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Inequalities for Compound Mean Iterations with Logarithmic Asymptotes*

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We consider the compound means arising as limits from the arithmetic–geometric mean iteration and related iterations. Each of these iterations possesses a logarithmic asymptote. We show that these limit means satisfy very precise inequalities. These can be deduced in a quite uniform fashion from a "comparison lemma" for compound means.

1. INTRODUCTION

We follow the terminology for mean iterations as used in Chap. 8 of [4]. More explicitly: $M(a, b)$ is a *mean* on the positive orthant in $\mathbb{R}^2$ if

$$\min\{a, b\} \leqslant M(a, b) \leqslant \max\{a, b\}$$

for each $a > 0$, $b > 0$. The mean is *strict* if $M(a, b) = b$ or a implies that $a = b$. Throughout we assume that all means are continuous and strict. Let $\Delta := \{(a, b) \mid a \geqslant b > 0\}$. We suppose that M and N are *comparable* means, that is $M(a, b) \geqslant N(a, b)$ on Δ.

We are interested in the *mean iteration*:

$$a_{n+1} := M(a_n, b_n) \qquad b_{n+1} := N(a_n, b_n) \tag{1.1}$$

where $a_0 := a$, $b_0 := b$ for positive numbers a and b.

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Then Theorem 8.2 in [4] insures that a_n and b_n converge to a common limit, denoted by $M \otimes N(a, b)$, which represents another continuous strict mean. Also, $M \otimes N$ is positively homogeneous, symmetric, or isotone whenever both M and N are.

It will also be convenient to recall the *classical hypergeometric function*

$$F(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n \cdot (b)_n}{(c)_n \cdot n!} x^n$$

where $(a)_n$ is the *rising factorial* $a(a+1) \cdots (a+n-1)$. We will also need the following special case of (15.3.10) in [1]:

$$F(a, 1-a; 1; 1-x) = \frac{\sin(\pi a)}{\pi} \sum_{n=0}^{\infty} \frac{(a)_n \cdot (1-a)_n}{(n!)^2} \times [2\psi(n+1) - \psi(n+a) - \psi(n+1-a) - \ln(x)] x^n. \tag{1.2}$$

Here ψ is the *digamma function* and satisfies

$$\psi(1) = -\gamma, \psi(x+1) = \psi(x) + \frac{1}{x}, \psi(mx) = \ln(m) + \sum_{k=0}^{m-1} \psi\left(x + \frac{k}{m}\right). \tag{1.3}$$

In each case we are satisfied to consider real x and positive a and b with $c = a + b = 1$. Related asymptotics for $c = a + b$, $a > 0$ and $b > 0$ can be found in [2] and [11].

EXAMPLES 1.1. (a) Let $M_2(a, b) := (a + b)/2$ $(:= A(a, b))$ and $N_2(a, b) := \sqrt{ab}$ $(:= G(a, b))$. Then $AG_2 := M_2 \otimes N_2$ is the *arithmetic-geometric mean* of Gauss and Legendre and convergence is quadratic. This is discussed at length in [4]. We record from [4, 5, 6] that, for $0 < x < 1$,

$$AG_2(1, x) = \frac{1}{F(1/2, 1/2; 1; 1-x^2)}.$$

(b) Let $M_3(a, b) := (a + 2b)/3$ and $N_3(a, b) := \sqrt[3]{b(a^2 + ab + b^2)/3}$. Then $AG_3 := M_3 \otimes N_3$ is the limit of a cubically convergent iteration. As was shown in [6, 8], for $0 < x < 1$

$$AG_3(1, x) = \frac{1}{F(1/3, 2/3; 1; 1-x^3)}.$$

(c) Let $M_4(a, b) := (a+3b)/4$ and $N_4(a, b) := \sqrt{b(a+b)/2}$. Then $AG_4 := M_4 \otimes N_4$ is the limit of a quadratically convergent iteration. As was shown in [6], for $0<x<1$

$$AG_4(1, x) = \frac{1}{F(1/4, 3/4; 1; 1-x^2)^2}.$$

(d) Let $M_5(a, b) := (a+3b)/4$ and $N_5(a, b) := (b+\sqrt{ab})/2$. Then $B_2 := M_5 \otimes N_5$ is *Borchardt's mean* and is the limit of a quadratically convergent iteration. As was shown in [7] and [9], B_2 has various expressions in terms of hypergeometric functions.

In each of these four cases it is easy to verify directly that the limit exists and that convergence is at the rate claimed, but identifying the limit is much harder. Additionally, each limit mean is homogeneous so that we have identified the limit on Δ. (In the case of AG_2 this extends by symmetry to the entire orthant.)

Formula (1.2) allows us to write down the precise asymptotic behaviour in cases (a) to (c) above.

PROPOSITION 1.2. *For* $0<x\leqslant 1$ *we have*

$$\text{(a)} \qquad AG_2(1, x) < \frac{\pi}{2}\Big/\ln\left(\frac{4}{x}\right),$$

$$\text{(b)} \qquad AG_3(1, x) < \frac{2\pi}{3\sqrt{3}}\Big/\ln\left(\frac{3}{x}\right),$$

$$\text{(c)} \qquad AG_4(1, x) < \left(\frac{\pi}{\sqrt{2}}\Big/\ln\left(\frac{8}{x}\right)\right)^2.$$

Moreover, in each case the two expressions are asymptotic as x decreases to zero.

Proof. The hypergeometric expressions given in Example 1.1 can be expressed in terms of (1.2). In each case the right-hand expression is the leading term in (1.2). This follows on using the properties of the digamma function listed in (1.3). Next, we observe that all other coefficients in (1.2) are positive for x in [0, 1]. (This again uses (1.3) and some elementary estimates.) ∎

The errors can be explicitly determined in each case. Thus

$$F(1/2, 1/2; 1; 1-x^2) = \frac{1}{AG_2(1, x)} \leqslant \frac{2}{\pi}\ln\left(\frac{4}{x}\right) + \frac{8x^2}{\pi}\left[8+\ln\left(\frac{1}{x}\right)\right] \qquad (1.4)$$

as follows from [4, p. 11]. We have no need of corresponding estimates for AG_3 or AG_4 which are of much the same form with the error of order $x^3 \ln(x)$ in the case of AG_3. Related inequalities can be found in [2].

2. Inequalities between Means

We now turn to a simple but powerful basic tool in establishing inequalities between compound means. We recall that a mapping $\Phi: \Delta \to \mathbb{R}$ is *diagonal* if $\Phi(x, x) = x$. All means are diagonal mappings.

Lemma 2.1 (Comparison Lemma). *Let $M \geqslant N$ be strict means on Δ. Let $\Phi: \Delta \to \mathbb{R}$ be continuous and diagonal. Suppose that*

$$\Phi(M(a, b), N(a, b)) \leqslant \Phi(a, b) \tag{2.1}$$

for (a, b) in Δ with $1 > b/a > \varepsilon \geqslant 0$. Then

$$M \otimes N(a, b) \leqslant \Phi(a, b)$$

for (a, b) in Δ with $1 > b/a > \varepsilon \geqslant 0$.

Proof. In the notation of (1.1) it follows inductively, because $M \geqslant N$ are comparable means on Δ, that $1 \geqslant b_{n+1}/a_{n+1} > \varepsilon$ whenever $1 \geqslant b_n/a_n > \varepsilon$. This in turn means that

$$\Phi(a_{n+1}, b_{n+1}) \leqslant \Phi(a_n, b_n) \leqslant \Phi(a, b)$$

for all n in $\mathbb{N}$. On moving to the limit we reach our conclusion because Φ is diagonal. ∎

Remarks 2.2. (i) In most applications ε is zero. (ii) We may replace '$\leqslant$' by '$<$' in both hypothesis and conclusion. (iii) We observe that since $-\Phi$ is diagonal whenever Φ is, we may apply the Comparison Lemma with '$\geqslant$' or '$=$' replacing '$\leqslant$'. In the equality case, we recapture a well known *Invariance Principle* ([4, Thm. 8.3]). We note that in the equality case (2.1) characterizes the limit mean, but in the inequality case (2.1) is a stronger inequality.

We illustrate this principle as follows: let L denote the *logarithmic mean*

$$\mathscr{L}(a, b) := (a - b) \Big/ \ln\left(\frac{a}{b}\right) \tag{2.2}$$

extended continuously so that $\mathscr{L}(a, a) := a$.

EXAMPLE 2.3. For $|r| < 1$ we consider the means

$$M_r(a, b) := r\frac{a^r(a-b)}{a^r - b^r} \quad \text{and} \quad N_r(a, b) := r\frac{b^r(a-b)}{a^r - b^r}.$$

We observe that $M_{-r} = N_r$ and that $\lim_{r \to 0} M_r(a, b) = \mathscr{L}(a, b)$. We claim that for $0 \leqslant r < 1$

$$M_r \otimes N_r(a, b) = \mathscr{L}(a, b). \tag{2.3}$$

This is trivial for $r = 0$. For strictly positive r, we verify that

$$\mathscr{L}(M_r(a, b), N_r(a, b)) = \mathscr{L}(a, b),$$

and then may apply the Invariance Principle (the equality form of the previous Comparison Lemma). This last equation holds since

$$\text{(i)} \qquad \frac{M_r(a, b)}{N_r(a, b)} = \frac{a^r}{b^r}$$

and

$$\text{(ii)} \qquad M_r(a, b) - N_r(a, b) = r(a - b).$$

Equation (ii) also shows that convergence is linear of order r^n. For values such as $r := 1/2,\ 1/3,\ 2/3$, the means simplify. In particular, for $r = 1/2$ we have

$$M_{1/2}(a, b) := \frac{a + \sqrt{ab}}{2} \quad \text{and} \quad N_{1/2}(a, b) := \frac{b + \sqrt{ab}}{2} \tag{2.4}$$

and we rederive a well known iteration, see [4].

We are now ready to prove our main results. Where convenient we will suppress variables.

THEOREM 2.4. *The following inequalities hold for all x such that* $0 < x < 1$:

$$\frac{\pi}{2}\mathscr{L}(1, x) > \frac{\pi}{2}\Big/\ln\left(\frac{4}{x}\right) > AG_2(1, x) > \mathscr{L}(1, x) > AG_4(1, x) > B_2(1, x).$$

Proof. (i) The first inequality is a simple calculus exercise.

(ii) The second inequality was established in Proposition 1.2(a).

(iii) $AG_2(1, x) > \mathscr{L}(1, x)$: We will show that, in the notation of Example 1.1(a),

$$\mathscr{L}(1, x) < \mathscr{L}(M_2(1, x), N_2(1, x)). \tag{2.5}$$

This will establish the strict form of (2.1) because all the means involved are positively homogeneous. To prove (2.5) we observe that on cross multiplying, substituting $x := y^2$, and factoring out $(y-1)$, it suffices to show that

$$y \ln(y) > \frac{y+1}{2} \ln\left(\frac{1+y^2}{2}\right) \qquad (0 < y < 1).$$

Let $g(y) := y \ln(y)$. Now $(1+y^2)/2 > ((1+y)/2)^2$ so that

$$\frac{y+1}{4} \ln\left(\frac{1+y^2}{2}\right) < \frac{y+1}{2} \ln\left(\frac{1+y}{2}\right) = g\left(\frac{y+1}{2}\right) < \frac{g(y)}{2} + \frac{g(1)}{2} = \frac{g(y)}{2}$$

since g is convex and $g(1) = 0$.

(iv) $\mathscr{L}(1, x) > AG_4(1, x)$: Arguing as in (iii), we need to show that

$$\mathscr{L}(1, x) > \mathscr{L}(M_4(1, x), N_4(1, x)). \tag{2.6}$$

Let

$$g(x) := \mathscr{L}(1, x) - \mathscr{L}\left(\frac{1+3x}{4}, \sqrt{x\,\frac{1+x}{2}}\right).$$

On rearranging, the numerator of $g(x)$ can be written as

$$4(1-x)\left\{\ln\left(\frac{1+3x}{4}\right) - \left[\frac{5}{4} - \frac{x+2}{2+\sqrt{2x(x+1)}}\right] \ln(x) - \frac{1}{2} \ln\left(\frac{1+x}{2}\right)\right\}.$$

Let

$$\alpha(x) := \frac{5}{4} - \frac{x+2}{2+\sqrt{2x(x+1)}}.$$

Thus it suffices to show that

$$\ln\left(\frac{1+3x}{4}\right) - \alpha(x) \ln(x) - \frac{1}{2} \ln\left(\frac{1+x}{2}\right) > 0.$$

Since ln is strictly concave increasing and since $\alpha(x) \leqslant 1/2$ on $[0, 1]$ this last inequality holds.

(v) $AG_4(1, x) > B_2(1, x)$: The Invariance Principle yields

$$AG_4 = AG_4(M_4, N_4) > AG_4(M_5, N_5) \tag{2.7}$$

for $0<b/a<1$, since $M_4=M_5$ and $N_4>N_5$ while AG_4 is monotone. Then the Comparison Lemma applies to (2.7). ∎

THEOREM 2.5. *The following inequalities hold for all x such that $0<x<1$:*

$$AG_2(1, x) > AG_3(1, x) > AG_4(1, x).$$

Proof. (i) $AG_3(1, x) > AG_4(1, x)$: This is very similar to the proof of $AG_4(1, x) > B_2(1, x)$. We observe, with a little manipulation that, for $0<b/a<1$, $M_3>M_4$ and $N_3>N_4$. Thus

$$AG_3 = AG_3(M_3, N_3) > AG_3(M_4, N_4) \tag{2.8}$$

and the Comparison Lemma applies again.

(ii) $AG_2(1, x) > AG_3(1, x)$: This is a little more subtle. Two iterations of AG_2 produce

$$AG_2 = AG_2(A_{1/2}, B_{1/2}),$$

where

$$A_{1/2}(a, b) := \left(\frac{a^{1/2}+b^{1/2}}{2}\right)^2 \quad \text{and} \quad B_{1/2}(a, b) := \sqrt{\sqrt{ab}\,\frac{a+b}{2}}.$$

[*Inter alia*, this shows that $A_{1/2} > AG_2 > B_{1/2}$ for (a, b) in Δ.] While $B_{1/2} > N_3$ for $0<b/a<1$, $A_{1/2} > M_3$ holds exactly for $1/25 < b/a < 1$. Thus, for $1/25 < b/a < 1$

$$AG_2 = AG_2(A_{1/2}, B_{1/2}) > AG_2(M_3, N_3).$$

The Comparison Lemma applies with $\varepsilon := 1/25$ and we deduce that $AG_2(1, x) > AG_3(1, x)$ for $1/25 < x < 1$. By Examples 1(a), (b) it remains to show that

$$F(1/2, 1/2; 1; 1-x^2) < F(1/3, 2/3; 1; 1-x^3) \qquad (0<x<1/25). \tag{2.9}$$

It will follow that (2.9) and (ii) hold on the entire interval $0<x<1$.

We establish (2.9) by observing that Proposition 1.2(b) implies that

$$F(1/3, 2/3; 1; 1-x^3) > \frac{3\sqrt{3}}{2\pi} \ln\left(\frac{3}{x}\right),$$

while (1.4) shows

$$F(1/2, 1/2; 1; 1-x^2) \leqslant \frac{2}{\pi} \ln\left(\frac{4}{x}\right) + \frac{8x^2}{\pi}\left[8 + \ln\left(\frac{1}{x}\right)\right].$$

Thus it suffices to verify that for $0 < x < 1/25$,

$$\frac{3\sqrt{3}}{2\pi}\ln\left(\frac{3}{4}\right) - \frac{2}{\pi}\ln\left(\frac{4}{x}\right) - \frac{8x^2}{\pi}\left[8 + \ln\left(\frac{1}{x}\right)\right] > 0. \tag{2.10}$$

It is easy to check that this function decreases on $0 < x < 1$ and is positive at 1/8. Thus (2.10) holds at least for $0 < x < 1/8$. ∎

We observe that AG_3 and $\mathscr{L}$ are not comparable (they switch over around 0.0063), so that the last two results are distinct. The implicit inequality $(\pi/2)\,\mathscr{L}(1, x) > AG_2(1, x)$ was first observed in [11], while the inequality $AG_2(1, x) > \mathscr{L}(1, x)$ was first established in [10]. The last inequality also follows from the next inequality which is of interest in its own right.

PROPOSITION 2.6. *For all* x *with* $0 < x < 1$,

$$\left(\frac{1+x}{2}\right) AG_2(1, x) < AG_2(1, x^2) \tag{2.11}$$

or equivalently,

$$\left(\frac{1+x}{2}\right) F(1/2, 1/2; 1; 1-x^4) < F(1/2, 1/2; 1; 1-x^2). \tag{2.12}$$

Proof. Example 1.1(a) establishes the equivalence of (2.11) and (2.12). To prove (2.11) we observe that the Invariance Principle shows that $AG_2(1, x) = AG_2((1+x)/2, \sqrt{x})$ so we need to show that

$$\left(\frac{1+x}{2}\right) AG_2\left(\frac{1+x}{2}, \sqrt{x}\right) < AG_2(1, x^2).$$

We now argue with theta functions. (See [4, Chap. 2], [3], or [12] for the relevant definitions.) Let $x := \theta_3(q)/\theta_4(q)$ for $0 < |q| < 1$ so that (2.11) becomes

$$\left(\frac{\theta_3(q)+\theta_4(q)}{2}\right) AG_2\left(\frac{\theta_3(q)+\theta_4(q)}{2}, \sqrt{\theta_3(q)\,\theta_4(q)}\right) < AG_2(\theta_4^2(q), \theta_3^2(q)).$$

It is a fundamental property of AG_2 that the right-hand quantity is identically 1, [4, p. 35]. Moreover, $(\theta_3(q)+\theta_4(q))/2 = \theta_3(q^4)$ and $\theta_3(q)\,\theta_4(q) = \theta_4^2(q^2)$, [4, p. 34], and so the desired inequality is

$$AG_2(\theta_3^2(q^4), \theta_3(q^4)\,\theta_4(q^2)) < 1 = AG_2(\theta_3^2(q^4), \theta_4^2(q^4)).$$

This holds since $\theta_3(q^4)\,\theta_4(q^2) < \theta_3(q^2)\,\theta_4(q^2) = \theta_4^2(q^4)$, (because $\theta_3(q)$ increases with q) and since AG_2 is monotone.

The Comparison Lemma may be applied to (2.11) to rederive $AG_2(1, x) > \mathscr{L}(1, x)$. ∎

We recall (see [4, Chap. 8]) that for any mean

$$M_p(a, b) := M(a^p, b^p)^{1/p}. \tag{2.13}$$

Thus, if A denotes the arithmetic mean, $A_{1/2}$ is $(A)_{1/2}$, the Hölder mean of exponent 1/2.

We finish with a proof of the following extremely sharp inequality recently discovered experimentally by Vuorinen (private correspondence).

PROPOSITION 2.7. $\mathscr{L}_{3/2}(1, x) > AG_2(1, x) > \mathscr{L}(1, x)$.

Proof. By now familiar arguments, it suffices to show that

$$\mathscr{L}_{3/2}(1, x) > \mathscr{L}_{3/2}(M_2(1, x), N_2(1, x)) \qquad (0 < x < 1)$$

and, on replacing x by $x^{2/3}$, we are finished if we show that

$$\mathscr{L}(A_{2/3}(1, x), G(1, x)) < \mathscr{L}(1, x) \qquad (0 < x < 1).$$

Written out explicitly, we wish to show that

$$\frac{\ln(x)}{x-1} < \ln\left(\left(\frac{1+x^{2/3}}{2}\right)^{3/2} \Big/ \sqrt{x}\right) \Big/ \left(\left(\frac{1+x^{2/3}}{2}\right)^{3/2} - \sqrt{x}\right) \quad (0 < x < 1). \tag{2.14}$$

This we can only establish—by a somewhat unsatisfactory computational route—which we now sketch. Substitute $x = y^6$ and normalize to get an equivalent inequality namely:

$$g(y) := \left\{(1-y^6)\left[\frac{3}{2}\log\left(\frac{1+y^4}{2}\right) - 3\log y\right] - 6y^3\log(y)\right\}^2$$
$$- (6\log(y))^2\left(\frac{1+y^4}{2}\right)^3 > 0 \qquad y \in (0, 1). \tag{2.15}$$

We write $g(y)$ as a quadratic in $\log((1+y^4)/2)$ and $\log(y)$ with coefficients which are polynomial in y. On replacing $-y^k\ln(y)$ by $-1/(ke)$ and the like (most easily with a symbol manipulation system) we find that

$$|yg'(y)| \leqslant 50 \qquad y \in [0, 1] \tag{2.16}$$

and that

$$g(y) \to \infty \qquad \text{as} \quad y \downarrow 0. \tag{2.17}$$

Indeed, easy estimates show $g(y) > 0$ on $[0, 1/1000]$. Let $y_0 := 1/1000$ and

$$y_{n+1} := y_n[1 + y_n g(y_n)/50]. \tag{2.18}$$

It follows from (2.16) and the Mean Value theorem that by checking the positivity of $g(y_n)$ until $y_n > 3/4$ we prove (with about 20,000 function evaluations) that $g(y) > 0$ on $(0, 3/4]$.

Furthermore g expanded exactly (symbolically) at 1 is of the form

$$g(y) = \tfrac{189}{5}(y-1)^8 + O((y-1)^9). \tag{2.19}$$

If we numerically evaluate the integral

$$\left(\int_{|z-1|=1/4} \frac{g'(z)}{g(z)}\, dz\right)\bigg/ 2\pi i \tag{2.20}$$

we discover that it is 8 to as many places as we compute. By the Argument Principle, g can only have 8 zeroes inside the disc $\{|z-1| < 1/4\}$. Thus they all lie at 1. In particular g has no real zeros in $[3/4, 1]$ and so is positive on $[0, 1]$.

Finally, to validate the numerical evaluation of (2.20), it is necessary to know that (i) g does not vanish on the contour and (ii) to have an estimate for $(g'(z)/g(z))'$. We estimate $|g'(z)| \leqslant 30$ on the contour and obtain from the Maximum Modulus Principle that $|g(1 + 0.25e^{2\pi i s})|$ has Lipschitz constant less than 96 on $[0, 1]$. A similar calculation to (2.18) proves (very tediously) that $|g(z)| > 1/500{,}000$ on the contour. We may also estimate $|g''(z)|$ and so get a derivative estimate for $g'(z)/g(z)$. ∎

Of course, a self-contained proof of (2.15) would be desirable.

We note that the first four terms of the Taylor series, for $x := 1$, on each side of (2.14) coincide (and the ratio remains very close for the first 50 terms):

$$\ln\left(\left(\frac{1+x^{2/3}}{2}\right)^{3/2}\bigg/\sqrt{x}\right)\bigg/\left(\left(\frac{1+x^{2/3}}{2}\right)^{3/2} - \sqrt{x}\right)$$

$$= 1 - \frac{1}{2}(x-1) + \frac{1}{3}(x-1)^2 - \frac{1}{4}(x-1)^3 + \frac{2075}{10368}(x-1)^4 + \cdots$$

$$\frac{\ln(x)}{x-1} = 1 - \frac{1}{2}(x-1) + \frac{1}{3}(x-1)^2 - \frac{1}{4}(x-1)^3 + \frac{1}{5}(x-1)^4 + \cdots.$$

This explains why, near 1, $\mathscr{L}_{3/2}(1, x) - AG_2(1, x)$ is of very small order.

References

1. M. Abramowitz and I. Stegun, "Handbook of Mathematical Functions," Dover, New York, 1964.
2. G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, Functional inequalities for hypergeometric functions and complete elliptic integrals, *SIAM J. Math.* **23** (1992), in press.
3. R. Bellman, "A Brief Introduction to Theta Functions," Holt, Reinhart, and Winston, New York, 1961.
4. J. M. Borwein and P. B. Borwein, "Pi and the AGM—A Study in Analytic Number Theory and Computational Complexity," Wiley, N.Y., 1987.
5. J. M. Borwein and P. B. Borwein, Quadratic mean iterations, monograph in preparation.
6. J. M. Borwein and P. B. Borwein, A cubic counterpart of Jacobi's identity and the AGM, *Trans. Amer. Math. Soc.* **323** (1991), 691–701.
7. J. M. Borwein and P. B. Borwein, On the mean iteration $(a, b) \to ((a+3b)/4, (\sqrt{ab}+b)/2)$, *Math. Comp.* **53** (1989), 311–326.
8. J. M. Borwein, P. B. Borwein, and F. Garvan, Some cubic modular identities of Ramanujan, *Trans. Amer. Math. Soc.*, in press.
9. J. M. Borwein, P. B. Borwein, and F. Garvan, Hypergeometric analogues of the AGM iteration, *Constr. Approx.*, in press.
10. B. C. Carlson and M. Vuorinen, An inequality on the AGM and the logarithmic mean, *SIAM Rev. Probl. Sect.*, in press.
11. M. K. Vamanamurthy and M. Vuorinen, Inequalities for the AGM and other means, preprint, University of Auckland, New Zealand, August 1991.
12. E. T. Whittaker and G. N. Watson, "A Course of Modern Analysis," 4th ed. Cambridge Univ. Press, London, 1927.