The Density of Rational Functions in Markov Systems: A Counterexample to a Conjecture of D.J. Newman

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Abstract

We construct a family of infinite Markov systems on $[-1,1]$ with the property that the rational functions from these systems are not dense in $C[-1,1]$. This gives counterexamples to a long standing conjecture of D.J. Newman.

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1 Introduction

The classical Muntz Theorem says that the system of functions

$$\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}, \quad \lambda_j \to \infty$$

is complete, meaning linear combinations are dense, in $C[0,1]$ if and only if

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty.$$

In the early seventies D.J. Newman conjectured that the completeness question for rational functions from the above system had a completely different answer. More precisely, for any real sequence $\{\lambda_j\}$ the rational functions of the form

$$\frac{a_o + \sum a_j x^{\lambda_j}}{b_0 + \sum b_j x^{\lambda_j}}$$

are always dense. This surprising and pretty result was proved by Somorjai [14] in the case that the sequence tends to infinity and in full generality by Bak and Newman [2]. This is all discussed in [9] where Newman writes

> "Apparently rational functions always want to be dense. There is something magical about performing that one division."

He goes on to conjecture that for *any* infinite Markov system (defined below) the rational functions from the system will be dense in the continuous functions. Though in fairness to Newman he calls this "a wild conjecture in search of a counterexample". Apart from the above example, Newman's conjecture holds for various other Markov systems, including

$$\left\{\frac{1}{x+\alpha_1}, \frac{1}{x+\alpha_2}, \frac{1}{x+\alpha_3}, \ldots\right\}$$

for arbitrary $\{\alpha_i\}$. These and related results may be found in [5], [6]. The point of this paper is to construct Markov systems for which the conjecture fails. These examples are "reasonably natural" and suggest that the class of Markov systems for which the conjecture holds may unfortunately be quite limited.

The definitions we need are the following. An infinite Markov system on $[a, b]$ is a sequence of functions $\{\varphi_0, \varphi_1, \ldots\}$ in $C[a, b]$ with the property that every "polynomial" of the form

$$a_0\varphi_0 + a_1\varphi_1 + \cdots + a_n\varphi_n \ (a_i \in \mathbf{R})$$

has at most n zeros on $[a, b]$. This is equivalent to demanding that each initial segment $\{\varphi_0, \varphi_1, \ldots, \varphi_n\}$ is a Chebyshev system. Chebyshev systems sit at the heart of approximation questions because they are the only systems that allow for both the existence and uniqueness of best approximants. By a rational function from such a system we mean any function of the form

$$\frac{a_0\varphi_0 + a_1\varphi_1 + \cdots + a_n\varphi_n}{b_0\varphi_0 + b_1\varphi_1 + \cdots + b_n\varphi_n}; \quad a_j, b_j \in \mathbf{R}$$

that is a ratio of "polynomials." Material on Markov systems, Chebyshev systems, and related matters may be found in [10].

2 Construction

We construct an infinite Markov system as follows. Consider non-negative *even* integers

$$0 = \mu_0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \lambda_n < \mu_n < \cdots$$

which are lacunary in the sense that, for some $q > 1$,

$$\frac{\mu_i}{\lambda_i} > q \quad \text{and} \quad \frac{\lambda_{i+1}}{\mu_i} > q \quad \text{for all} \quad i.$$

Let $\varphi_k \in C[-1,1]$ be defined by

$$\varphi_0 := 1, \quad \varphi_{2k}(x) := x^{\mu_k}$$

and

$$\varphi_{2k+1}(x) := \begin{cases} x^{\lambda_k}; & x \geq 0 \\ -x^{\lambda_k}; & x \leq 0. \end{cases}$$

Proposition 1. *$\{\varphi_0, \varphi_1, \ldots\}$ is a Markov system on $[-1,1]$.*

Proof. If

$$p(x) = \sum_{k=0}^{n} a_k \varphi_k(x)$$

then

$$p(x) = a_0 x^{\mu_0} + a_1 x^{\lambda_1} + a_2 x^{\mu_1} + a_3 x^{\lambda_2} + \cdots \quad \text{for} \quad x \in [0,1]$$

while

$$p(x) = a_0 x^{\mu_0} - a_1 x^{\lambda_1} + a_2 x^{\mu_1} - a_3 x^{\lambda_2} + \cdots \quad \text{for} \quad x \in [-1,0].$$

Observe that if the sequence

$$\{a_0, a_1, a_2, \ldots, a_n\}$$

has k sign changes (indices for which $a_i a_{i+1} \leq 0$) then the sequence

$$\{a_0, -a_1, +a_2, \ldots, (-1)^n a_n\}$$

has $n-k$ sign changes. So by Descarte's rule of signs [10, p. 15] $p(x)$ has at most $n-k$ zeros in $(0,1]$ and at most k zeros in $[-1,0)$ (recall that all the exponents are even so we may apply the rule on $[-1,0)$). There is a zero at zero only if $a_0 = 0$, and p is really a "polynomial" of "degree" at most

$n-1$. Thus we have proved that $p(x)$ has at most n zeros on $[-1,1]$ and have verified that we have a Markov system. ■

We make two simple observations. First, we may make each $\varphi_i \in C^k[-1,1]$ just by choosing $\lambda_i \geq k$. Secondly, while we need lacunarity for the counterexample we only need an increasing sequence of exponents for the construction in the previous proposition.

We now formulate the main result.

Theorem 2. *Let $\{\varphi_n\}$ be as above. Then the rational functions of the form*

$$\frac{\sum a_j \varphi_j}{\sum b_j \varphi_j} \quad a_j, b_j \in \mathbf{R}$$

are not dense in $C[-1,1]$.

The proof of this theorem is based on the following two lemmas. In the remainder of the paper all the summations are to infinity and all norms are the uniform (supremum) norm on the indicated intervals.

Lemma 3. (cf. [3], [11]) *There exists a constant $K > 0$ depending only on the lacunarity constant q such that*

$$\|\sum \alpha_j x^{\lambda_j} + \sum \beta_j x^{\mu_j}\|_{[0,1/2]} \leq K \|\sum \alpha_j x^{\lambda_j} + \sum \beta_j x^{\mu_j}\|_{[1/2,1]}$$

for all choices of $\alpha_j, \beta_j \in \mathbf{R}$.

Lemma 4. *There exists a constant $c > 0$ depending only on q such that*

$$\|\sum \alpha_j x^{\lambda_j} + \sum \beta_j x^{\mu_j}\|_{[0,1]} \leq c \|\sum \alpha_j x^{\lambda_j} - \sum \beta_j x^{\mu_j}\|_{[0,1]}$$

for all $\alpha_j, \beta_j \in \mathbf{R}$.

Proof. By the lacunarity condition of the exponents the system $\{x^{\lambda_j}, x^{\mu_j}\}$ is a basic sequence in $C[0,1]$ (cf. [7]). In particular this implies (cf. [12, pp. 53-54]) the existence of a constant $d > 0$ such that

$$\|\sum \alpha_j x^{\lambda_j} - \sum \beta_j x^{\mu_j}\|_{[0,1]} \geq \frac{d}{2} \max \left\{ \|\sum \alpha_j x^{\lambda_j}\|_{[0,1]}, \|\sum \beta_j x^{\mu_j}\|_{[0,1]} \right\}.$$

Hence

$$\begin{aligned} \|\sum \alpha_j x^{\lambda_j} - \sum \beta_j x^{\mu_j}\|_{[0,1]} &\geq d \left(\|\sum \alpha_j x^{\lambda_j}\| + \|\sum \beta_j x^{\mu_j}\| \right) \\ &\geq d \|\sum \alpha_j x^{\lambda_j} + \sum \beta_j x^{\mu_j}\| \end{aligned}$$

and the inequality in Lemma 4 follows. ■

The result of Gaurii and Macaev [7] can be also found in [13, pp. 141-148], [1], and [4]. Lemma 3 is explicit in [3] but follows from [4] and a version may be found in [11].

Proof of Theorem 2. Consider $f \in C[-1,1]$ defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in [-1,-1/2] \\ 0 & \text{if} \quad x \in [0,1] \\ -2x & \text{if} \quad x \in [-1/2, 0]. \end{cases}$$

We show that f is not uniformly approximable.

Suppose that

$$\left\| \frac{\sum a_j \varphi_j}{\sum b_j \varphi_j} - f \right\|_{[-1,1]} < \varepsilon < 1.$$

This implies

$$\left\| \frac{\sum \alpha_j x^{\lambda_j} + \sum \beta_j x^{\mu_j}}{\sum \alpha'_j x^{\lambda_j} + \sum \beta'_j x^{\mu_j}} \right\|_{[0,1]} < \varepsilon \tag{1}$$

and

$$\left\| \frac{\sum \alpha_j x^{\lambda_j} - \sum \beta_j x^{\mu_j}}{\sum \alpha'_j x^{\lambda_j} - \sum \beta'_j x^{\mu_j}} - 1 \right\|_{[1/2,1]} < \varepsilon \tag{2}$$

for some $\alpha_j, \alpha'_j, \beta_j, \beta'_j$.

Without loss of generality we may normalize to assume that

$$\|\sum \alpha'_j x^{\lambda_j} + \sum \beta'_j x^{\mu_j}\|_{[0,1]} = 1. \tag{3}$$

This assumption implies, together with (1), that

$$\|\sum \alpha_j x^{\lambda_j} + \sum \beta_j x^{\mu_j}\|_{[0,1]} < \varepsilon; \tag{4}$$

and by Lemma 4 it also implies that

$$\|\sum \alpha_j x^{\lambda_j} - \sum \beta_j x^{\mu_j}\|_{[0,1]} < c\varepsilon. \tag{5}$$

Also (3) and Lemma 4 give

$$\|\sum \alpha'_j x^{\lambda_j} - \sum \beta'_j x^{\mu_j}\|_{[0,1]} \geq \frac{1}{c}.$$

By Lemma 3 there exists a point $x_0 \in [1/2, 1]$ with

$$|\sum \alpha'_j x_0^{\lambda_j} - \sum \beta'_j x_0^{\mu_j}| \geq \frac{1}{Kc}. \tag{6}$$

So (6) and (5) now imply the right hand side of

$$1 - \varepsilon < \frac{\sum \alpha_j x_0^{\lambda_j} - \sum \beta_j x_0^{\mu_j}}{\sum \alpha'_j x_0^{\lambda_j} - \sum \beta'_j x_0^{\mu_j}} < Kc^2\varepsilon \tag{7}$$

while (2) implies the left hand side. Hence

$$\varepsilon > \frac{1}{Kc^2 + 1}$$

and we are done. ■

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