The Unreasonable Efficacy of Symbolic Computation

—or—

Imagine if Gauss had Maple.

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May 2000

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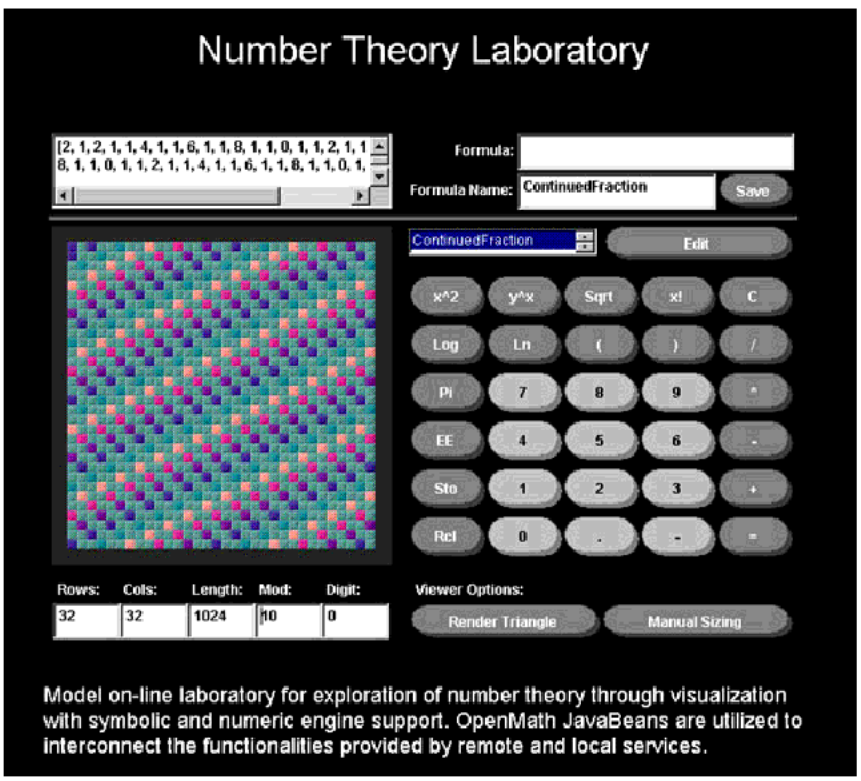


Abstract: Modern symbolic computation packages such as Maple have been far more successful then even their inventors could have imagined twenty years ago.

They play a critical role in how large parts of modern mathematics grow and evolve and can be anticipated to play an even larger role as they become increasingly more sophisticated.

It is not too grandiose to talk about a paradigm shift in mathematics with such tools at its core.

Symbolic algebra packages (Maple and Mathematica) have over the last fifteen years reached a remarkable degree of sophistication.

Rather difficult problems, like exact integration of elementary functions have been significantly attacked.

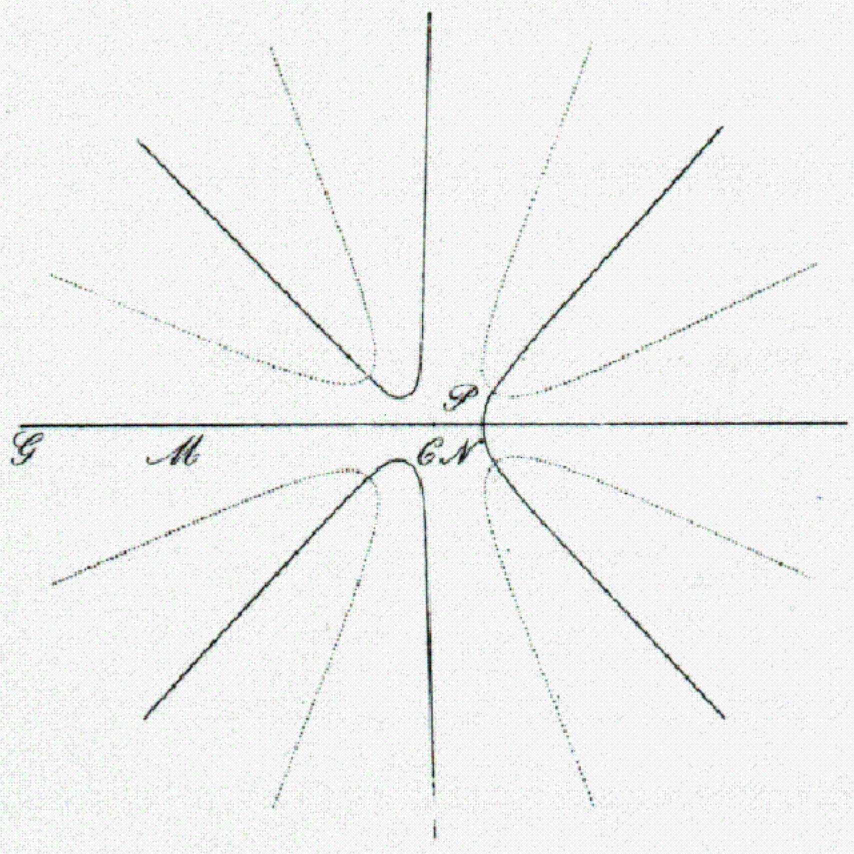
A number of the most important algorithms of the twentieth century like FFT's and LLL are centrally incorporated.

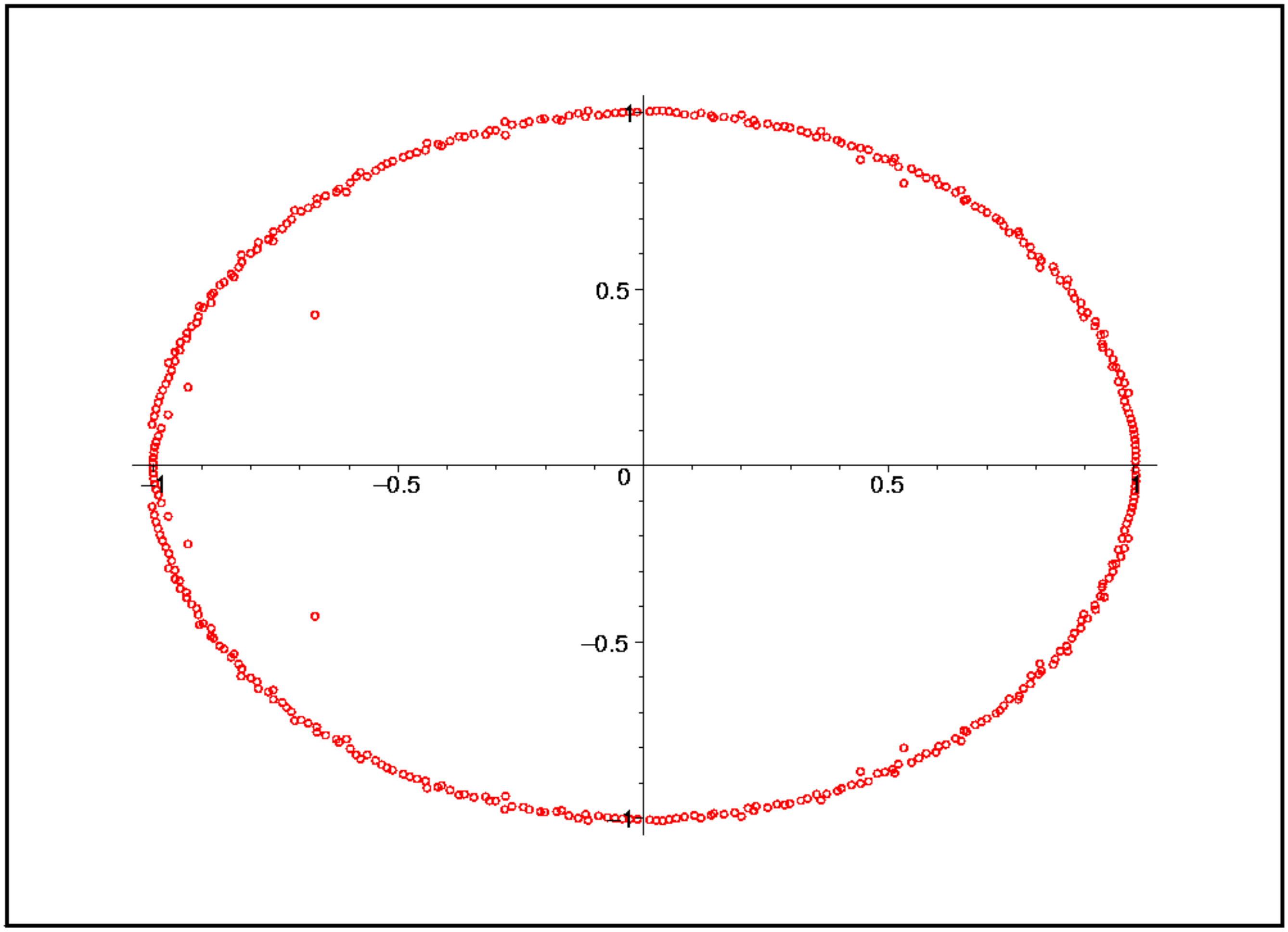
The great success of the symbolic algebra packages has been their mathematical generality and ease of use.

"You do not see from Fig. 4 in my first paper of 1799 , how much work was required for a proper drawing of that curve."

K. F. Gauss (1777-1855)

Fig. 4.





All zeros of $1/2 + x + x^4 + \ldots + x^{400}$

These packages deal most successfully with algebraic problems while many (perhaps most) serious applications require analytic objects such as definite integrals, series and differential equations.

All the elementary notions of analysis, like continuity and differentiability have to be given precise computational meaning.

Many of the difficult problems involve automatic simplification of complicated analytic formulae and recognition of when two very different such expressions represent the same object.

The packages can now substantially deal with large parts of the standard mathematics curriculum (and can outperform most of our undergraduates).

There is a coherent argument that they are the most significant part in a paradigm shift in how mathematics is done and certainly they have become a central research tool in many subareas of mathematics both from an exploratory and from a formal point of view.

$$\int (log(x)^7)dx =$$

$$(\ln(x))^7 x - 7x(\ln(x))^6 + 42x(\ln(x))^5$$

$$-210x(\ln(x))^4 + 840x(\ln(x))^3$$

$$-2520x(\ln(x))^2 + 5040x\ln(x) - 5040x$$

$$\int (exp(x) * log(x))dx =$$

$$e^x \ln(x) + \mathrm{Ei}(1, -x)$$

$$\int (exp(x)*log(x)^2)dx = \int (exp(x)*log(x)^2)dx$$

(It is acceptable now to see a line in a proof that says something like "by a large calculation in Maple we see".)

The first objective of symbolic algebra packages was to do as much exact mathematics as possible.

$$x^{198} - x^{197} - x^{181} + x^{174} + x^{168} - x^{157} -$$

$$x^{133} + x^{115} + x^{112} - x^{86} - x^{83} + x^{65} +$$

$$x^{41} - x^{30} - x^{24} + x^{17} + x - 1$$

Maple will factor this in a few seconds. (This polynomial has $(1-x)^9$ as a factor.)

$$\sum_{i=1}^{\infty} i^{-6} = \frac{1}{945}\pi^6$$

$$\sum_{i=1}^{\infty} \frac{1}{i^6 (i+1)^6} =$$

$$-462 + \frac{2}{945}\pi^6 + \frac{7}{15}\pi^4 + 42\pi^2$$

$$\sum_{i=1}^{\infty} \frac{1}{i^3+1} = -1/3 + 1/3\,\gamma +$$

$$1/3\,\left(1/2 - 1/2\sqrt{-3}\right)\Psi(1/2+1/2\sqrt{-3})$$

$$+1/3\,\left(1/2 + 1/2\sqrt{-3}\right)\Psi(1/2-1/2\sqrt{-3})$$

A second objective is do it fast and to deal (in an arbitrary precision environment) with the more usual algorithms of analysis.

Basically one would like to be able to incorporate the usual methods of numerical analysis into an exact environment or at least into an infinite precision environment.

The problems are obvious and hard.

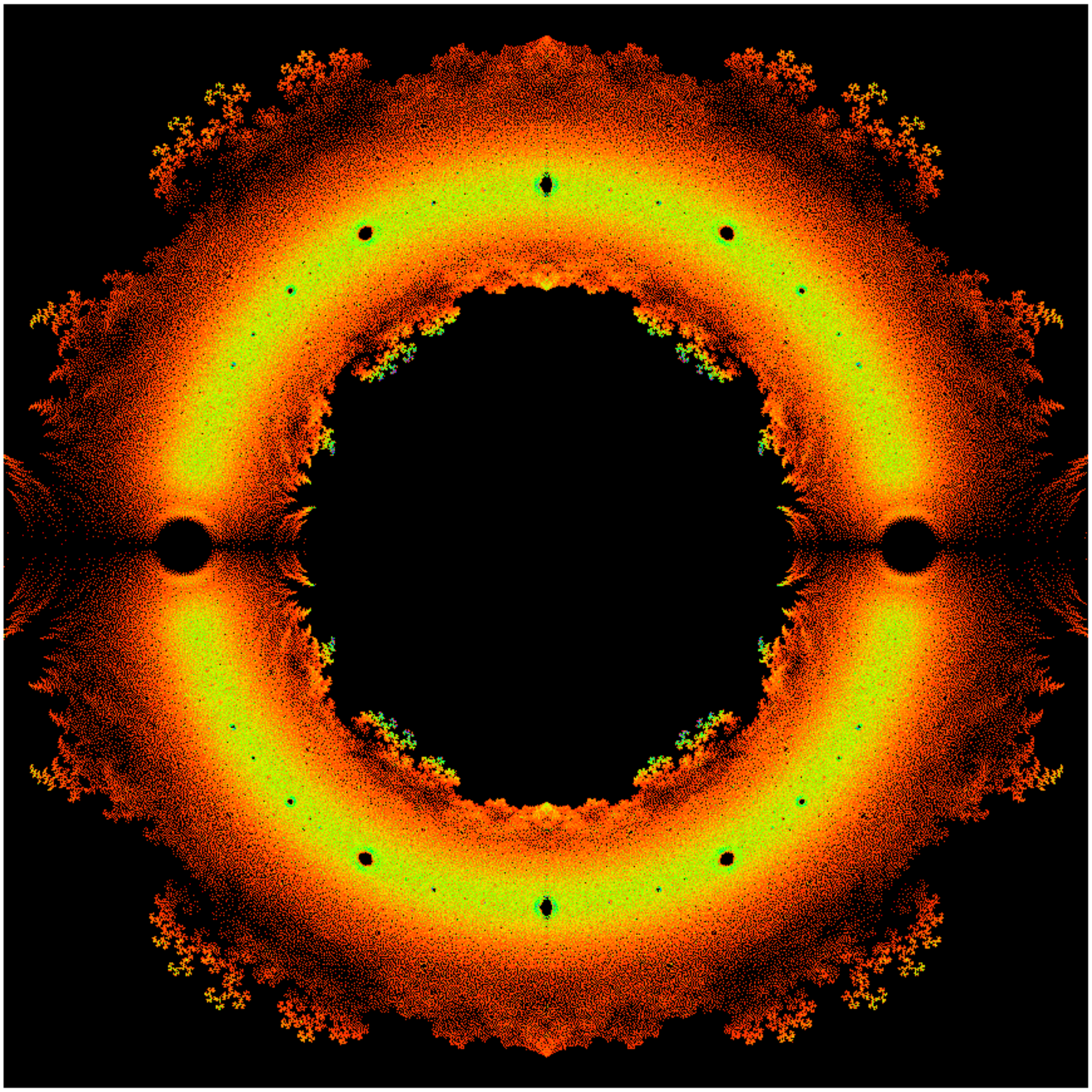
For example how does one do arbitrary precision numerical quadrature?

When does one switch methods with precision required or with different analytic properties of the integrand?

How does one deal with branch cuts of analytic functions? How does one even deal consistently with log (even this isn't completely worked out)?

More ambitiously how does one do a similar analysis for differential equations?

The goal is to marry the algorithms of analysis with symbolic and exact computation and to do this with as little loss of speed as possible. (SNAP)



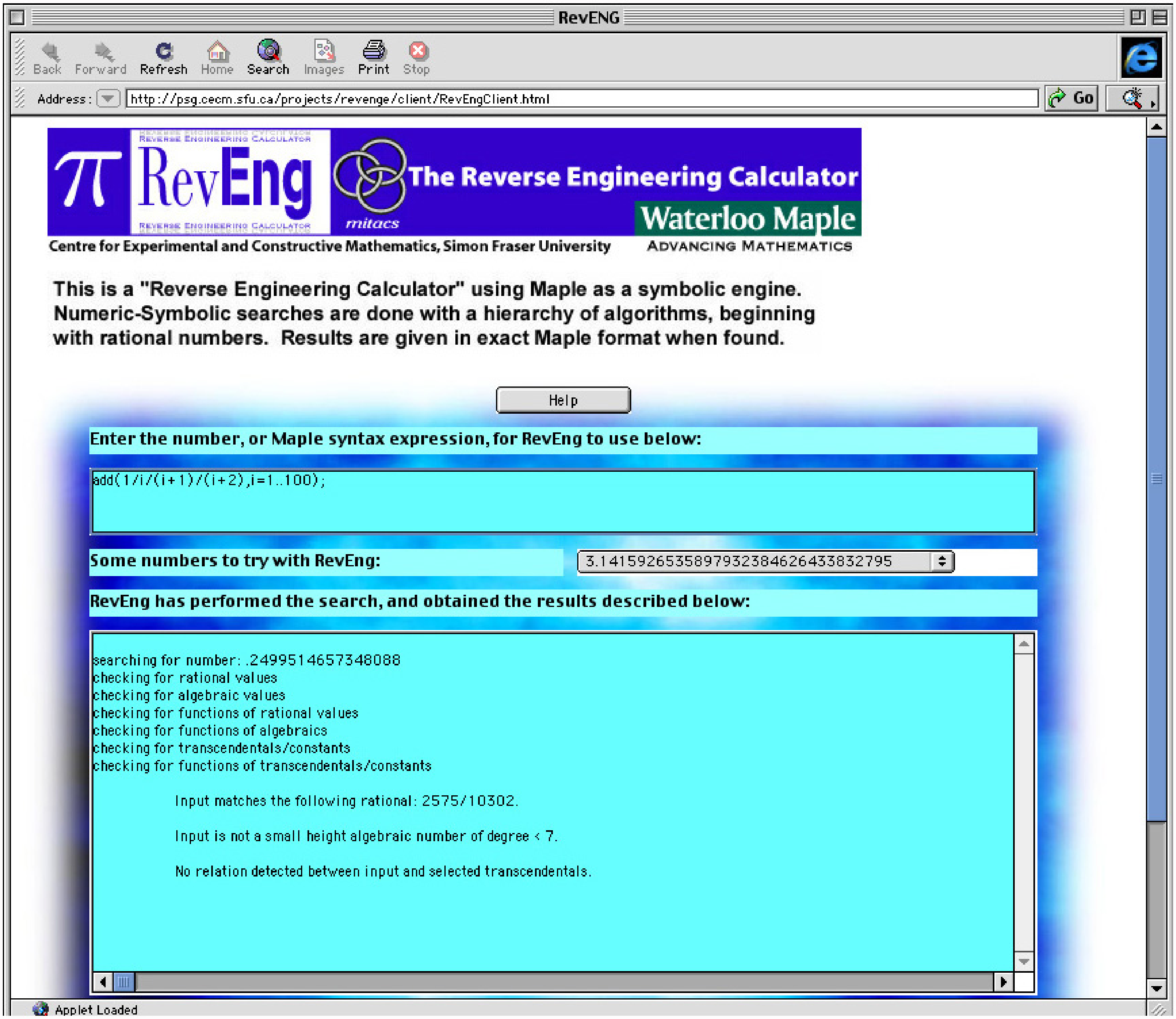
Within this context a number of very interesting problems concerning the visualization of mathematics arise (how does one actually "see" what one is doing.

It has been argued that Cartesian graphing was the most important invention of the last millennium.

Certainly it changed how we though about mathematics – the subsequent development of differential calculus rested on it.

More subtle and complicated graphics, like those of fractals, allow for a kind of exploration that was previously impossible.

There are many issues to be worked out here that live at the interface of mathematics, pedagogy and even psychology but are very timely to get right. (Think of how one visualizes the human genome and its patterns – which is after all just a particular several billion digit number base four.)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ADVANCING MATHEMATICS
This is a "Reverse Engineering Calculator" using Maple as a symbolic engine. Numeric-Symbolic searches are done with a hierarchy of algorithms, beginning with rational numbers. Results are given in exact Maple format when found.
Help
Enter the number, or Maple syntax expression, for RevEng to use below:
add(1/i/(i+1)/(i+2),i=1..100);
Some numbers to try with RevEng:
3.14159265358979323846264338327950
RevEng has performed the search, and obtained the results described below:
searching for number: .2499514657348088
checking for rational values
checking for algebraic values
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checking for functions of algebraics
checking for transcendentals/constants
checking for functions of transcendentals/constants
Input matches the following rational: 2575/10302.
Input is not a small height algebraic number of degree < 7.
No relation detected between input and selected transcendentals.
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