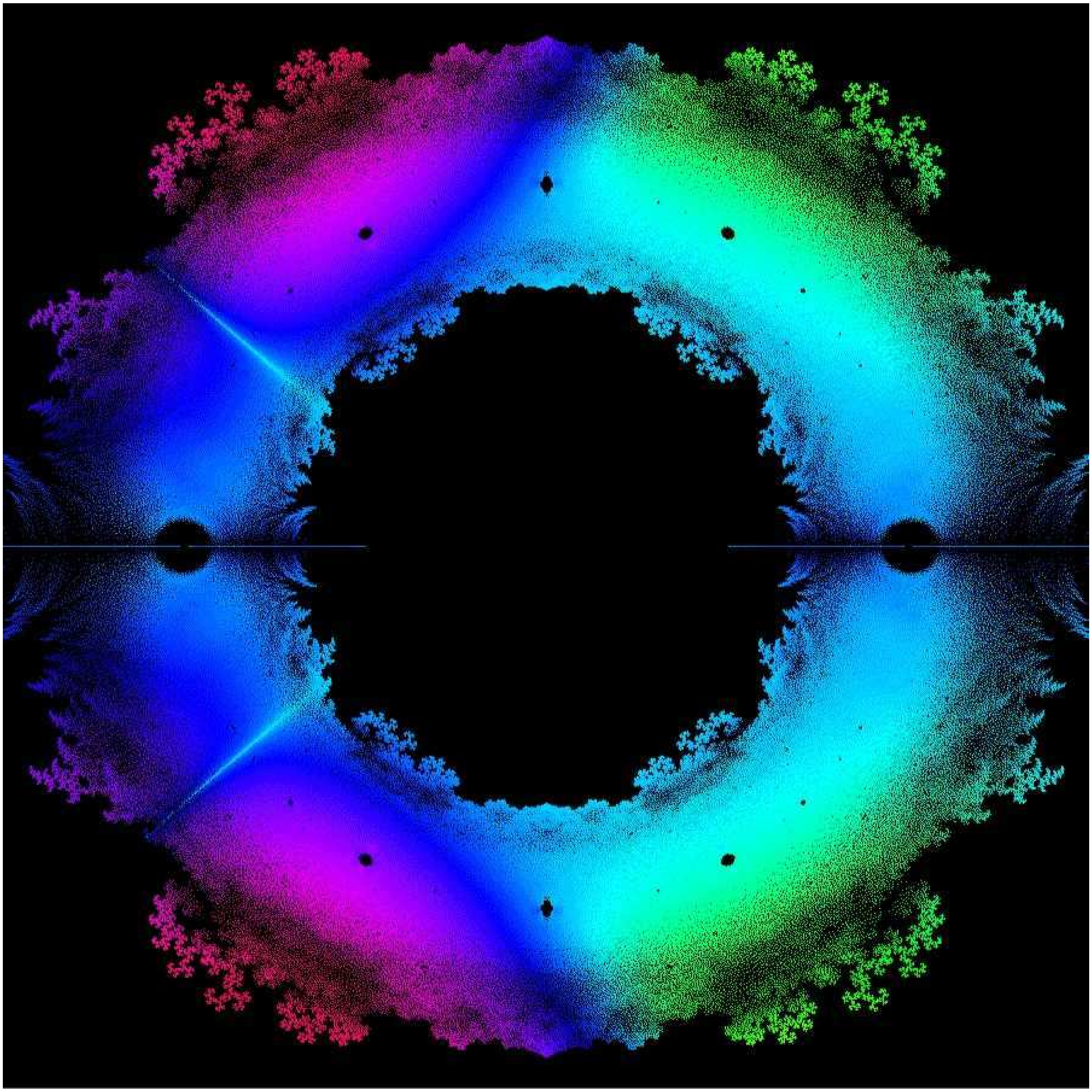
The Riemann Hypothesis

Peter Borwein




RESEARCH INSTITUTE
UNANSWERED QUESTIONS
UNQUESTIONED ANSWERS

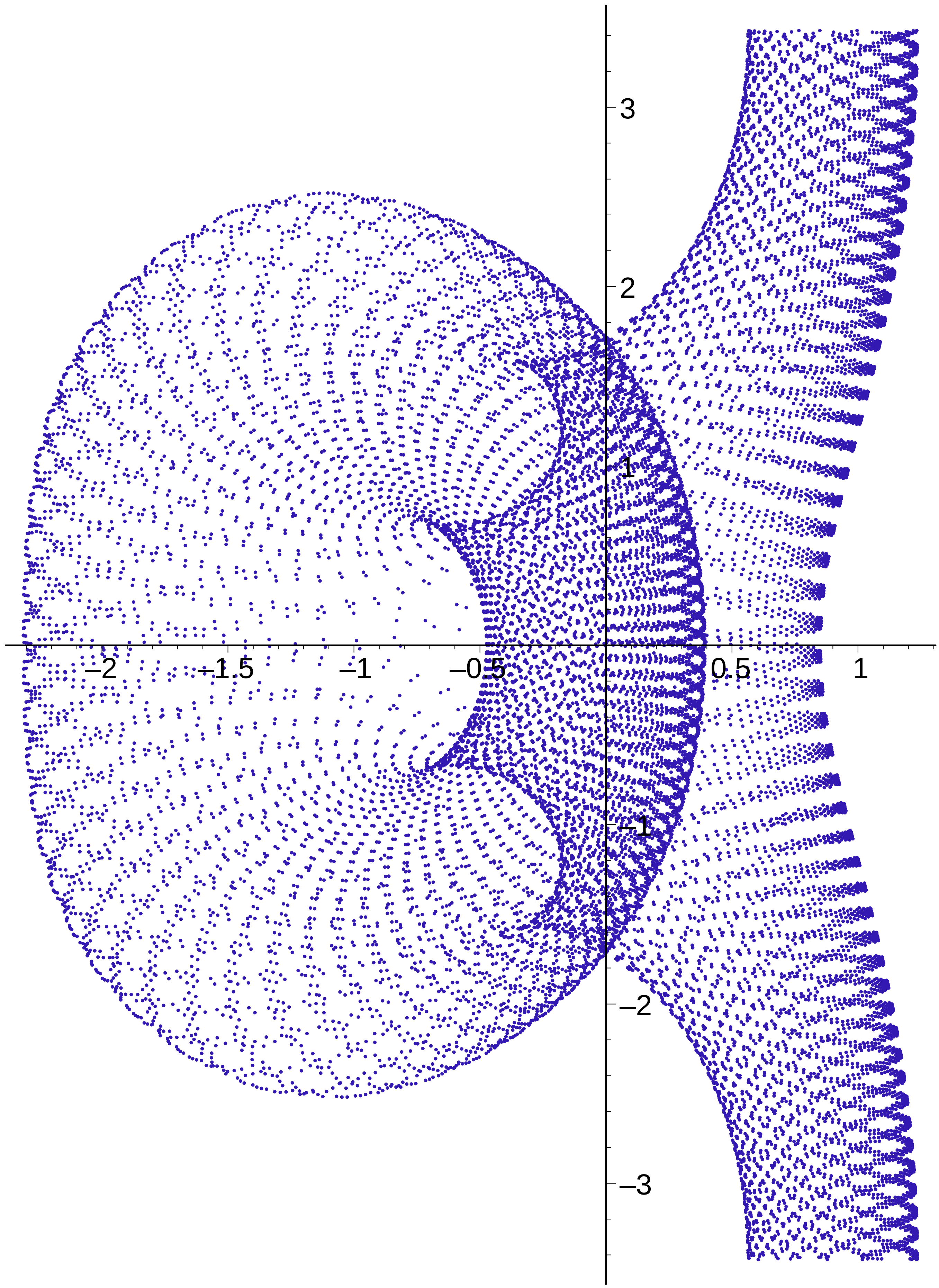


"Curiosity is part of human nature. Unfortunately, the established religions no longer provide the answers that are satisfactory, and that translates into a need for certainty and truth.

And that is what makes mathematics work, makes people commit their lives to it.

It is the desire for truth and the response to the beauty and elegance of mathematics that drives mathematicians"

Landon Clay (Wealthy mutual fund magnate and Harvard English graduate.)



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Millennium Prize Problems

“To celebrate mathematics in the new millennium, CMI identifies seven old and important mathematics questions that resisted all past attempts to solve them. Clay Mathematics Institute designates the $7 million prize fund for their solution, with $1 million allocated to each Millennium Prize Problem.

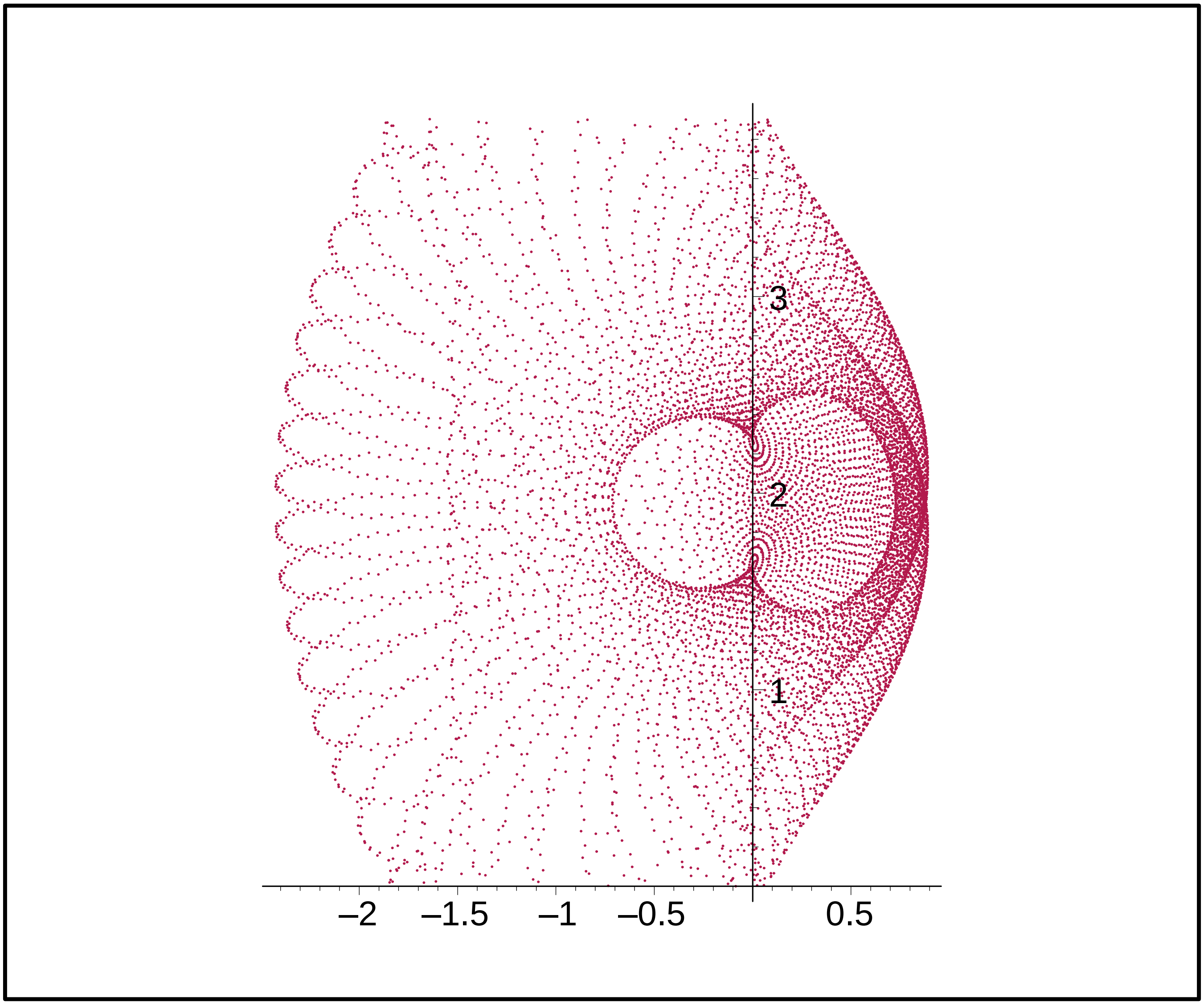
The Clay Mathematics Institute (CMI) is a private, non-profit foundation, dedicated to increase and to disseminate mathematical knowledge. The formation of CMI grew from the vision of Boston businessman Landon T. Clay working together with mathematician Arthur M. Jaffe: mathematics embodies the quintessence of human knowledge; mathematics reaches into every field of human endeavor; and the frontiers of mathematical understanding evolve today in deep and unfathomable ways.

Fundamental advances in mathematical knowledge go hand in hand with discoveries in all fields of science.

Technological applications of mathematics underpin our daily life, including our ability to communicate and to travel, our health and well-being, our security, and our global prosperity.

The evolution of mathematics today will remain a central ingredient in shaping our world tomorrow. To appreciate the scope of mathematical truth challenges the capabilities of the human mind.

CMI attempts to further the beauty, the power, and the universality of mathematical thought. Toward this end, CMI currently pursues a series of programs."



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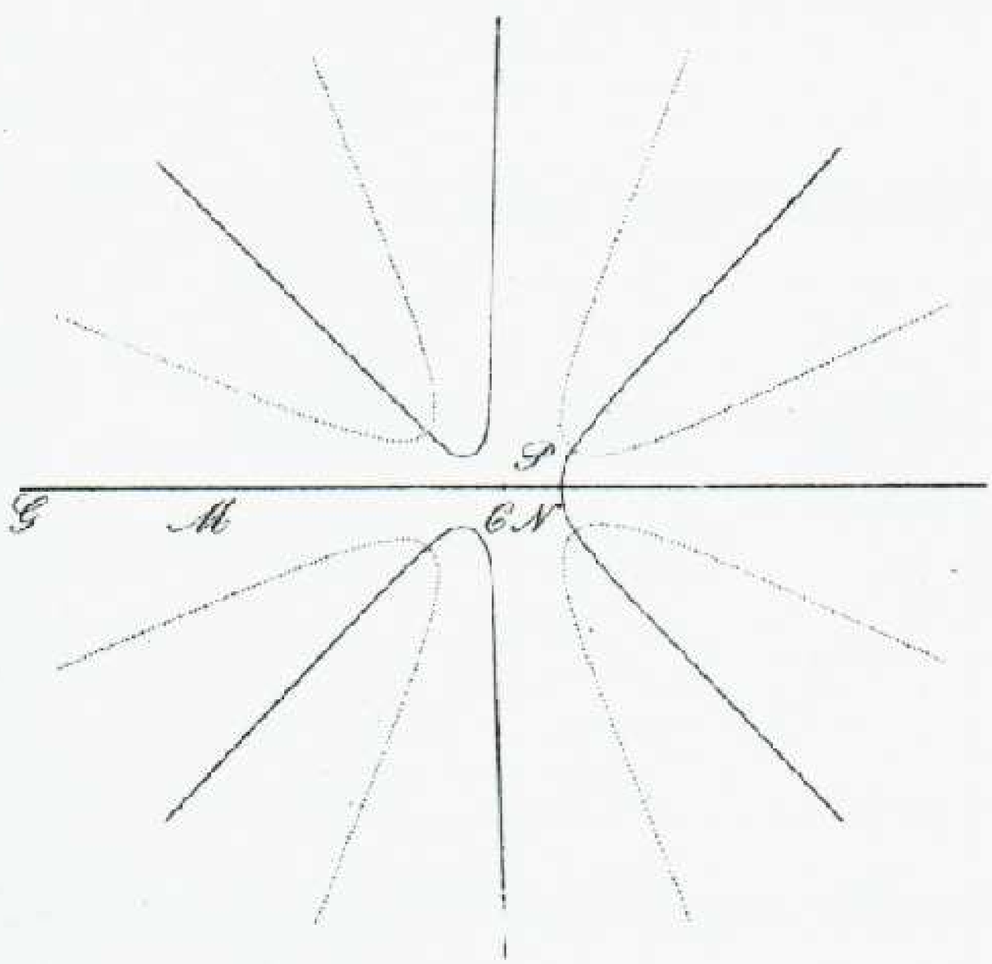
“It still remains true that, with negative theorems such as this, transforming personal convictions into objective ones requires deterringly detailed work. To visualize the whole variety of cases, one would have to display a large number of equations by curves; each curve would have to be drawn by its points, and determining a single point alone requires lengthy computations. You do not see from Fig. 4 in my first paper of 1799 , how much work was required for a proper drawing of that curve.”

K. F. Gauss (1777-1855)

Fig. 4.



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> Sum(1/n^6+1/n^8,n=1..infinity);
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$$\sum_{n=1}^{\infty} \left(\frac{1}{n^6} + \frac{1}{n^8} \right)$$

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> sum(1/n^6+1/n^8,n=1..infinity);
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$$\frac{1}{945}\pi^6 + \frac{1}{9450}\pi^8$$

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> Int(exp(x)*sin(x),x);
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$$\int e^x \sin(x)\, dx$$

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> int(exp(x)*sin(x),x);
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$$-\frac{1}{2}e^x \cos(x) + \frac{1}{2}e^x \sin(x)$$

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>
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"I hate getting all these Canadian coins, but I guess that's the price of living in Toronto."

6. RIEMANN HYPOTHESIS

Some numbers have the special property that they cannot be expressed as the product of two smaller numbers, e.g., 2, 3, 5, 7, etc. Such numbers are called prime numbers, and they play an important role, both in pure mathematics and its applications.

The distribution of such prime numbers among all natural numbers does not follow any regular pattern, however the German mathematician G.F.B. Riemann (1826 - 1866) observed that the frequency of prime numbers is very closely related to the behavior of an elaborate function $\zeta(s)$ called the Riemann Zeta function.

The Riemann hypothesis asserts that all interesting solutions of the equation

$$\zeta(s) = 0$$

lie on a straight line. This has been checked for the first 1,500,000,000 solutions. A proof that it is true for every interesting solution would shed light on many of the mysteries surrounding the distribution of prime numbers.

More about the RIEMANN HYPOTHESIS

Ask any professional mathematician to name the most important unsolved problem of mathematics and the answer is virtually certain to be, “the Riemann Hypothesis.”

Keith Devlin – The Millennium Problems – 2002

On the Number of Prime Numbers less than a Given Quantity.

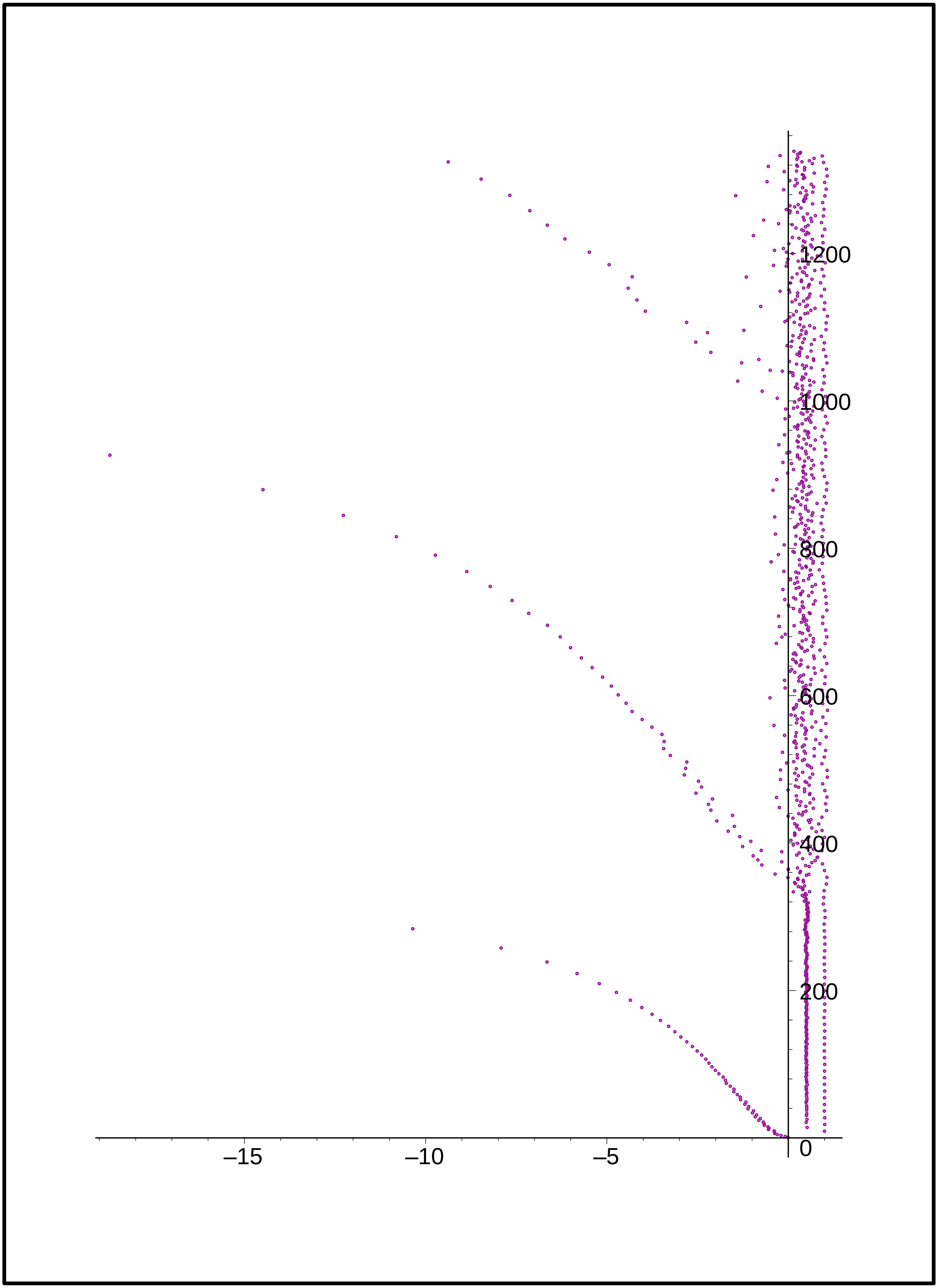
(Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.)

Bernhard Riemann

[Monatsberichte der Berliner Akademie, November 1859.]

Translated by David R. Wilkins

”One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation.”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The Holy Grail

The Holy Grail in mathematics is the Riemann Hypothesis. The problem, formulated in 1859 by Bernard Riemann, one of the extraordinary mathematical talents of the 19th century, makes a very precise connection between two seemingly unrelated objects, and if solved, would tell us something profound about the nature of mathematics and, in particular, prime numbers.

Why is the Riemann Hypothesis so important? Why is it the problem that mathematicians would make a pact with the devil to solve?

There are a number of great old unsolved problems in mathematics but none of them have quite the stature of the Riemann Hypothesis – for a variety of reasons both mathematical and cultural.

In common with the other old great unsolved problems, the Riemann Hypothesis is clearly very hard. It has resisted solution for 150 years and has been attempted by many of the greatest minds in mathematics.

David Hilbert one of the seminal figures in mathematical history re raised the problem at the 1900 International Congress of Mathematics, a conference held every 4 years and the most important international mathematics meeting. Hilbert, who by that time was the pre eminent mathematician of his generation, raised 23 problems that he thought would shape 20th century mathematics, and in large this proved to be true. This was somewhat self-fulfilling as solving a Hilbert problem was a guarantee of instant fame and perhaps local riches. Many of Hilbert's problems have been now been solved. The most notable recent example being the Fermat problem solved by Andrew Wiles in 1993–5.

Being one of Hilbert's 23 problems was enough to guarantee the Riemann problem being central. There is now also a million dollar bounty in the form of a so called "Millennium Prize Problem" of the Clay Mathematics Institute of Cambridge. (There are seven such mathematical problems each with a million dollar prize associated with their solution.)

Solving one of the great unsolved problems in mathematics is akin to the first ascent of Everest. It is a formidable achievement but after the conquest there is sometimes nowhere to go but down. Some of the great problems proved to be isolated mountain peaks not connected to any others.

The Riemann Hypothesis is quite different in this regard. There is a large body of mathematics that would instantly become proved if the Riemann Hypothesis was solved. We know many statements of the form ”if the Riemann Hypothesis then the following interesting mathematical statement” and this is rather different from the solution of problems such as the Fermat problem.

The Riemann Hypothesis can be formulated in many diverse and seemingly unrelated ways, this is one of its beauties.

The most common formulation is that certain numbers, the zeros of the "Riemann Zeta function" all lie in on certain place (precise definitions later) and this formulation can to some extent be checked numerically.

In one of the largest calculations ever done to date, it was checked that the first hundred billion of these zeros lie where they are supposed to lie. So there are a hundred billion pieces of evidence indicating that the Riemann Hypothesis is true and not a single piece of evidence indicating that it is false. A physicist might be overwhelmingly pleased with this much evidence in its favor but to the mathematical experts this is hardly evidence at all.

Though it is interesting ancillary information.

A proof is required that all of these numbers lie in the right place, not just the first hundred billion, and until the proof is provided the Riemann Hypothesis cannot be incorporated into the corpus of mathematical facts and accepted as true by mathematicians. (Even though it is probably true!) This is not just pedantic fussiness.

A feature of the mathematics related to the Riemann Hypothesis is that certain phenomena that appear likely true and that can be tested in part computationally are false but only false past computational range.

Accept for a moment that the Riemann Hypothesis is the greatest unsolved problem in mathematics and that the greatest achievement any young graduate student could aspire to is to solve it. Why isn't it better known? Why hasn't it permeated public consciousness? (The way black holes and unified field theory have, at least to some extent.)

Part of the reason for this is it is hard to state precisely. It requires most of an undergraduate degree in mathematics to be familiar with enough of the mathematical objects to even accurately state the Riemann Hypothesis. Our suspicion is that only a minority of professional mathematicians –perhaps a quarter – can state the Riemann Hypothesis if asked.

The Liouville λ function and RH

For $\Re s > 1$, the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

The Riemann Hypothesis is usually given as: the nontrivial zeros of the Riemann zeta function lie on the line $\Im s > 1$.

(There is already, of course, the problem that the above series doesn't converge on this line so one is already talking about an analytic continuation.)

Our immediate goal is to give as simple an (equivalent) statement of the Riemann Hypothesis as we can.

Loosely the statement is " the number of integers with an even number of prime factors is the same as the number of integers with an odd number of prime factors."

This is made precise in terms of the Liouville Function.

The Riemann Hypothesis is equivalent to the statement that an integer has equal probability of having an odd number or an even number of distinct prime factors, a statement with some intuitive appeal.

The Liouville Function gives the parity of the number of prime factors.

The **Liouville Function** is defined by

$$\lambda(n) = (-1)^{\omega(n)}$$

where $\omega(n)$ is the number of distinct prime factors in n with multiple factors counted multiply.

So

$$\lambda(1) = \lambda(2) = \lambda(5) = \lambda(7) = \lambda(8) = -1$$

and

$$\lambda(4) = \lambda(6) = \lambda(9) = \lambda(10) = 1.$$

(Alternatively one can define λ as the completely multiplicative function with $\lambda(p) = -1$ for any prime p.)

The connection between the Liouville function and the Riemann Hypothesis were explored by Landau in his doctoral thesis of 1899.

Theorem 1 *The Riemann Hypothesis is equivalent to*

$$\gamma(n) := \lambda(1) + \lambda(2) + \cdots + \lambda(n) \ll n^{1/2+\epsilon},$$

for every positive ϵ.

This is saying that the sequence

$$\{\lambda i\}_{i=1} := \{1, -1, -1, 1, -1, \ldots\}$$

behaves more-or-less like a random sequence of of plus and minus ones in that the difference between the number of plus one's and minus ones is not much larger that the square root of the number of terms.

The proof of the equivalence is relatively easy. We give the proof that the growth of $\gamma(n)$ implies the Riemann Hypothesis.

proof It is well known (Hardy and Wright p 255) and not very hard that

$$\frac{\zeta(2s)}{\zeta(s)} = \frac{1}{\prod_{p\,prime}(1+p^{-s})}$$

$$= 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s} - \frac{1}{5^s} - \cdots$$

$$= \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}.$$

and we have

$$\frac{\zeta(2s)}{\zeta(s)} := \int_0^{\infty} x^{-s} dW(x)$$

where W is the Stieltjes measure defined as follows. W is a step function, $W(0) = 0$ and W has a jump of $\lambda(n)$ at n. Also

$$\frac{W(n-\epsilon) + W(n+\epsilon)}{2} = \frac{1}{2}\lambda(n) + \sum_{j=1}^{n-1} \lambda(j).$$

Now, if, for some $\delta > 0$

$$|W(x)| \ll x^{\delta}$$

then the above integral actually converges for $\Re(s) > \delta$. So

$$\frac{\zeta(2s)}{\zeta(s)} = s \int_0^{\infty} W(x) x^{-s-1} dx$$

continues analytically for $\Re(s) > \delta$ and thus $\zeta(s)$ can't vanish here.

Landau in his doctoral thesis of 1899 also proved the following.

Theorem 2 (Landau)

$$\frac{\lambda(1)+\lambda(2)+\cdots+\lambda(n)}{n} \to 0$$

is equivalent to the Prime Number Theorem

This can be made the basis for an elementary (though not easy)proof of the Prime Number Theorem.

One also has

Theorem 3 (Landau)

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n}$$

converges is equivalent to the the Prime Number Theorem.

Turán's conjecture

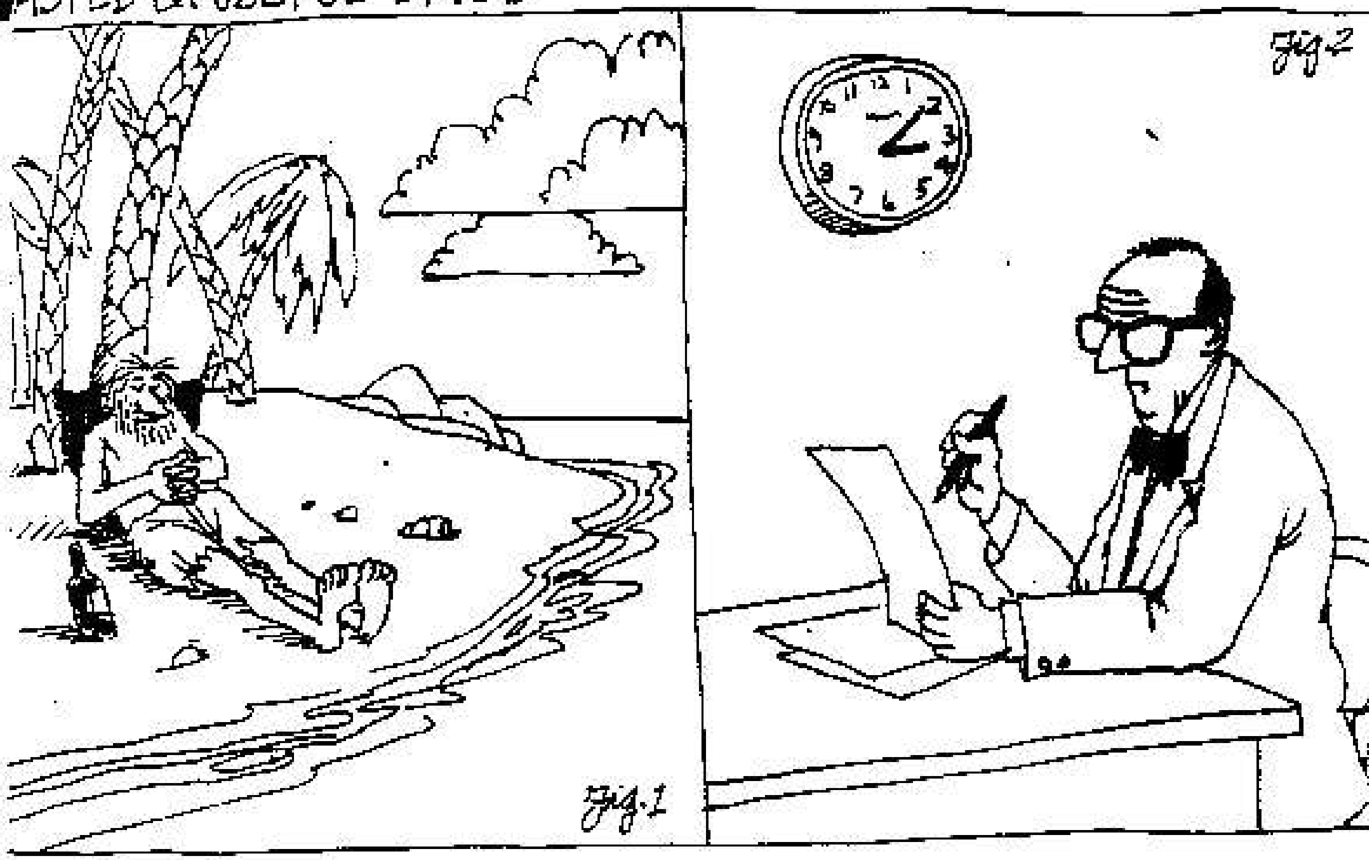
Turán conjectured that that for all n

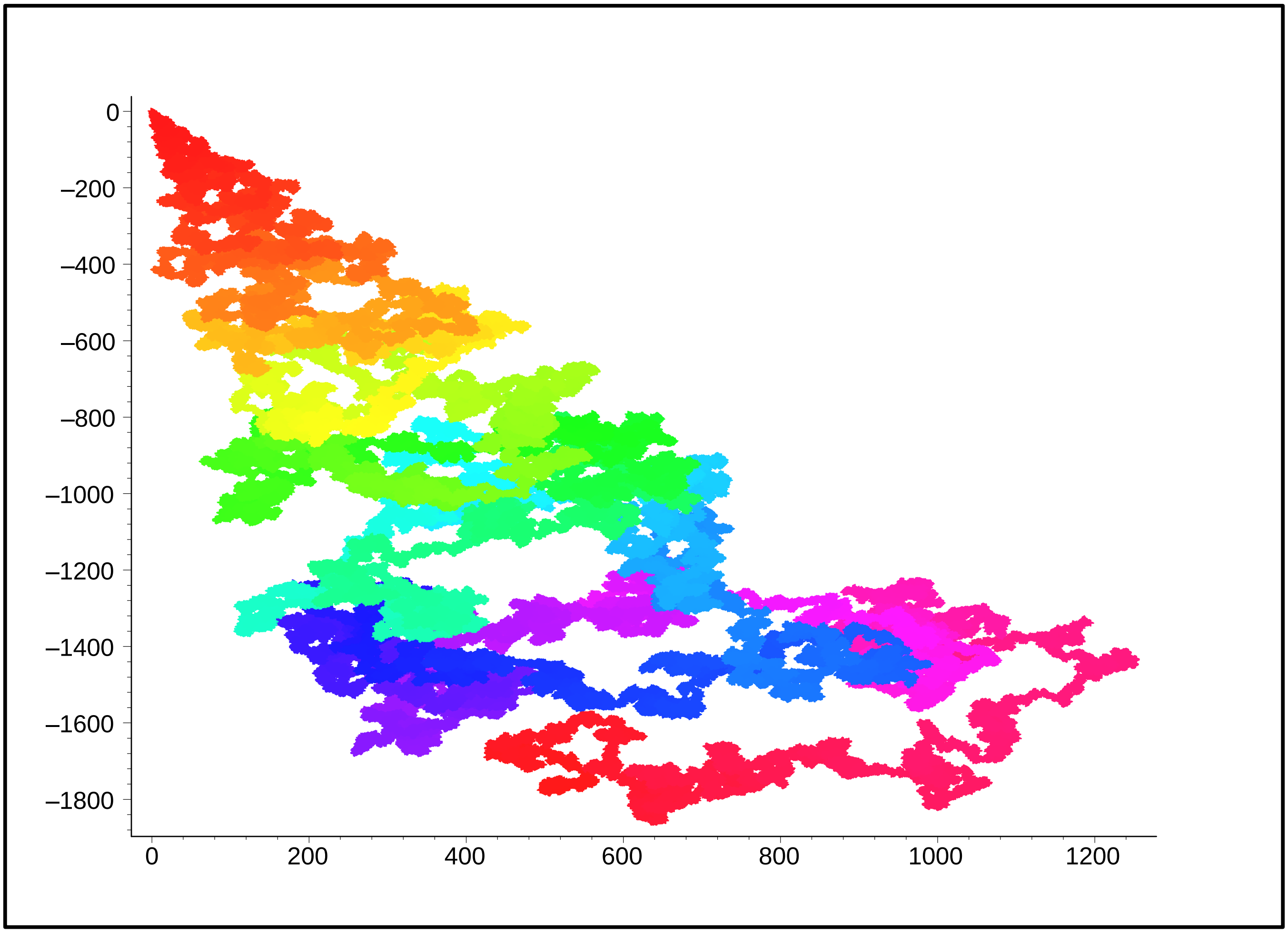
$$\sum_{i=1}^{n} \frac{\lambda(i)}{i} > 0.$$

This would imply the Riemann Hypothesis. However it is provably false.

Though no actual counterexample is known. It is true at least up to $n = 10^{15}$ and is a cautionary example on not trusting the numbers.

WASTED & USEFUL LIVES





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