Reprinted from JOURNAL OF APPROXIMATION THEORY
All Rights Reserved by Academic Press, New York and London

Vol. 34, No. 1, January 1982
Printed in Belgium

On a Method of Newman and a Theorem of Bernstein

PETER B. BORWEIN

Mathematics Department, Dalhousie University, Halifax, Nova Scotia B3H 4H8, Canada

Communicated by E. W. Cheney

Received September 19, 1980

INTRODUCTION

D. J. Newman, in [2], derives precise upper bounds for uniform rational approximations to e^x on $[-1, 1]$. He writes

$$e^x = e^{z/2} e^{\bar{z}/2} \sim R_{n,m}(z/2)\, R_{n,m}(\bar{z}/2),$$

where

$$x = \tfrac{1}{2}(z + \bar{z}) \qquad \text{and} \qquad |z| = 1$$

and where $R_{n,m}$ is the (n, m) Padé approximate to e^z. The critical observation is that the approximant $R_{n,m}(z/2)\, R_{n,m}(\bar{z}/2)$ is a rational function of type (n, m) in the variable $x = (\frac{1}{2})(z + \bar{z})$. (See also Szabados [4].) It is our intention to further examine this approach.

Let E_ρ, $\rho > 1$, be the closed ellipse in the complex plane with foci at ± 1 and with semiaxes $\frac{1}{2}(\rho \pm \rho^{-1})$. Suppose that f is analytic and non-zero on a neighbourhood of E_ρ. As in [2, p. 25], for

$$z = x + iy \qquad \text{and} \qquad x^2 + y^2 = 1,$$

we have

$$f(x) = F(z)\, F(\bar{z}),$$

where

$$\log \left(f \left(\frac{z + z^{-1}}{2} \right) \right) = \log F(z) + \log F(\bar{z}).$$

Furthermore, we may assume that $F(z)$ is analytic on $D_\rho = \{z : |z| \leqslant \rho\}$.

Let Π_n denote the set of algebraic polynomials of degree at most n. We

0021-9045/82/010037-05$02.00/0

say that a rational function $p(z)/q(z)$ is of type (n, m) if $p \in \Pi_n$ and $q \in \Pi_m$. The normal (n, m) Padé approximant to a function g analytic in a neighbourhood of zero is the (n, m) rational function p_n/q_m if it exists, that satisfies

$$g(z)\, q_m(z) - p_n(z) = z^{m+n+1} h(z),$$

where h is analytic in a neighbourhood of zero and where $q_m(0) \neq 0$.

The following theorem generalizes a well-known result of S. N. Bernstein [1, p. 76] and reduces to his result in the polynomial case. Let $\|\cdot\|_I$ denote the supremum norm on the set I.

THEOREM. *Suppose that f is analytic and non-zero in a neighbourhood of E_ρ. Let F be defined as above and suppose that R is the normal (n, m) Padé approximant to F. If*

$$\|F(z) - R(z)\|_{D_\rho} \leqslant A \quad \textit{and} \quad \|F(z)\|_{D_\rho} \leqslant B \tag{$*$}$$

then

$$\|f(x) - S(x)\|_{[-1,1]} \leqslant \frac{3A(A+B)\rho}{\rho^{n+m}(\rho-1)^2},$$

where $S(x) = R(z/2)\, R(\bar{z}/2)$ is a rational function of type (n, m).

Proof. If $|z| = 1$ then

$$\begin{aligned}
&F(z)\, F(\bar{z}) - R(z)\, R(\bar{z}) \\
&\quad = F(z)\, F(1/z) - R(z)\, R(1/z) \\
&\quad = \frac{z^{n+m+1}[F(z) - R(z)]\, F(1/z)}{z^{n+m+1}} + \frac{z^{-(n+m+1)}[F(1/z) - R(1/z)]\, R(z)}{z^{-(n+m+1)}} \\
&\quad = \frac{z^{n+m+1}}{2\pi i} \int_{\alpha_1} \frac{(F(\zeta) - R(\zeta))\, F(1/\zeta)\, d\zeta}{\zeta^{n+m+1}(\zeta - z)} \\
&\quad\quad + \frac{z^{n+m+1}}{2\pi i} \int_{\alpha_2} \frac{(F(\zeta) - R(\zeta))\, F(1/\zeta)\, d\zeta}{\zeta^{n+m+1}(\zeta - z)} \\
&\quad\quad + \frac{z^{-(n+m+1)}}{2\pi i} \int_{\alpha_1} \frac{(F(1/\zeta) - R(1/\zeta))\, R(\zeta)\, d\zeta}{\zeta^{-(n+m+1)}(\zeta - z)} \\
&\quad\quad + \frac{z^{-(n+m+1)}}{2\pi i} \int_{\alpha_2} \frac{(F(1/\zeta) - R(1/\zeta))\, R(\zeta)\, d\zeta}{\zeta^{-(n+m+1)}(\zeta - z)} \\
&\quad = I_1(z) + I_2(z) + I_3(z) + I_4(z),
\end{aligned}$$

where α_1 is the circle of radius ρ taken counter-clockwise and α_2 is the circle of radius $1/\rho$ taken clockwise. It is easily verified from the definitions that

$$\frac{F(z)-R(z)}{z^{n+m+1}}$$

is analytic on the annulus $\{z\colon 1/\rho \leqslant |z| \leqslant \rho\}$ and hence, that the preceding application of Cauchy's integral formula is valid. For $|z|=1$,

$$|I_1(z)| \leqslant \frac{1}{2\pi}\int_{\alpha_1} \frac{AB\, d\xi}{\rho^{n+m+1}(\rho-1)} \leqslant \frac{AB}{\rho^{n+m}(\rho-1)}.$$

Similarly, for $|z|=1$,

$$|I_4(z)| \leqslant \frac{A(A+B)}{\rho^{n+m+1}(\rho-1)}.$$

We now estimate $I_2(z)$ and $I_3(z)$. First, for $|w|<\rho$,

$$F(w)-R(w) = \frac{w^{n+m+1}}{2\pi i}\int_{\alpha_1} \frac{(F(\zeta)-R(\zeta))\, d\zeta}{\zeta^{n+m+1}(\zeta-w)}$$

and for $w \in \alpha_2$,

$$|F(w)-R(w)| \leqslant \frac{A}{\rho^{2n+2m+1}(\rho-\rho^{-1})}.$$

Thus, for $|z|=1$,

$$|I_2(z)| \leqslant \frac{AB}{\rho^{n+m}(\rho-\rho^{-1})(\rho-1)}$$

and

$$|I_3(z)| \leqslant \frac{A(A+B)\rho}{\rho^{n+m}(\rho-\rho^{-1})(\rho-1)}.$$

Combining the above estimates yields, for $|z|=1$,

$$|F(z)\,F(\bar{z}) - R(z)\,R(z)| \leqslant \frac{3A(A+B)}{\rho^{n+m-1}(\rho-1)^2}$$

whence the result follows.

Condition $(*)$ of the above theorem is always satisfied in the polynomial case. In the general case Szabados [4] shows that there exists a function f analytic in E_ρ so that

$$\lim_{n\to\infty} \sup(R_{n,n}(f))^{1/n} = 1/\rho,$$

where

$$R_{n,n}(f) = \inf_{p_n, q_n \in \pi_n} \|p_n/q_n - f\|_{[-1,1]}.$$

Thus, we cannot hope to omit assumption $(*)$ completely from the theorem.

Approximating $(x-\rho)^{1/2}$

Assume that p/q is the normal (n, n) Padé approximant at the point $z = 1$ to the function $z^{1/2}$. Then,

$$p(z) - q(z)\sqrt{z} = (1-z)^{2n+1}h(z)$$

and

$$p(z^2) - q(z^2)z = (1-z)^{2n+1}(1+z)^{2n+1}h(z^2),$$

where $h(z)$ is analytic on $C - (-\infty, 0]$. Since $p(z^2) - q(z^2)z$ is a polynomial of degree $2n+1$ with $2n+1$ roots at 1, it follows that

$$p(z) - q(z)\sqrt{z} = (1-\sqrt{z})^{2n+1}, \tag{1}$$

where q is suitably normalized. Expanding (1) and comparing coefficients yields

$$\begin{aligned} q(z) &= \sum_{k=0}^{n} \binom{2n+1}{2k+1} z^k \\ &= \frac{1}{2\sqrt{z}}\left[(1+\sqrt{z})^{2n+1} - (1-\sqrt{z})^{2n+1}\right] \end{aligned} \tag{2}$$

and

$$p(z) = \tfrac{1}{2}\left[(1+\sqrt{z})^{2n+1} + (1-\sqrt{z})^{2n+1}\right]. \tag{3}$$

(That we may assume the existence of the normal (n, n) Padé approximant is now clear from (1), (2) and (3).) From (2) one deduces that q has only real-negative roots and that for $|z-1| \leqslant 1$,

$$\left| \frac{p(z)}{q(z)} - \sqrt{z} \right| \leqslant \frac{|(1-\sqrt{z})^{2n+1}|}{q_n(0)} \leqslant \frac{1}{2n+1}.$$

If $p_\rho(z)/q_\rho(z)$ is the normal (n, n) Padé approximant to $(\rho - z)^{1/2}$ then

$$\|p_\rho(z)/q_\rho(z) - (\rho - z)^{1/2}\|_{D_\rho} \leqslant \frac{\rho^{1/2}}{2n+1}.$$

Thus, by the Theorem, there exists $S_{n,n}$ a rational function of type (n, n) so that

$$\left\| S_{n,n}(x) - \frac{(\rho^2 + 1 - 2\rho x)^{1/2}}{(2\rho)^{1/2}} \right\|_{[-1,1]} \leqslant \frac{3\rho^2}{n\rho^{2n}(\rho - 1)^2}.$$

If we set $\alpha = (\rho^2 + 1)/2\rho$ we get

$$\| S_{n,n}(x) - (\alpha - x)^{1/2} \|_{[-1,1]} \leqslant \left(\frac{12\alpha^2}{n(\alpha^2 - 1)} \right) \cdot \frac{1}{(\alpha + \sqrt{\alpha^2 - 1})^{2n}}.$$

These types of rational approximations to $(\alpha - x)^{1/2}$ converge at least as fast as $(\alpha + \sqrt{\alpha^2 - 1})^{-2n}$ while polynomial approximations only behave like $(\alpha + \sqrt{\alpha^2 - 1})^{-n}$. See [3 p. 437] for a more general discussion of the derivation of Padé approximations to $x^{1/2}$ and related functions.

References

1. G. G. Lorentz, "Approximation of Functions," Holt Rinehart & Winston, New York, 1966.
2. D. J. Newman, "Approximation with Rational Functions," Regional Conference Series in Mathematics, No. 41, Amer. Math. Soc., Providence, R. I., 1979.
3. O. Perron, "Die Lehre von den Kettenbrüchen," Chelsea, New York, 1950.
4. J. Szabados, Rational approximation to analytic functions on an inner part of the domain of analyticity, *in* "Approximation Theory" (A. Talbot, Ed.), Academic Press, London/New York, 1970.

Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium