INTEGER CHEBYSHEV POLYNOMIALS

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

- Five old chestnuts.

- All involve Chebyshev type problems for polynomials with integer coefficients.

- All very hard.

- All have a highly non-trivial computational component.

- All have accessible partials?

- All very interesting.

A. The Integer Chebyshev Problem of Hilbert and Fekete.

1. Problem. Find

$$C_N[\alpha,\beta] :=$$

$$\left(\min_{a_i \in z, a_N \neq 0} \|a_0 + a_1 x + \ldots + a_N x^N\|_{[\alpha,\beta]} \right)^{\frac{1}{N}} .$$

We will restrict to $\beta - \alpha \leq 4$.

2. One can show that

$$C[\alpha,\beta] := \lim_{N\to\infty} C_n[\alpha,\beta]$$

exists. This is the integer Chebyshev constant for the interval or the integer transfinite diameter.

3. From the non-integer case

$$C[\alpha, \beta] \geq \frac{\beta - \alpha}{4}$$

and

$$C[0, 4] = 1.$$

In no other case is $C[\alpha, \beta]$ known.

4. Hilbert, Fekete (refinements by Kashin)

$$\frac{\beta - \alpha}{4} \leq C[\alpha, \beta] \leq \left(\frac{\beta - \alpha}{4}\right)^{1/2}.$$

5. Sanoy, Aparisio (1939, 1979)

$$\frac{1}{2.3768} \leq C[0, 1] \leq \frac{1}{2.3307}.$$

6. (Gelfond)

$$\frac{1}{e} \leq C[0,1].$$

Proof.

$$\|p_n\|^2_{[0,1]} \geq \int_0^1 p_n^2(x)dx$$
$$\geq \frac{1}{LCM(1,\ldots,2n+1)} \geq \frac{1}{e^{2n(1+\delta)}}.$$

7. We (T. Erdélyi and P.B.) show

$$\frac{1}{2.3768-\in} \leq C[0,1] \leq \frac{1}{2.360}.$$

8. The upper bound comes first from *LLL* lattice basis reduction. Followed by refinement using the simplex method. To use *LLL* one converts to the disc and proceeds incrementally.

9. If $p(x) =$

$$x^{67}$$

$$(x-1)^{67}$$

$$(2x-1)^{24}$$

$$(5x^2-5x+1)^9$$

$$(13x^3-19x^2+8x-1)$$

$$(13x^3-20x^2+9x-1)$$

$$(29x^4-59x^3+40x^2-11x+1)^3$$

$$(31x^4-61x^3+41x^2-11x+1)$$

$$(31x^4-63x^3+44x^2-12x+1)$$

$$(941x^8-3764x^7+6349x^6-5873x^5$$

$$+3243x^4-1089x^3+216x^2-23x+1)$$

Then

$$\|p(x)\|_{[0,1]} = (2.3543...)^{-210}.$$

10. The lower bound comes from

Lemma. *Suppose*

$$q_m(x) = a_m x^m + \ldots + a_0, \qquad a_m \in Z$$

has all its roots in $(0,1)$*. (That is:* $q_m \in TR(0,1)$*). Then, provided* $(q_m, p_n) = 1$

$$\|p_n\|_{[0,1]}^{1/n} \geq \frac{1}{a_m^{1/m}}.$$

• So finding such q_m with small lead coefficients either gives factors of each Chebyshev polynomial or gives a lower bound.

• All the factors in the minimal example satisfy $a_m^{1/m} \leq 2.6$ so they are <u>all</u> factors of <u>all</u> large integer Chebyshev polynomials on $[0,1]$.

11. We can slightly strengthen the Aparisio/Sanoy lower bound by proving that for large n an integer Chebyshev polynomial has as a factor on $[0, 1]$

$$x^{n/4}(1-x)^{n/4}.$$

12. The Small Interval Problem.

a] For $n \leq m/2e$

$$C_n[0, 1/m] = 1/m$$

and the nth integer Chebyshev polynomial on $[0, 1/m]$ is just x^n.

b] However in the limit

$$1/(m+2) \leq C[0, 1/m] < 1/(m+1)$$

c] What is $C[0, 1/m]$?

B. The Schur, Siegel, Smyth Trace Problem.

1. Conjecture. Suppose

$$p_n(z) = a_n z^n + \ldots + a_0, a_i \in Z$$

has all real, positive roots and is irreducible. Then

$$a_{n-1} \geq 2n - 1.$$

2. Partials. Except for finitely many (explicit) exceptions

$a_{n-1} \geq e^{1/2} n$ Schur (1918)

$a_{n-1} \geq (1.733..) n$ Siegel (1943)

$a_{n-1} \geq (1.771..) n$ Smyth (1983).

3. The Relationship to the Small Interval Problem.

Lemma. *If*

$$C[0, 1/m] \leq 1/(m + \delta)$$

then, for totally positive polynomials

$$a_{n-1} \geq \delta n$$

(with finitely many explicit exceptions).

Corollary. $\delta > 1.744$

Proof. By example on $C[0, 1/100]$.

C. Prouhet-Tarry-Escott Problem.

1. Conjecture.

For any N there exists $p \in Z[x]$ (a polynomial with integer coefficients) so that

$$p(x) = (x-1)^N q(x) = \Sigma a_k x^k$$

and

$$S(p) := \Sigma |a_k| = 2N.$$

Almost equivalently (though not quite obviously)

$$\|p\|_{L^2\{|z|=1\}} = \sqrt{2N}.$$

2. The Basis for the Conjecture.

$$x^{\alpha_1}+\ldots+x^{\alpha_N}-x^{\beta_1}-\ldots-x^{\beta_N}=0((x-1)^N).$$

For $N=2,\ldots,10$ with

$$[\alpha_1,\ldots,\alpha_N] \quad \text{and} \quad [\beta_1,\ldots,\beta_N]$$

$[0,3]=[1,2]$

$[1,2,6]=[0,4,5]$

$[0,4,7,11]=[1,2,9,10]$

$[1,2,10,14,18]=[0,4,8,16,17]$

$[0,4,9,17,22,26]=[1,2,12,14,24,25]$

$[0,18,27,58,64,89,101]$

$=[1,13,38,44,75,84,102]$

$[0, 4, 9, 23, 27, 41, 46, 50]$
$= [1, 2, 11, 20, 30, 39, 48, 49]$

$[0, 24, 30, 83, 86, 133, 157, 181, 197]$
$= [1, 17, 41, 65, 112, 115, 168, 174, 198]$

$[0, 3083, 3301, 11893, 23314, 24186, 35607,$
$44199, 44417, 47500] =$
$[12, 2865, 3519, 11869, 23738, 23762, 35631,$
$43981, 44635, 47488]$

- The size 10 example illustrates the problems inherent with searching for a solution.

3. Partial History.

- Euler
- Prouhet (1851)
- Tarry (1910) - Small Examples
- Escott (1910) - Small Examples
- Letac (1941) - Size 9 and 10
- Gloden (1946) - Size 9 and 10
- Smyth (Math Comp. 1991) - Size 10 generalized.

4. Diophantine Form

Find distinct integers $[\alpha_1, \ldots, \alpha_N]$ and $[\beta_1, \ldots, \beta_N]$ so that

$$
\begin{aligned}
&\alpha_1 + \ldots + \alpha_N = \beta_1 + \ldots + \beta_n \\
&\alpha_1^2 + \ldots + \alpha_N^2 = \beta_1^2 + \ldots + \beta_n^2 \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\alpha_1^{N-1} + \ldots + \alpha_N^{N-1} = \beta_1^{N-1} + \ldots + \beta_N^{N-1}
\end{aligned}
$$

5. Open Questions.

- The problem is completely open for $N \geq 11$.

- We computed extensively on $N = 11$ to show no (symmetric) solutions of degree ≤ 745.

D. The Weak Prouhet-Tarry-Escott Problem.

1. Problem. For fixed N find $p \in Z[x]$

$$p(x) = (x-1)^N q(x) = \Sigma a_k x^k$$

that minimizes

$$S(p) = \Sigma |a_i|$$

or

$$S^2(p) = (\Sigma |a_i|^2)^{1/2}$$

2. Solving $S(p) = |S^2(p)|^2 = 2N$ is the Prouhet-Tarry-Escott-Problem and is the big prize.

3. Showing that there exist

$$\{p_N\} = \{(x-1)^N q(x)\}$$

so that

$$S(p_N) = o(N \log N)$$

is also a big prize.

• This shows that the "Easier Waring Problem" is easier than the "Waring Problem" (At the moment.)

• That is: it requires essentially fewer powers to write every integer as sums and differences of Nth powers than just as sums of Nth powers. (Fuchs and Wright, Quart. J. Math. 1936).

4. It is known that

$$S((x-1)^N q(x)) \leq \frac{N^2}{2}$$

is possible.

Any improvement would be a major step.

5. If we demand that p has a zero of order N but not $N+1$ at 1 then

$$S(p) = 0((\log N)N^2)$$

is possible (Hua).

Any improvement would be interesting.

E. Problem of Erdös and Szekeres (1958).

1. Problem. Minimize over $\{\alpha_1, \dots, \alpha_N\}$

$$S\left(\prod_{k=1}^{N}(1 - x^{\alpha_i})\right)$$

Call this minimum S_N^π.

2. Conjecture. $S_N^\pi \gg N^k$ for any k.

3. From the P-T-E problem

$$S_N^\pi \geq 2N$$

4. Examples.

N	$\|f\|_1$	$\{\alpha_1, \ldots, \alpha_N\}$
1	2	$\{1\}$
2	4	$\{1, 2\}$
3	6	$\{1, 2, 3\}$
4	8	$\{1, 2, 3, 4\}$
5	10	$\{1, 2, 3, 5, 7\}$
6	12	$\{1, 1, 2, 3, 4, 5\}$
7	16	$\{1, 2, 3, 4, 5, 7, 11\}$
8	16	$\{1, 2, 3, 5, 7, 8, 11, 13\}$
9	20	$\{1, 2, 3, 4, 5, 7, 9, 11, 13\}$
10	24	$\{1, 2, 3, 4, 5, 7, 9, 11, 13, 17\}$
11	28	$\{1, 2, 3, 5, 7, 8, 9, 11, 13, 17, 19\}$
12	36	$\{1, \ldots, 9, 11, 13, 17\}$
13	48	$\{1, \ldots, 9, 11, 13, 17, 19\}$

5. Conjecture. Except for $N = 1, 2, 3, 4, 5, 6$ and 8

$$S_N^\pi \geq 2N + 2.$$

6. Problem. Show that

$$S((1 - x^{\alpha_1})(1 - x^{\alpha_2}) \ldots (1 - x^{\alpha_7})) \neq 14$$

(or even make this algorithmic).

7. Partials.

$S_N^\pi \ll N^{0(N^{1/2})}$ (Atkinson, Dobrowolski)

$S_N^\pi \ll N^{0(\log N N^{1/3})}$ (Odlyzko)

(could equally well use $\| \ \|_{L^2(D)}$.)

8. Proposition. *Let β_i be the sequence formed by taking the set $\{2^n - 2^m : n > m \geq 0\}$ in increasing order. Then for all N*

$$\left\| \prod_{i=1}^{N} (1 - z^{\beta_i}) \right\| \leq (32N)^{\sqrt{N/8}}.$$

Lemma. *Let $1 \leq \beta_1 < \beta_2 < \ldots$ and let*

$$W_n(z) = \prod_{1 \leq i < j \leq n} (1 - z^{\beta_j - \beta_i})$$

then

$$\|W_n(z)\| \leq n^{\frac{n}{2}}.$$

Proof. We can explicitly evaluate the Vandermonde determinant

$$D_n := \prod_{1\le i<j\le n} (z^{\beta_j} - z^{\beta_i})$$

$$= \begin{vmatrix} 1 & z^{\beta_1} & \dots & z^{(n-1)\beta_1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & z^{\beta_n} & \dots & z^{(n-1)\beta_n} \end{vmatrix}$$

and by Hadamard's inequality, since each entry of the matrix has modulus at most one in the unit disk,

$$\|D_n\| \le n^{n/2}.$$

Thus

$$\left\| \prod_{1\le i<j\le n} (1 - z^{\beta_j - \beta_i}) \right\|$$
$$= \left\| \prod_{1\le i<j\le n} (z^{\beta_j} - z^{\beta_i}) \right\|$$
$$\le n^{n/2}.$$

□

So this constructs an infinite product with all partial products growing at most like $0(N^{c\sqrt{N}})$.

9. Theorem. *(P.B. JNT). If* $(p, \alpha_i) = 1$*, p prime then*

$$\left\| \prod_{i=1}^{N} (1 - x^{\alpha_i}) \right\| \geq p^{N/(p-1)}$$

and this is best possible for $p = 2, 3, 5, 7, 11, 13$ *by*

$$\prod_{\substack{n=1 \\ (p,n)=1}}^{\infty} (1 - x^n)$$

10. Problem. For each n write

$$(1 - x)(1 - x^2)(1 - x^4)(1 - x^5)\ldots$$
$$(1 - x^{3n+1})(1 - x^{3n+2}) = \Sigma a_i x^i$$

then $a_i \geq 0$ if and only if 3 divides i.

This would give an exact bound in the above theorem for $p := 3$. A similar result should hold for $p := 5$.