CHEBYCHEV PROBLEMS FOR POLYNOMIALS WITH INTEGER COEFFICIENTS

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Introduction.

Let

$$\mathcal{Z}_n := \{\sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{Z}\}$$

denote the polynomials of degree n with integer coefficients.

Let

$$(*) \qquad \Omega_n[a,b] := \left(\inf_{0 \neq p \in \mathcal{Z}_n} \|p\|_{[a,b]} \right)^{1/n}$$

and let

$$\Omega[a,b] := \inf_n \{\Omega_n[a,b]\} = \lim_{n\to\infty} \Omega_n[a,b].$$

Any polynomial satisfying $(*)$ above is called an n-th **integer Chebyshev polynomial** on $[a,b]$.

A main (hard) problem is to determine $\Omega[0,1]$.

The above limit exists and equals the inf mostly because

$$(\Omega_{n+m}[a,b])^{n+m} \leq (\Omega_n[a,b])^n\,(\Omega_m[a,b))^m\,.$$

We have from the unrestricted case the trivial inequality

$$\Omega[a,b] \geq \frac{b-a}{4}.$$

We also have

$$\Omega[a,b] \leq \Omega_n[a,b]$$

for any particular n.

Thus good upper bounds can be achieved by computation (although the computation to any degree of accuracy is hard).

The limit $\Omega[a,b]$ may be thought an integer version of the **transfinite diameter**.

Hilbert showed that there exists an absolute constant c so that

$$\inf_{0 \neq p \in \mathcal{Z}_n} \|p\|_{L_2[a,b]} \leq c n^{1/2} \left(\frac{b-a}{4}\right)^{1/2}$$

and Fekete showed that

$$(\Omega_n[a,b])^n \leq 2^{1-2^{-n-1}} (n-1) \left(\frac{b-a}{4}\right)^{n/2}$$

There are many refinements.

From the above it follows that

$$\frac{b-a}{4} \leq \Omega[a,b] \leq \left(\frac{b-a}{4}\right)^{1/2}$$

We restrict to $b - a \leq 4$.

There is a pretty argument due to Gelfond to see that integer coefficients really are a restriction on $[0,1]$.

If $0 \neq p_n \in \mathcal{Z}_n$ then

$$\|p_n\|^2_{[0,1]} \geq \|p_n\|^2_{L_2[0,1]} = \int_0^1 p_n^2(x)dx$$

$$= \frac{m}{\mathrm{LCM}(1,2,\ldots,2n+1)} \neq 0$$

where LCM denotes the least common multiple.

Now $\mathrm{LCM}(1,2,\ldots,n))^{1/n} \sim e$, by the prime number theorem and it follows that

$$\Omega[0,1] \geq 1/e.$$

This is not however the right lower bound.

The best previous bounds, due to Amoroso and Gorshkov,

$$\frac{1}{(2.37686\dots)} \leq \Omega[0,1] \leq \frac{1}{(2.3541\dots)}.$$

The upper bound comes by example.

We work hard to improve this result in the third digit in the upper bound and in the twenty seventh digit in the lower bound.

For the above lower bound we use

Lemma. *Suppose $p_n \in \mathcal{Z}_n$ and suppose $q_k(z) := a_k z^k + \cdots + a_0 \in \mathcal{Z}_k$ has all its roots in $[a, b]$. If p_n and q_k do not have common factors then*

$$\left(\|p_n\|_{[a,b]}\right)^{1/n} \geq |a_k|^{-1/k}.$$

Proof. Let $\beta_1, \beta_2, \ldots, \beta_k$ be the roots of q_k. Then

$$|a_k|^n p_n(\beta_1) p_n(\beta_2) \cdots p_n(\beta_k)$$

is a non-zero integer and the result follows. $\square$

There exist infinitely many relatively prime polynomials $q_k \in \mathcal{Z}_k$ with all roots in $(0, 1)$, and with lead coefficients satisfying $a_k^{1/k} \leq 2.37686\ldots$. This comes from iterating $(x - 1/x)$ on $(-\infty, \infty)$.

This number (2.376...) was conjectured (by the Chudnovsky's) to be $\Omega[0, 1]$.

It is also conjectured (by H. Montgomery) that the best bound given by the lemma is $\Omega[0, 1]$.

Computing Integer Chebyshev Polynomials.

We restrict our attention to the interval $[0, 1]$. Though we observe in passing that

$$(\Omega[-1, 1])^4 = (\Omega[0, 1])^2 = \Omega[0, 1/4]$$

as a consequence of the changes of variable $x \to x^2$ and $x \to x(1 - x)$ and symmetry.

The dependence of the constant $\Omega[a, b]$ and the minimal polynomials on $[a, b]$ is interesting and is explored a little further later.

Even computing low degree examples is complicated. There is no good algorithm and getting examples of say degree 100 seems intractable.

Very Small Examples.

n	n-th integer Chebyshev poly on [0,1]
1	x or $(1-x)$ or $(2x-1)$
2	$x(1-x)$
3	$x(1-x)(2x-1)$
4	$x^2(1-x)^2$ or $x(1-x)(2x-1)^2$
5	$x^2(1-x)^2(2x-1)$
6	$[x(1-x)(2x-1)]^2$

Note that we do not have uniqueness, though it is open as to whether we have uniqueness for n sufficiently large.

Examples in $L_2[0,1]$. which minimize $\|p_n\|_{L_2[0,1]}$
This was done in pari by using **minum**.

Degree 13

$$(5z^2-5z+1)(2z-1)^2z^4(z-1)^5$$

$$(5z^2-5z+1)(2z-1)^2(z-1)^4z^5$$

$$(5z^2-5z+1)(2z-1)^3(z-1)^4z^4$$

$$(2z-1)(5z^2-5z+1)(z-1)^5z^5$$

$$(2z-1)(5z^2-5z+1)^2(z-1)^4z^4$$

$$(2z-1)^3(z-1)^5z^5$$

$$(2z-1)(z-1)^4z^4(29z^4-58z^3+40z^2-11z+1)$$

Degree 14

$$(5z^2-5z+1)(2z-1)^2(z-1)^5z^5$$

Degree 16

$(5z^2 - 5z + 1)(2z - 1)^2(z - 1)^6 z^6$

Degree 17

$(5z^2 - 5z + 1)(2z - 1)^3(z - 1)^6 z^6$

Degree 18

$(2z - 1)^2(5z^2 - 5z + 1)(z - 1)^6 z^6$

Degree 19

$(5z^2 - 5z + 1)(2z - 1)^3(z - 1)^7 z^7$

Degree 20

$(5z^2 - 5z + 1)(29z^4 - 58z^3 + 40z^2 - 11z + 1)(2z - 1)^2(z - 1)^6 z^6$

Let

$$
\begin{aligned}
p_0(x) :=& x \\
p_1(x) :=& 1 - x \\
p_2(x) :=& 2x - 1 \\
p_3(x) :=& 5x^2 - 5x + 1 \\
p_4(x) :=& 13x^3 - 19x^2 + 8x - 1 \\
p_5(x) :=& 13x^3 - 20x^2 + 9x - 1 \\
p_6(x) :=& 29x^4 - 59x^3 + 40x^2 - 11x + 1 \\
p_7(x) :=& 31x^4 - 61x^3 + 41x^2 - 11x + 1 \\
p_8(x) :=& 31x^4 - 63x^3 + 44x^2 - 12x + 1 \\
p_9(x) :=& 941x^8 - 3764x^2 + 6349x^6 - 5873x^5 \\
& 3243x^4 - 1089x^3 + 216x^2 - 23x + 1
\end{aligned}
$$

We have

Proposition. *Let*

$$P_{210} := p_0^{67} \cdot p_1^{67} \cdot p_2^{24} \cdot p_3^{9} \cdot p_4 \cdot p_5 \cdot p_6^{3} \cdot p_7 \cdot p_8 \cdot p_9$$

then

$$\left(\|P_{210}\|_{[0,1]}\right)^{1/210} = \frac{1}{(2.3543\dots)}$$

and hence

$$\Omega[0,1] \leq \frac{1}{(2.3543\dots)}.$$

Proof. This proof is obviously just a computational verification. It is the algorithm for finding P_{210} which is of some interest. It is based on LLL lattice basis reduction in the following way.

a] Lattice basis reduction finds a short vector in a lattice. If we construct a lattice of the form

$$p(z) \cdot \sum_{k=0}^{n} \alpha_k z^k = \sum_{k=0}^{m} \beta_k z^k$$

where p is a fixed polynomial and the set

$$\{(\alpha_0, \alpha_1, \dots, \alpha_n)\}$$

is a lattice then the set

$$\{(\beta_0, \beta_1, \dots, \beta_m)\}$$

is also a lattice, and LLL will return a short vector in the sense of $\sum_{k=0}^{m} |\beta_k|^2$ being relatively small.

Observe that $\left(\sum_{k=0}^{m} |\beta_k|^2\right)^{1/2}$ is just the L_2 norm on the unit disk of the polynomial $\sum_{k=0}^{m} \beta_k z^k$.

So LLL lets us find polynomials of small L_2 norm (and hence small sup norm) on the disk, and we can do this while preserving divisibility by a fixed p.

b] Convert the problem from the interval $[\alpha, \beta]$ to the disk. This is easy. One first maps $[\alpha, \beta]$ to $[-2, 2]$ by a linear change of variables. One then lets $x := z + 1/z$. This maps a polynomial in x on $[-2, 2]$ to a polynomial in z and $1/z$ on the boundary of the unit disk.

c] Attack the problem incrementally by using a] and b]. That is, at the k-th stage find a polynomial q_k of degree kN divisible by q_{k-1} of degree $(k-1)N$ using LLL on a lattice of size $N+1$. This allows us to keep the size of LLL fairly small and uses the fact that integer Chebyshev polynomials tend to have (of necessity) many repeat factors. We used $N = 10$ in the actual computation and started with $q_0 := 1$. $\square$

We can computationally refine the above.

Proposition. *The inequality*

$$\Omega[0,1/4] \leq \frac{1}{(5.5723\ldots)}$$

and hence

$$\Omega[0,1] \leq \frac{1}{(2.3605\ldots)}.$$

holds.

This is done by minimizing over

$$P_1^{\alpha_1} P_2^{\alpha_2} \ldots P_9^{\alpha_9}$$

which is a linear problem.

Corollary. *Let k be a positive integer, and let P_{210} be as in the penultimate Proposition. Then $(P_{210})^k$ divides all the n-th integer Chebyshev polynomials on $[0,1]$ provided n is sufficiently large.*

Proof. Each p_i, $i = 0, 1, \cdots, 9$ is irreducible and satisfies

$$p_i(x) = a_k x^k + a^{k-1} x^{k-1} + \cdots + a_0$$

with

$$|a_k|^{1/k} < 2.36.$$

Each p_i also has all roots in $[0, 1]$. It follows now by the first Lemma that if Q is a polynomial of degree n with integer coefficients, and

$$\left(\|Q_n\|_{[0,1]}\right)^{1/n} \leq \frac{1}{2.3605}$$

then p_i divides Q.

Markov's inequality gives the arbitrarily high multiplicity eventually. $\square$

We deduce immediately as above.

Corollary. *The polynomials*

$$p_0, p_1, \cdots, p_9$$

are the only irreducible polynomials with all their roots in $[0,1]$ *of the form*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

with

$$|a_n|^{1/n} < 2.36.$$

3. Finer Structure.

The exact dependence of $\Omega[a,b]$ on the interval $[a,b]$ is interesting and complicated. If we let

$$\Omega(x) := \Omega[0,x]$$

Then clearly Ω is a non-decreasing function on $(0,\infty)$. Obviously

$$\lim_{x \to 0} \Omega(x) = 0$$

(consider x^m on $[0,\delta]$). So $\Omega(x)$ maps $[0,4]$ onto $[0,1]$.

It is an exercise to show that Ω is in fact continuous. This follows mostly from a theorem of Chebyshev that gives

$$\|p_n\|_{[0,\delta+\epsilon]} \le (1+k_{\epsilon,\delta})^n \|p_n\|_{[0,\delta]}$$

for every $p_n \in \mathcal{P}_n$.

What is less obvious is that $\Omega(x)$ is locally flat on many intervals. Indeed it is conceivable that the derivative of Ω is almost everywhere zero.

Theorem. *Let $T_n := T_n\{[0,1]\}$ be an n-th integer Chebyshev polynomial on* $[0,1]$. *Then T_n is of the form*

$$T_n(x) = x^k(1-x)^k S_{n-2k}(x)$$

where $(0.26)n < k$ *if n is large enough.*

As a consequence, there exists an absolute constant $\delta > 0$ (independent of n) so that T_n is an n-th integer Chebyshev polynomial on larger intervals $[-a, 1+a]$ for every $a \in (0, \delta]$.

Another consequence is that the Chudnovsky lower bound conjecture is false. Essentially because the integer Chebyshev polynomials are geometrically smaller near the endpoints than over the whole interval.

Likely, for the same reason, Montgomery's conjecture is also false.

The Schur-Siegel Trace Problem.

Let $\alpha := \alpha_1$ be an algebraic number with conjugate roots $\alpha_2, \cdots, \alpha_n$. We say that α is **totally real (positive)** if all the α_i are real (positive). The **trace** of a totally positive algebraic integer is

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

Except for finitely many explicit exceptions, if α is a **totally real algebraic integer** then then

$$\frac{\alpha_1 + \alpha_2 + \cdots + \alpha_d}{d} \geq 1.648, \quad \text{Schur (18)}$$

$$\frac{\alpha_1 + \alpha_2 + \cdots + \alpha_d}{d} \geq 1.733, \quad \text{Siegel (43)}$$

$$\frac{\alpha_1 + \alpha_2 + \cdots + \alpha_d}{d} \geq 1.771, \quad \text{Smyth (83).}$$

Note that $4\cos^2(\pi/p)$ is a totally positive algebraic integer of degree $(p-1)/2$ and trace $p-2$ for p prime. So the best constant in the above theorem is less than 2.

Connection to the integer Chebyshev Problem is

Proposition. *If*

$$\Omega[0, 1/m] < \frac{1}{m+\delta}.$$

Then, with finitely many exceptions,

$$\frac{\alpha_1 + \alpha_2 + \cdots + \alpha_d}{d} \geq \delta$$

for every totally positive algebraic integer α_1 *of degree* $d > 1$ *with conjugates* $\alpha_2, \ldots, \alpha_d$.

Proof. Mostly an application of the original lemma and the Arithmetic-Geometric mean inequality. □

Corollary. *If* α_1 *is a totally positive algebraic integer of degree* $d > 1$ *with conjugates* $\alpha_2, \ldots, \alpha_d$ *then*

$$\frac{\alpha_1 + \alpha_2 + \cdots + \alpha_d}{d} > 1.752$$

with at most finitely many exceptions. (No exceptions of degree greater than 8.)

This is not as good as Smyth's result. It, however, follows immediately from a computation, as in Section 2, which shows that

$$\Omega[0, 1/200] < \frac{1}{201.752}$$

and gives the factors of an example which yields the above upper bound.

$\{-1, 0, 1\}$ **Polynomials.**

We examine a number of problems concerning polynomials with $\{-1, 0, 1\}$ coefficients. We are particularly interested in how small such polynomials can be on the interval $[0, 1]$ and in what kind of Markov inequality such polynomials can satisfy.

Let

$$\mathcal{P}_n := \{\sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{R}\}$$

Let

$$\mathcal{Z}_n := \{\sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{Z}\}$$

Let

$$\mathcal{F}_n := \{\sum_{i=0}^{n} a_i x^i : a_i \in \{-1, 0, 1\}\}$$

Let

$$\mathcal{A}_n := \{\sum_{i=0}^{n} a_i x^i : a_i \in \{0, 1\}\}$$

So obviously

$$\mathcal{A}_n \subset \mathcal{F}_n \subset \mathcal{Z}_n \subset \mathcal{P}_n$$

Theorem A (Chebyshev Problem for $\mathcal{F}_n$). *There are absolute constants $c_1 > 0$ and $c_2 > 0$ so that*

$$\exp\left(-c_1\sqrt{n}(\log(n+1))\right)$$

$$\leq \inf_{0 \neq P \in \mathcal{F}_n} \|P\|_{[0,1]}$$

$$\leq \exp\left(-c_2\sqrt{n}(\log(n+1))^{-1/2}\right).$$

This should be compared with

$$\min_{p_n := x^n + \ldots \in \mathcal{P}_n} \|p_n\|_{[0,1]}^{1/n} = \frac{2^{1/n}}{4}$$

and also with

$$\frac{1}{2.376..} < \min_{0 \neq p_n \in \mathcal{Z}_n} \|p_n\|_{[0,1]}^{1/n} < \frac{1+o(1)}{2.3605}$$

Small norm polynomials from $\mathcal{P}_n$ are characterized by equioscillation and are given by the Chebyshev polynomials explicitly.

Small norm polynomials from $\mathcal{Z}_n$ is closely related to finding irreducible polynomials with all their roots in $[0, 1]$

The construction of small norm polynomials from $\mathcal{F}_n$ is governed by how large a zero such a polynomial can have at 1.

The polynomials of smallest supremum norm on $[0, 1]$ from $\mathcal{P}_n$ have coefficients that alternate in sign. This also appears to be true for the analogous polynomials from $\mathcal{Z}_n$ (though this is only conjectural and probably quite hard to prove). In both case this is because all the zeros should lie in the interval $[0, 1]$.

There are many related results on heights of polynomials with prescribed zeros at 1 (Amoroso, Bombieri, Mignotte, Vaaler...) and many related results on minimizing $\{0, +1, -1\}$ polynomials (Kahane, Littlewood, Newman...).

The story for $\mathcal{F}_n$ is different again. Here polynomials with alternating sign coefficients (that is $p(-x) \in \mathcal{A}_n$) give very much worse rate of approximation.

Theorem B (Chebyshev Problem for $\mathcal{A}_n$). *There are absolute constants $c_1 > 0$ and $c_2 > 0$ so that*

$$\exp\left(-c_1 \log^2(n+1)\right)$$

$$\leq \inf_{0 \neq P \in \mathcal{A}_n} \|P(-x)\|_{[0,1]}$$

$$\leq \exp\left(-c_2 \log^2(n+1)\right)$$

We also have a Markov type inequality

Inequality. *There is an absolute constant* $c > 0$ *so that*

$$\|P'\|_{[0,1]} \le cn \log^2(n+1)\|P\|_{[0,1]}$$

for every $P \in \mathcal{F}_n$.

The Markov factor $cn \log^2(n+1)$ should be compared to the factor $4n^2$ which is best possible over the classes $\mathcal{P}_n$ and $\mathcal{Z}_n$ (in this case $\mathcal{Z}_n$ behaves identically to $\mathcal{P}_n$ because the Chebyshev polynomial has rational coefficients and is extremal for this Markov inequality).

2. Minimal $\{-1, 0, 1\}$ Polynomials on $[0, 1]$.

The proof of the estimate for $\mathcal{F}_n$ requires a sequence of lemmas.

Lemma. *There is an absolute constant $c_3 > 0$ so that for every $n \in \mathbb{N}$ there is a $P \in \mathcal{F}_n$ having at least $c_3\sqrt{(n/\log(n+1)}$ zeros at 1.*

Proof. Let $\mathcal{F}_n^*$ denote the set of polynomials of degree at most n with coefficients from $\{0, 1\}$. The number of different outputs of the map

$$M(P) := \left(P(1), P'(1), \dots, P^{(k-1)}(1)\right), \qquad P \in \mathcal{F}_n^*$$

is at most

$$\prod_{j=0}^{k-1} \left((n+1)n^j\right) \le (n+1)^{k(k+1)/2}.$$

There are 2^{n+1} different elements of $\mathcal{F}_n^*$. So if

$$(n+1)^{k(k+1)/2} \le 2^{n+1}$$

then there are two different $P_1 \in \mathcal{F}_n^*$ and $P_2 \in \mathcal{F}_n^*$ so that

$$P_1^{(j)}(1) = P_2^{(j)}(1), \qquad j = 0, 1, \ldots, k$$

that is $0 \neq P_1 - P_2 \in \mathcal{F}_n$ has at least k zeros at 1. Note that

$$k \leq \sqrt{\frac{(2\log 2)(n+1)}{\log(n+1)}} - 1$$

implies

$$(n+1)^{k(k+1)/2} \leq 2^{n+1}$$

which finishes the proof. $\square$

Proof of the upper bound of Theorem A. Let

$$P_n(x) := x^{4n} S_n(x) \in \mathcal{F}_{5n}$$

where $S_n \in \mathcal{F}_n$ is of the form

$$S_n(x) = (x-1)^k Q_{n-k}(x), , \qquad Q_{n-k} \in \mathcal{P}_{n-k}$$

with $k \geq c_3 \sqrt{(n/\log(n+1)}$ (see the Lemma).

Then

$$\|P_n\|_{[0,1]}$$

$$\leq \|x^{2n}(1-x)^k\|_{[0,1]} \|Q_{n-k}\|_{[0,1]}$$

$$\leq \left(\frac{4n}{4n+k}\right)^{4n} \left(\frac{k}{4n+k}\right)^k (n+1)^3 \left(\frac{en}{k}\right)^k$$

$$\leq (n+1)^3 \left(\frac{e}{4}\right)^k = \exp(-k\log(4/e) + 3\log(n+1))$$

$$\leq \exp\left(-c_2\sqrt{n}(\log(n+1))^{-1/2}\right)$$

□

For the lower bound we use

Lemma. *If* $n \in \mathbb{N}$, $|a_0|, |a_1|, \dots, |a_{n-1}| \le 1$, *and* $a_n = 1$, *then the multiplicity of the zero of*

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

at 1 *is at most* $8\sqrt{n}$.

Proof. If p has a zero at 1 of multiplicity l, then for every polynomial f of degree less than l, we have

$$a_0 f(0) + a_1 f(1) + \cdots + a_n f(n) = 0. \tag{1}$$

We construct a polynomial f of degree at most $8\sqrt{n}$, for which

$$f(n) > |f(0)| + |f(1)| + \cdots + |f(n-1)|. \tag{2}$$

Equality (1) cannot hold with this f, so the multiplicity of the zero of p at 1 is at most the degree of f.

Let T_ν be the ν-th Chebyshev polynomial, $k \in \mathbb{N}$, and let

$$g(x) := T_0(x) + T_1(x) + \cdots + T_k(x).$$

This is a polynomial of degree k; $g(1) = k+1$; and if $0 < y \le \pi$ then

$$g(\cos y) = 1 + \cos y + \cos 2y + \cdots + \cos ky$$

$$= \frac{\sin(k+\frac{1}{2})y + \sin\frac{1}{2}y}{2\sin\frac{1}{2}y}$$

hence, for $-1 \le x < 1$,

$$|g(x)| \le \frac{2}{2\sqrt{\frac{1-x}{2}}} = \frac{\sqrt{2}}{\sqrt{1-x}}.$$

Let $f(x) := g^4(\frac{2x}{n} - 1)$. Then $f(n) = g^4(1) = (k+1)^4$ and

$$|f(0)| + \ldots + |f(n-1)|$$

$$\le \sum_{j=1}^{n} \frac{4}{(\frac{2j}{n})^2} < n^2 \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} n^2 < 4n^2.$$

If $k := [2\sqrt{n}]$ then (2) holds. In this case the degree of f is $4k \le 8\sqrt{n}$. $\square$

Lemma of Halász. *For every $k \in \mathbb{N}$, there exists a polynomial h of degree k so that*

$$h(0) = 1, \quad h(1) = 0, \quad |h(z)| < e^{\frac{2}{k}}, \quad \text{if} \quad |z| \le 1.$$

The proof of the lemma can be found in "Studies in Pure Mathematics / To the Memory of Paul Turán", pp. 264–265.

Lemma. *For every $n \in \mathbb{N}$ there exists a polynomial*

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

so that $a_n = 1$, $|a_0|, \ldots, |a_{n-1}| < 1$ and p has a zero at 1 with multiplicity at least $\frac{1}{6}\sqrt{n}$.

Proof. We may assume that $n > 36$, otherwise the statement is trivial.

Let $k := \left[\sqrt{8n}\right] + 1$ and $l := \left[\frac{n}{k}\right]$. Let h be a polynomial given by the lemma, that is the degree of h is k, $h(0) = 1$, $h(1) = 0$, and if $|z| \le 1$ then $|h(z)| < e^{\frac{2}{k}}$. Let

$$f(x) := \big(h(x)\big)^l = b_0 + b_1 x + b_2 x^2 + \cdots + b_{kl} x^{kl}.$$

The degree of the polynomial f is $kl \le n$; the multiplicity of the zero of f at 1 is at least l because of the choice of h; $f(0) = b_0 = 1$; and for $|z| \le 1$, $|f(z)| \le e^{\frac{2l}{k}}$. The last inequality, together with the Parseval formula, implies that

$$|b_0|^2 + |b_1|^2 + \cdots + |b_{kl}|^2$$

$$= \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^2 dt \le e^{\frac{4l}{k}} < e^{\frac{4n}{k^2}} < e^{\frac{1}{2}} < 2.$$

Since $b_0 = 1$, it follows that each of $b_1, b_2, \ldots, b_{kl}$ has modulus less than 1.

□

Proof of the lower bound of Theorem A. Suppose $0 \neq P \in \mathcal{F}_n$ has exactly k zeros at 1. Then, using the Lemma and Markov's Inequality, we obtain

$$\|P\|_{[0,1]} \geq (2n)^{-2k}|P^{(k)}(1)| \geq (2n)^{-c_3\sqrt{n}}$$

(note that $|P^{(k)}(1)|$ is a positive integer, hence at least 1), and the result follows $\square$

Minimal $\{0,1\}$ Polynomials on $[-1,0]$.

Approximation with positive coefficients on $[-1,0]$ is equivalent to approximation with alternating coefficients on $[0,1]$. The main result of this section shows that this kind of Chebyshev problem for $\mathcal{A}_n$ gives much worse rate of decrease than the corresponding Chebyshev problem for $\mathcal{F}_n$

Theorem B (Chebyshev Problem for $\mathcal{A}_n$). *There are absolute constants $c_1 > 0$ and $c_2 > 0$ so that*

$$\begin{aligned} &\exp\left(-c_1 \log^2(n+1)\right) \\ &\leq \inf_{0 \neq P \in \mathcal{A}_n} \|P(-x)\|_{[0,1]} \\ &\leq \exp\left(-c_2 \log^2(n+1)\right) \end{aligned}$$

The key lemma is the following.

Lemma. *There is an absolute constant $c_3 > 0$ so that every $0 \neq P \in \mathcal{A}_n$ has at most $c_3 \log n$ zeros at -1.*

Proof. Suppose $P \in \mathcal{A}_m$ has n zeros at -1 and suppose

$$P(x) := \sum_{i=0}^{m} a_i x^i.$$

It follows on a consideration of the derivatives of P at -1 that the elementary symmetric functions in the coefficients $\{a_i\}$ vanish up to order n. Thus

$$(1+x)^n \text{ divides } \sum_{i=0}^{m} x^{a_i}.$$

On evaluating the above at 1 we see that $a_m \geq 2^m - 1$ from which the result follows. $\square$

Proof of Theorem B. The lower bound comes follows from the above Lemma exactly as in the proof of Theorem A.

The upper bound follows from the following example. Let

$$Q_n(x) := \prod_{i=1}^{n} x^{3^i}(1 + x^{3^i})$$

Then the degree of Q is less than 6^{n+1} and its supremum norm on $[-1, 0]$ is bounded by $1/c^{n^2}$ for some $c > 1$ from which the result follows. To see the above estimate observe that for $x \in [0, 1]$,

$$\begin{aligned}
& x^{3^n} \prod_{k=0}^{n} \left(1 - x^{3^k}\right) \\
& \leq \left(x^{3^n}(1-x)^{n+1}\right) \prod_{k=0}^{n} \left(\sum_{j=0}^{3^k-1} x^j\right) \\
& \leq \left(\frac{n+1}{3^n+n+1}\right)^{n+1} \left(1 - \frac{n+1}{3^n+n+1}\right)^{3^n} \prod_{k=0}^{n} 3^k \\
& \leq \left(\frac{n+1}{3^n+n+1}\right)^{n+1} 3^{n(n+1)/2} \leq \exp\left(-c(n+1)^2\right)
\end{aligned}$$

with an absolute constant $c > 0$. □

Let R_n be defined by

$$R_n(x) := \prod_{i=1}^{n} (1 + x^{a_i})$$

where $a_1 := 1$ and a_{i+1} is the smallest odd integer that is greater than $\sum_{k=1}^{i} a_k$.

It is tempting to speculate that R_n is the lowest degree polynomial with coefficients $\{0, 1\}$ and a zero of order n at -1. This is true for $n := 1, 2, 3, 4, 5$ but fails for $n := 6$ and hence for all larger n.