The Monic Integer Chebyshev Problem

Peter Borwein

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http://www.cecm.sfu.ca/~pborwein.

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The Integer Chebyshev Problems

Find a polynomial with integer coefficients of minimum supnorm on an interval.

Problem 1.1 *For any interval* $[\alpha, \beta]$ *find*

$$\Omega[\alpha, \beta] := \lim_{N \to \infty} \Omega_N[\alpha, \beta]$$

where

$$\Omega_N[\alpha, \beta] :$$

$$= \left(\min_{a_i \in \mathcal{Z}, a_N \neq 0} \|a_0 + a_1 x + \ldots + a_N x^N\|_{[\alpha, \beta]} \right)^{\frac{1}{N}}$$

One can show that

$$\Omega[\alpha, \beta] := \lim_{N\to\infty} \Omega_N[\alpha, \beta]$$

exists. This quantity is called the *integer Chebyshev constant* for the interval or *the integer transfinite diameter*.

On $[-2, 2]$ (or any interval with integer endpoints of length 4) this problem is solvable because the usual Chebyshev polynomials normalized to have lead coefficient 1 have integer coefficients and supnorm 2.

So $\Omega[-2, 2] = 1$.

There are no other intervals were the explicit value is known.

Since

$$\Omega[a,b] \leq \Omega_n[a,b]$$

for any particular n upper bounds can be derived computationally from the computation of any specific $\Omega_n[a,b]$.

$$p_0(x) := x, \quad p_1(x) := 1 - x,$$

$$p_2(x) := 2x - 1, \quad p_3(x) := 5x^2 - 5x + 1,$$

$$p_4(x) := 13x^3 - 19x^2 + 8x - 1,$$

$$p_5(x) := 13x^3 - 20x^2 + 9x - 1,$$

$$p_6(x) := 29x^4 - 58x^3 + 40x^2 - 11x + 1,$$

$$p_7(x) := 31x^4 - 61x^3 + 41x^2 - 11x + 1,$$

$$p_8(x) := 31x^4 - 63x^3 + 44x^2 - 12x + 1,$$

$$p_9(x) := 941x^8 - 3764x^7 + 6349x^6 - 5873x^5$$

$$+ 3243x^4 - 1089x^3 + 216x^2 - 23x + 1.$$

Proposition 1.1 *Let*

$$P_{210} := p_0^{67} \cdot p_1^{67} \cdot p_2^{24} \cdot p_3^{9} \cdot p_4 \cdot p_5 \cdot p_6^{3} \cdot p_7 \cdot p_8 \cdot p_9;$$

then

$$\left(\|P_{210}\|_{[0,1]}\right)^{1/210} = \frac{1}{2.3543\ldots},$$

and hence

$$\Omega[0,1] \leq \frac{1}{2.3543\ldots}.$$

Refinements on the method

$$\Omega[0,1] \leq \frac{1}{2.3612\ldots}.$$

Of course when the coefficients of the polynomials above are not required to be integers this reduces to the usual problem of constructing Chebyshev polynomials and the the limit (provided $a_N = 1$) gives the usual transfinite diameter. From the unrestricted case we have the obvious inequality

$$\Omega_n[a,b] \geq 2^{1/n}\frac{b-a}{4},$$

However inspection of the above example shows that the integer Chebyshev polynomial doesn't look anything like a usual Chebyshev polynomial.

In particular it has many multiple roots and indeed this must be the case since we have the following lemma.

Lemma 1.3 *Suppose* $p_n \in \mathcal{Z}_n$ *(the polynomials of degree n with integer coefficients) and suppose* $q_k(z) := a_k z^k + \cdots + a_0 \in \mathcal{Z}_k$ *has all its roots in* $[a, b]$. *If* p_n *and* q_k *do not have common factors, then*

$$\left(\|p_n\|_{[a,b]}\right)^{1/n} \geq |a_k|^{-1/k}.$$

From this lemma and the above mentioned bound we see that all of p_1 through p_9 must occur as high order factors of integer Chebyshev polynomials on $[0, 1]$ for sufficiently large n.

There is a sequence of polynomials that Montgomery calls the Gorshkov–Wirsing polynomials that arise from iterating the rational function

$$u(x) := \frac{x(1-x)}{1-3x(1-x)}.$$

These are defined inductively by

$$q_0(x) := 2x - 1, \quad q_1(x) := 5x^2 - 5x + 1$$

and

$$q_{n+1} := q_n^2 + q_n q_{n-1}^2 - q_{n-1}^4.$$

It transpires that

$$u^{(n)} = \frac{q_{n-1}^2 - q_n}{2q_{n-1}^2 - q_n}.$$

Each q_k is a polynomial of degree 2^k with all simple zeros in $(0, 1)$ and if b_k is the lead coefficient of q_k then

$$\lim b_k^{1/2^k} = 2.3768417062\ldots.$$

Wirsing proves these polynomials irreducible. If follows now from Lemma 1.3 that

$$\Omega[0,1] \geq \frac{1}{2.3768417062\ldots}.$$

Montgomery conjectured that if s is the least limit point of $|a_k|^{-1/k}$ (as in in Lemma 1.3) over polynomials with all their roots in $[0,1]$, then $\Omega[0,1] = s$. Chudnovsky further conjectured that the minimal s arises from the Gorshkov–Wirsing polynomials and so s would equal $(2.3768417062\ldots)^{-1}$.

We show that

$$\Omega[0,1] \geq \frac{1}{2.3768417062\ldots} + \epsilon.$$

This shows that either Montgomery's conjecture is false or the the Gorshkov–Wirsing polynomials do not give rise to the minimal s. This leads us to ask

Conjecture 1.4 *The minimal s arising in Lemma 1.3 does not give the right value for* $\Omega[0,1]$.

Habsieger and Salvy show that Integer Chebyshev polynomials on $[0,1]$ need not have all real roots. The first non totally real factor occurs for $n = 70$.

This is a non-trivial computation and is quite likely NP hard.

Monic Integer Chebyshev Problem.

Minimize the supremum norm by monic polynomials with integer coefficients.

Let $\mathcal{M}_n(\mathbb{Z})$ denote the monic polynomials of degree n with integer coefficients.

A *monic integer Chebyshev polynomial* $M_n \in \mathcal{M}_n(\mathbb{Z})$ satisfies

$$\|M_n\|_E = \inf_{P_n \in \mathcal{M}_n(\mathbb{Z})} \|P_n\|_E.$$

The *monic integer Chebyshev constant* is then defined by

$$\Omega^*(E) := \lim_{n\to\infty} \|M_n\|_E^{1/n}.$$

The monic integer Chebyshev problem is quite different from the classical integer Chebyshev problem where the polynomials are not required to be monic.

Conjecture *Suppose* $[a_2/b_2, a_1/b_1]$ *is an interval whose endpoints are consecutive Farey fractions. This is characterized by* $(a_1 b_2 - a_2 b_1) = 1$. *Then*

$$\Omega^*[a_2/b_2, a_1/b_1] = \max(1/b_1, 1/b_2).$$

Our first result shows that the monic integer Chebyshev constant coincides with the regular Chebyshev constant (capacity) for sufficiently large sets.

Theorem *If* E *is* $\mathbb{R}$*-symmetric and* $cap(E) \geq 1$*, then*

$$\Omega^*(E) = cap(E).$$

An argument going back to Kakeya gives

Theorem *Let* $E \subset \mathbb{C}$ *be a compact* $\mathbb{R}$*-symmetric set. If* $cap(E) < 1$ *then* $\Omega^*(E) < 1$.

Perhaps, the most distinctive feature of $\Omega^*(E)$ is that it may be different from zero even for a single point. For example suppose that $m, n \in \mathbb{Z}$, where $n \geq 2$ and $gcd(m, n) = 1$. Then

$$\Omega^*\left(\left\{\frac{m}{n}\right\}\right) = \frac{1}{n}.$$

On the other hand, if $a \in \mathbb{R}$ is irrational, then

$$\Omega^*(\{a\}) = 0.$$

This result has several interesting consequences. Consider

$$E_n := \{z : z^n = 1/2\}, \ n \in \mathbb{N}.$$

It is easy that $cap(E_n) = t_{\mathbb{C}}(E_n) = 0$ for any $n \in \mathbb{N}$. However, as $n \to \infty$

$$\Omega^*(E_n) = 2^{-1/n} \to 1.$$

Thus no uniform upper estimate of $\Omega^*(E)$ in terms of $cap(E)$ is possible, in contrast with results of of Hilbert and Fekete

$$\Omega(E) \leq \sqrt{cap(E)}$$

Note that

$$\Omega^*(\{1/\sqrt{2}\}) = \Omega^*(\{-1/\sqrt{2}\}) = 0$$

while

$$\Omega^*(\{1/\sqrt{2}\} \cup \{-1/\sqrt{2}\}) = 1/\sqrt{2}$$

This shows that another well known property of capacity is not valid for $\Omega^*(E)$. Namely, capacity (Chebyshev constant) for the union of two sets of zero capacity is still zero.

Theorem *For* $n > 2$

$$\Omega^*\left(\left[0, \frac{1}{n}\right]\right) = \Omega^*\left(\left[\frac{n-1}{n}, 1\right]\right) = \frac{1}{n}$$

$$\Omega^*\left(\left[-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right]\right) = \sqrt{\Omega^*\left(\left[0, \frac{1}{n}\right]\right)} = \frac{1}{\sqrt{n}}$$

$$\Omega^*\left(\left[n - \frac{1}{2}, n\right]\right) = \Omega^*\left(\left[0, \frac{1}{2}\right]\right) = \frac{1}{2},$$

$$\Omega^*([0, 1]) = \sqrt{\Omega^*\left(\left[0, \frac{1}{4}\right]\right)} = \frac{1}{2}$$

and

$$\Omega^*([-1,1]) = \sqrt{\Omega^*([0,1])} = \frac{1}{\sqrt{2}}.$$

Also, if $E \subset [(1-\sqrt{2})/2, (1+\sqrt{2})/2]$ *and* $\{1/2\} \in E$, *then*

$$\Omega^*(E) = \Omega^*\left(\left[\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}\right]\right) = \frac{1}{2}.$$

It is worth remembering that finding the value of $\Omega([0,1])$ is a notoriously difficult problem, where we do not even have a current conjecture.

Finite Sets of Points

Theorem *For any k rational points*

$$\frac{a_i}{b_i}, \quad (a_i, b_i) = 1, \;\; i = 1, ..., k$$

there is a monic integer polynomial $f(x)$ of some degree n with

$$f\left(\frac{a_i}{b_i}\right) = \frac{1}{b_i^n}, \quad i = 1, ..., k.$$

Corollary *If $E = \left\{\frac{a_1}{b_1}, ..., \frac{a_k}{b_k}\right\}$ with the a_i/b_i rationals written in their lowest terms and $b_i \geq 2$, then*

$$\Omega^*(E) = \max_{i=1,..,k} \frac{1}{b_i}.$$

Two consecutive Fareys: If $n \geq 2$

$$\frac{a_2}{b_2} < \frac{a_1}{b_1}, \quad (a_1 b_2 - a_2 b_1) = 1,$$

and

$$a_i^n \equiv A_i \mod b_i, \quad i = 1, 2,$$

then

$$f(x) = x^n + \left(\frac{A_1 - a_1^n}{b_1}\right)(b_2 x - a_2)^{n-1} +$$

$$\left(\frac{A_2 - a_2^n}{b_2}\right)(a_1 - b_1 x)^{n-1}$$

has

$$f(a_i/b_i) = A_i/b_i^n, \quad i = 1, 2$$

.

Theorem *Suppose that* $S = \{\alpha_1, ..., \alpha_k\}$ *is a set of* k *numbers, with the* α_i *transcendental or algebraic of degree more than* k. *Suppose that if* α_i *is complex then its complex conjugate is also in* S.

Then for any $1 > \varepsilon > 0$ *and* $n > k^2$ *there is a monic integer polynomial* F *of degree* n *with* $|F(\alpha_i)| < \varepsilon$, $i = 1, ..., k$.

Intervals of Consecutive Farey Nos

Conjecture *Suppose* $[a_2/b_2, a_1/b_1]$ *is an interval whose endpoints are consecutive Farey fractions. This is characterized by* $(a_1b_2 - a_2b_1) = 1$. *Then*

$$\Omega^*[a_2/b_2, a_1/b_1] = \max(1/b_1, 1/b_2).$$

Since

$$\Omega^*[a_2/b_2, a_1/b_1] \geq \max(1/b_1, 1/b_2)$$

the conjecture holds on intervals of the form $[0, 1/n]$.

We give enough solutions to fill in all Farey intervals with denominator less than 15. (The conjecture is verified to 23.) On the remaining intervals x works or the symmetry $x \to m \pm x$ works. The computations for the table are done with LLL. For certain n, we find a p of degree n that satisfies $p(a_2/b_2) = 1/b_2^n$ and $p(a_1/b_1) = 1/b_1^n$. One now constructs a basis

$$B := \big(p(x), (b_1 x - a_1)(b_2 x - a_2),$$

$$x(b_1 x - a_1)(b_2 x - a_2), \ldots ,$$

$$x^{n-3}(b_1 x - a_1)(b_2 x - a_2)\big)$$

and we reduce the basis with respect to the norm

$$\left(\int_{a_2/b_2}^{a_1/b_1} p(x)^2\,dx\right)^{1/2}.$$

We then search the reduced basis for solutions of the conjecture. These calculations were done in Maple using an LLL implementation that can accommodate reduction with respect to any positive definite quadratic form. (This was implemented by Kevin Hare and we would like to thank him for his code.)

$$T(1/3, 2/5) = x^2 - 3\,x + 1$$

$$T(1/4, 2/7) = x^2 - 4\,x + 1$$

$$T(2/5, 3/7) = x^4 - 716\,x^3 + 890\,x^2 - 369\,x + 51$$

$$T(1/3, 3/8) = x^2 - 6\,x + 2$$

$$T(3/8, 2/5) = x^2 - 3\,x + 1$$

$$T(1/5, 2/9) = -x^3 - 20\,x^2 + 9\,x - 1$$

$$T(2/7, 3/10) = -x^6 + 1151931\,x^5 - 1691236\,x^4 \quad 993150\,x^3 - 291587\,x^2 + 42802\,x - 2513$$

$$T(1/6, 2/11) = -x^3 - 30\,x^2 + 11\,x - 1$$

$$T(1/4, 3/11) = x^2 - 4\,x + 1$$

$$T(3/11, 2/7) = x^6 + 2359829\,x^5 - 3291253\,x^4 - 1836029\,x^3 - 512089\,x^2 + 71410\,x - 3983$$

$$T(1/7, 2/13) = -x^3 - 42\,x^2 + 13\,x - 1$$

$$T(2/9, 3/13) = -x^3 - 20\,x^2 + 9\,x - 1$$

$$T(1/5, 3/14) = x^2 - 5\,x + 1$$

$$T(3/14, 2/9) = -x^3 + 106\,x^2 - 46\,x + 5$$