PURE MATHEMATICS IN THE PRESENCE OF COMPUTERS

Peter Borwein

Simon Fraser University Centre for Constructive and Experimental Mathematics

http://www.cecm.sfu.ca/~pborwein

Typeset by $\mathcal{AMS}$-TEX

"It would appear that we have reached the limits of what it is possible to achieve with computer technology, although one should be careful with such statements, as they tend to sound pretty silly in 5 years."

John Von Neumann (ca. 1949)

★★★★★★

- Von Neumann was in good company....

"Fooling around with alternating current is just a waste of time. Nobody will use it, ever."

Thomas Edison

"I think there is a world market for about five computers."

T.J. Watson, CEO, IBM, 1947

"I have traveled the length and breadth of this country and talked with the best people, and I can assure you that data processing is a fad that won't last out the year."

Chief business ed., Prentice Hall , 1957

• Typical of the caution were quotes like

"In spite of its powers, the analyser cannot solve problems the mathematician cannot do himself"

despite the title of the 1949 Newsweek article "The Great Electro-Mechanical Brain"

• Aitken did speculate in Time (1950)

"it might be possible to "program" a machine to beat the stock market"

He should have added "into the ground."

• By and large predictions were mundane and extraordinarily cautious.

Conjectures people should have made

★★★★★★

“ Computers will solve the four colour conjecture”

★★★★★★

“ Computers will prove the non-existence of a projective plane of order ten”

★★★★★★

“ Computers will generate far more mathematical questions than answers ”

Soothsayer Borwein's Irrefutable Predictions For The Next Fifty Years

So far he is batting 1.000.

★★★★★★

" Computers will solve the Riemann Hypothesis"

Extensive experimentation will uncover a family of positive definite operators whose eigenvalues are the (suitably normalized) zeros of the Zeta function.

★★★★★★

★★★★★★

“ $e + \pi$ will be proved transcendental”

Extensive experimentation will uncover a family of generalized Padé type approximants whose rate of convergence is fast enough to prove this.

★★★★★★

“ It will still not be known whether the digits of π are normal”

“ no one will have computed the 10^{50}th digit ”

★★★★★★

★★★★★★

“ Computers will not solve the Riemann Hypothesis”

★★★★★★

"Computers are useless. They can only give you answers."

Pablo Picasso (1881-1973)

★★★★★★

• Picasso is wrong and this is the key to why the face of pure mathematics will change completely over the next 50 years.

• Consider $P \neq NP$

• Consider Factoring Integers. This problem was entirely answered 50 years ago only we asked the wrong question.

• Consider Fractals.

The Safest Prediction of All

★★★★★★

“ The undergraduate curriculum will be completely rewritten”

★★★★★★

• and it is about time!

• There will be no such thing as a non-computationally adept mathematician.

• Maple already outperforms the average undergraduate in many courses.

More Irrefutable Predictions For The Next Fifty Years

★★★★★★

“ The process of doing mathematics will profoundly change”

★★★★★★

• Or perhaps more precisely change back.

★★★★★★

“ mathematics will emerge as a (partially) experimental endevour”

★★★★★★

"God does not care about our mathematical difficulties. He integrates empirically."

Albert Einstein (1879-1955)

• In future God should use Maple (or Mathematica or ...)

"It still remains true that, with negative theorems such as this, transforming personal convictions into objective ones requires deterringly detailed work. To visualize the whole variety of cases, one would have to display a large number of equations by curves; each curve would have to be drawn by its points, and determining a single point alone requires lengthy computations.

You do not see from Fig. 4 in my first paper of 1799, how much work was required for a proper drawing of that curve."

K. F. Gauss (Letter to Schumacher 1836)

On The Experimental Nature of Math.

An Example:

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)^4 (k+1)^{-5}$$

$$= -\frac{29}{2}\zeta(9) + \frac{37}{2}\zeta(4)\zeta(5)$$

$$+\frac{33}{4}\zeta(3)\zeta(6) - \frac{8}{3}\zeta^3(3) - 7\zeta(2)\zeta(7)$$

- It is true - it is not (yet) proven.

- The proof of the pudding is in the eating.

Another Example:

$$\sqrt{\pi} \doteq \frac{1}{10^5} \sum_{n=-\infty}^{\infty} e^{-n^2/10^{10}}$$

This is correct to over 42 billion digits but not to 43 billion digits.

- Coincidence?

- "Messing" around only works if you know where to look.

Inverse Symbolic Calculation

- What is 1.198140234735592207$5$?

If $a_0 := 1, b_0 := \sqrt{2}$ and

$$a_{n+1} := \frac{a_n + b_n}{2}, \quad b_{n+1} := \sqrt{a_n b_n}$$

Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{\pi/2}{\int_0^1 \frac{dt}{\sqrt{1-t^4}}}$$

$$= 1.198140234735592207 5...$$

In 1799, Gauss observed this purely numerically and wrote that this result

“will surely open a whole new field of analysis.”

- What is 3.14159265358979323857?
- What is 2.71828182845904523547?
- What is 14.86134161768747018627?
- What is $(e^{\pi\sqrt{163}} - 744)^{\frac{1}{3}}$?

$(e^{\pi\sqrt{163}} - 744)^{\frac{1}{3}} = 640320$ to 30 digits

- However to thirty five places it equals

640319.99999999999999999999999999939032

Theorem.

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right).$$

Proof. This is equivalent to:

$$\pi = \int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1 - x^8} dx.$$

which on substituting $y := \sqrt{2}x$ becomes

$$\pi = \int_0^1 \frac{16\,y - 16}{y^4 - 2\,y^3 + 4\,y - 4} dy\,.$$

The equivalence follows from the identity

$$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1 - x^8}\,dx = \int_0^{1/\sqrt{2}} \sum_{i=0}^{\infty} x^{k-1+8i}\,dx$$

$$= \frac{1}{\sqrt{2}^k} \sum_{i=0}^{\infty} \frac{1}{16^i(8i+k)}$$

□

Littlewood's Other Conjecture.

Conjecture (1966). There is some

$$p(z) := \sum_{n=0}^{N} c_n z^n \qquad c_i \pm 1$$

so that for all z on the boundary of the unit disc

$$C_1 < \frac{|p(z)|}{\sqrt{n}} < C_2.$$

• Littlewood, in part, based his conjecture on computations of all such polynomials up to degree twenty.

• Odlyzko has now done 200 MIPS years of computing on this problem

• In Advice to a Young Scientist, P.B. Medawar defines four different kinds of experiment: the Kantian, Baconian, Aristotelian, and the Galilean. Mathematics has always participated deeply in the first three categories but has somehow managed to avoid employing the Galilean model.

“ (the Baconian experiment) is the consequence of ‘trying things out’ or even of merely messing about.”

“(the) Galilean Experiment is a critical experiment – one that discriminates between possibilities and, in doing so, either gives us confidence in the view we are taking or makes us think it in need of correction.”

Is Proof Dead?

- Data driven pure mathematics?

- The presentation of data as an end in itself (yes please).

- Computer assisted proof (certainly).

- Computer generated proof (maybe). The spell checker model.

- What constitutes certainty? Truth vs. probabilistic truth.

- There is knowledge without proof (there is also proof without knowledge).

Is Publication Dead?

★★★★★★

"Change is scientific, progress is ethical; change is indubitable, whereas progress is a matter of controversy."

Bertrand Russell (1872-1970)

★★★★★★

"It seems to me that the mine is already almost too deep, and unless we discover new seams we shall sooner or later have to abandon it. ... It is not impossible that the mathematical positions in the Academies will one day become what the University chairs in Arabic are now."

J. L. Lagrange (1736-1813)

★★★★★★

“Progress is measured funeral by funeral”