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# Class number three Ramanujan type series for $1/\pi$

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*Abstract*

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There is a class of remarkable series for $1/\pi$ of the form

$$\frac{\sqrt{-C^3}}{\pi}=\sum_{n=0}^{\infty}\frac{A+nB}{C^{3n}}\frac{(6n)!}{(3n)!\,(n!)^3},$$

where $A$, $B$, $C$ are certain algebraic numbers. These were first examined by Ramanujan. These examples arise by computing singular invariants for $j$ and the constants involved will have degree equal to the class number of the associated imaginary quadratic field. Our intention is compute all such class number three examples. The largest such example, with discriminant $-907$, adds 37 additional digit accuracy per term. A class number four example with discriminant $-1555$ gives 50 digits per term.

*Keywords:* Ramanujan; series; pi; class number.

## 1. Introduction

Ramanujan produced a number of remarkable series for $1/\pi$; for example,

$$\frac{1}{\pi}=\frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{4^{4n}(n!)^4}\frac{[1103+26390\,n]}{99^{4n}}.$$

This series adds roughly eight digits per term. Such series exist because various modular invariants are rational (which is more or less equivalent to identifying imaginary quadratic fields

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with class number one). The larger the discriminant of such a field, the greater the rate of convergence. Thus with $d = -163$ we have the largest of the class number one examples:

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} (-1)^n \frac{(6n)!}{(n!)^3(3n)!} \frac{13\,591\,409 + 545\,140\,134\,n}{(640\,320^3)^{n+1/2}},$$

a series first discovered by the Chudnovsky's in [4]. Quadratic versions of these series correspond to class number two imaginary quadratic fields. The most spectacular and largest example is $d = -427$ and

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n(6n)!}{(n!)^3(3n)!} \frac{(A + nB)}{C^{n+1/2}},$$

where

$$A := 212\,175\,710\,912\sqrt{61} + 1\,657\,145\,277\,365,$$

$$B := 13\,773\,980\,892\,672\sqrt{61} + 107\,578\,229\,802\,750,$$

$$C := \left[5280\,(236\,674 + 30\,303\sqrt{61})\right]^3.$$

The series adds roughly 25 digits per term, $\sqrt{C}/(12A)$ already agrees with pi to 25 places. One also has a conjugate series

$$\frac{1}{\pi} = 7.12 \sum_{n=0}^{\infty} \frac{(-1)^n(6n)!}{(n!)^3(3n)!} \frac{\bar{A} + n\bar{B}}{\bar{C}^{n+1/2}},$$

where $\bar{A}$, $\bar{B}$ and $\bar{C}$ are the conjugate quadratic numbers

$$\bar{A} := 212\,175\,710\,912\sqrt{61} - 1\,657\,145\,277\,365,$$

$$\bar{B} := 13\,773\,980\,892\,672\sqrt{61} - 107\,578\,229\,802\,750,$$

$$\bar{C} := \left[5280\,(236\,674 - 30\,303\sqrt{61})\right]^3.$$

In this case convergence is much slower: less than one digit per term.

For a further discussion of these, see [3] where many such quadratic examples are considered. One might note that various of the recent record setting calculations of pi have been based on these series. In particular, the Chudnovsky's computed over two billion digits of π using the second series above.

There is an unlimited number of such series with increasingly more rapid convergence. The price one pays is that one must deal with more complicated algebraic irrationalities. Thus a class number $p$ field will involve $p$th-degree algebraic integers as the constants $A$, $B$ and $C$ in the series. The largest class number three example corresponds to $d = -907$ and gives 37 or 38 digits per term. It is

$$\frac{\sqrt{-C^3}}{\pi} = \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!\,(n!)^3} \frac{A + nB}{C^{3n}},$$

where

$$C = (4320)2^{2/3}\,3^{1/3}\left(-4\,711\,544\,446\,661\,617\,873\,062\,970\,863 + 52\,735\,595\,419\,633\,(2721)^{1/2}\right)^{1/3}$$
$$-(4320)\,2^{2/3}\,3^{1/3}\left(4\,711\,544\,446\,661\,617\,873\,062\,970\,863 + 52\,735\,595\,419\,633\,(2721)^{1/2}\right)^{1/3}$$
$$-\,16\,580\,537\,033\,280,$$

$$A = 27\,136\left(2\,581\,002\,591\,670\,714\,650\,084\,289\,323\,501\,202\,067\,163\,298\,721 + 99\,780\,432\,501\,542\,041\,707\,016\,500\,(2721)^{1/2}\right)^{1/3}$$
$$-\,27\,136\left(-2\,581\,002\,591\,670\,714\,650\,084\,289\,323\,501\,202\,067\,163\,298\,721 + 99\,780\,432\,501\,542\,041\,707\,016\,500\,(2721)^{1/2}\right)^{1/3}$$
$$+\,37\,222\,766\,169\,818\,947\,772,$$

$$B = 193\,019\,904\,(907)^{1/3}\left(6\,696\,886\,031\,513\,505\,648\,275\,135\,384\,091\,973\,612 + 22\,970\,050\,316\,722\,125\,(2721)^{1/2}\right)^{1/3} - 193\,019\,904\,(907)^{1/3}$$
$$\times\left(-\,6\,696\,886\,031\,513\,505\,648\,275\,135\,384\,091\,973\,612 + 22\,970\,050\,316\,722\,125\,(2721)^{1/2}\right)^{1/3}$$
$$+\,35\,217\,794\,936\,040\,022\,065\,512.$$

The largest discriminant series we computed was the class number four example for $d = -1555$, with

$$C = -214\,772\,995\,063\,512\,240 - 96\,049\,403\,338\,648\,032 * 5^{1/2}$$
$$-\,1296 * 5^{1/2}(10\,985\,234\,579\,463\,550\,323\,713\,318\,473 + 4\,912\,746\,253\,692\,362\,754\,607\,395\,912 * 5^{1/2})^{1/2},$$

$$A = 63\,365\,028\,312\,971\,999\,585\,426\,220 + 28\,337\,702\,140\,800\,842\,046\,825\,600 * 5^{1/2}$$
$$+\,384 * 5^{1/2}\,(10\,891\,728\,551\,171\,178\,200\,467\,436\,212\,395\,209\,160\,385\,656\,017 + 4\,870\,929\,086\,578\,810\,225\,077\,338\,534\,541\,688\,721\,351\,255\,040 * 5^{1/2})^{1/2},$$

$$B = 7\,849\,910\,453\,496\,627\,210\,289\,749\,000 + 3\,510\,586\,678\,260\,932\,028\,965\,606\,400 * 5^{1/2}$$
$$+\,2\,515\,968 * 3110^{1/2}\,(6\,260\,208\,323\,789\,001\,636\,993\,322\,654\,444\,020\,882\,161 + 2\,799\,650\,273\,060\,444\,296\,577\,206\,890\,718\,825\,190\,235 * 5^{1/2})^{1/2}.$$

The above series with these constants gives 50 additional digits per term.

Our intention in this note is to exhibit all such series that correspond to class number three fields and to using the absolute invariant $J$. (Some examples based on other modular functions and forms are discussed in [2,3]. Since $J$ is rational in these invariants, the algebraic complexity of these series will always be at least as complicated and usually more so.)

Ramanujan's seminal paper on the subject is [11] (see also [8]). For the necessary theory of elliptic modular functions, and singular invariants, see [1,2,4–6,9,13]. Some of the special function theory may be found in [2,7,14]. References [10,12] contain some related results.

## 2. The series

All the series are of the form

$$\sum_{n=0}^{\infty}(a(t)+nb(t))\frac{(6n)!}{(3n)!\,(n!)^3}\frac{1}{(j(t))^n}=\frac{\sqrt{-j(t)}}{\pi}, \tag{*}$$

where

$$b(t)=(t(1728-j(t)))^{1/2}, \qquad a(t)=\tfrac{1}{6}b(t)\left(1-\frac{E_4(t)}{E_6(t)}\left(E_2(t)-\frac{6}{\pi\sqrt{t}}\right)\right),$$

$$j(t)=\frac{1728E_4^3(t)}{E_4^3(t)-E_6^2(t)}.$$

Here

$$E_2(q)=1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \qquad E_4(q)=1+240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n},$$

$$E_6(q)=1-504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n} \quad\text{and}\quad q=-\exp(-\pi\sqrt{t}).$$

So $j$ is the absolute invariant, and $E_2$, $E_4$ and $E_6$ are Eisenstein series, $E_4$ and $E_6$ are entire modular forms of weight four and six, respectively (in $q:=e^{2\pi iz}$), while $E_2$ satisfies

$$\frac{1}{z^2}E_2\left(-\frac{1}{z}\right)=E_2(z)+\frac{12}{2\pi iz}.$$

Table 1

| $t$ | Number of correct digits | $t$ | Number of correct digits |
|---|---|---|---|
| 23 | 3/4 | 307 | 20/21 |
| 31 | 4/5 | 331 | 21/22 |
| 59 | 7/8 | 379 | 23/24 |
| 83 | 9/10 | 499 | 27/28 |
| 107 | 10/11 | 547 | 28/29 |
| 139 | 12/13 | 643 | 31/32 |
| 211 | 16/17 | 883 | 37/38 |
| 283 | 19/20 | 907 | 37/38 |

Table 2

$d = -23$

$PJ = 23375 + 650\,x + 155\,x^2 + x^3$

$PA = -511488 + 32256\,x - 648\,x^2 + x^3$

$PB = -417146653 + 3384381\,x - 9338\,x^2 + x^3$

$CJ = \frac{1}{3}\{5\,(-57377 + 2793\,(69)^{1/2})^{1/3} - 5\,(57377 + 2793\,(69)^{1/2})^{1/3} - 155\,(2)^{1/3}\}/2^{1/3}$

$CA = 4\,(2)^{2/3}\,(3)^{1/3}\,(8919 + 125\,(69)^{1/2})^{1/3} - 4\,(2)^{2/3}\,(3)^{1/3}\,(-8919 + 125\,(69)^{1/2})^{1/3} + 216$

$CB = \frac{1}{3}\{7\,(23)^{1/3}\,(171802157 + 1332375\,(69)^{1/2})^{1/3} - 7\,(23)^{1/3}\,(-171802157 + 1332375\,(69)^{1/2})^{1/3} + 9338\,(2)^{1/3}\}/2^{1/3}$

$d = -31$

$PJ = 116127 + 837\,x + 342\,x^2 + x^3$

$PA = -5184000 - 158400\,x - 1920\,x^2 + x^3$

$PB = -6866276769 - 23367366\,x - 34317\,x^2 + x^3$

$CJ = \{9\,(-4093 + 117\,(93)^{1/2})^{1/3} - 9\,(4093 - 117\,(93)^{1/2})^{1/3} - 114\,(2)^{1/3}\}/2^{1/3}$

$CA = 20\,(2)^{2/3}\,(9857 + 81\,(93)^{1/2})^{1/3} - 20\,(2)^{2/3}\,(-9857 + 81\,(93)^{1/2})^{1/3} + 640$

$CB = \{9\,(31)^{1/3}\,(144597919 + 634491\,(93)^{1/2})^{1/3} - 9\,(31)^{1/3}\,(-144597919 + 634491\,(93)^{1/2})^{1/3} + 11439\,(2)^{1/3}\}/2^{1/3}$

$d = -59$

$PJ = 720896 + 68608\,x + 3136\,x^2 + x^3$

$PA = -129816000 - 3639600\,x - 55380\,x^2 + x^3$

$PB = -213282707968 + 665365184\,x - 1335288\,x^2 + x^3$

$CJ = \frac{32}{3}\,(-911942 + 1209\,(177)^{1/2})^{1/3} - \frac{32}{3}\,(911942 + 1209\,(177)^{1/2})^{1/3} - \frac{3136}{3}$

$CA = 640\,(23869 + 6\,(177)^{1/2})^{1/3} - 640\,(-23869 + 6\,(177)^{1/2})^{1/3} + 18460$

$CB = \{2048\,(59)^{1/3}\,(1563264 + 143\,(177)^{1/2})^{1/3} - 2048\,(59)^{1/3}\,(-1563264 + 143\,(177)^{1/2})^{1/3} + 445096\,(3)^{2/3}\}/3^{2/3}$

$d = -83$

$PJ = 8192000 + 128000\,x + 13920\,x^2 + x^3$

$PA = -6228004032 + 62625456\,x - 522372\,x^2 + x^3$

$PB = -16671685100032 + 16634384576\,x - 14948632\,x^2 + x^3$

$CJ = \{80\,(2)^{2/3}\,(-437715 + 247\,(249)^{1/2})^{1/3} - 80\,(2)^{2/3}\,(437715 + 247\,(249)^{1/2})^{1/3} - 4640\,(3)^{2/3}\}/3^{2/3}$

$CA = 128\,(2514765509 + 144750\,(249)^{1/2})^{1/3} - 128\,(-2514765509 + 144750\,(249)^{1/2})^{1/3} + 174124$

$CB = \frac{2048}{3}\,(83)^{1/2}\,(4683720046 + 92625\,(249)^{1/2})^{1/3} - \frac{2048}{3}\,(83)^{1/3}\,(-4683720046 + 92625\,(249)^{1/2})^{1/3} + \frac{14948632}{3}$

$d = -107$

$PJ = 69632000 - 3532800\,x + 50560\,x^2 + x^3$

$PA = -164029358016 + 795365424\,x - 3626484\,x^2 + x^3$

$PB = -632259939437056 + 276956903104\,x - 117844664\,x^2 + x^3$

$CJ = \frac{80}{3}\,(2)^{2/3}\,(-63501923 + 4557\,(321)^{1/2})^{1/3} - \frac{80}{3}\,(2)^{2/3}\,(63501923 + 4557\,(321)^{1/2})^{1/3} - \frac{50560}{3}$

$CA = 256\,(105258081014 + 1534125\,(321)^{1/2})^{1/3} - 256\,(-105258081014 + 1534125\,(321)^{1/2})^{1/3} + 1208828$

$CB = \frac{14336}{3}\,(2)^{2/3}\,(107)^{1/3}\,(1297666361 + 6375\,(321)^{1/2})^{1/3} - \frac{14336}{3}\,(2)^{2/3}\,(107)^{1/3}\,(-1297666361 + 6375\,(321)^{1/2})^{1/3} + \frac{117844664}{3}$

Table 2 (continued)

$d = -139$

$PJ = 40697856 + 2350080\, x + 230112\, x^2 + x^3$

$PA = -76937688000 + 1974769200\, x - 35134260\, x^2 + x^3$

$PB = -180497249413632 + 508393029312\, x - 1301330232\, x^2 + x^3$

$CJ = 144\, (2)^{2/3}\, (-37776337 + 513\, (417)^{1/2})^{1/3} - 144\, (2)^{2/3}\, (37776337 + 513\, (417)^{1/2})^{1/3} - 76704$

$CA = 3200\, (-49020259609 + 13284\, (417)^{1/2})^{1/3} - 3200\, (-49020259609 + 13284\, (417)^{1/2})^{1/3} + 11711420$

$CB = 18432\, (139)^{1/3}\, (93770454593 + 5814\, (417)^{1/2})^{1/3} - 18432\, (139)^{1/3}\, (-93770454593 + 5814\, (417)^{1/2})^{1/3} + 433776744$

$d = -211$

$PJ = 1744699293 - 138322944\, x + 4038624\, x^2 + x^3$

$PA = -64145792376000 - 696663896400\, x - 2583483780\, x^2 + x^3$

$PB = -237149887414583808 - 280005501996864\, x - 117895404312\, x^2 + x^3$

$CJ = 432\, (2)^{2/3}\, (-7565573297 + 4557\, (633)^{1/2})^{1/3} - 432\, (2)^{2/3}\, (7565573297 + 4557\, (633)^{1/2})^{1/3} - 1346208$

$CA = 640\, (2436204469367795509 + 15881726862\, (633)^{1/2})^{1/3} - 640\, (-2436204469367795509 + 15881726862\, (633)^{1/2})^{1/3} + 861161260$

$CB = 129024\, (211)^{1/3}\, (133916146295164 + 181917\, (633)^{1/2})^{1/3} - 129024\, (211)^{1/3}\, (-133916146295164 + 181917\, (633)^{1/2})^{1/3} + 39298468104$

$d = -283$

$PJ = 5861376000 - 117504000\, x + 44749440\, x^2 + x^3$

$PA = -1391460588462528 + 22392564106032\, x - 95286543636\, x^2 + x^3$

$PB = -3766509569857632768 + 8367427548603072\, x - 5035871775096\, x^2 + x^3$

$CJ = 720\, (2)^{2/3}\, (-2223011372695 + 100719\, (849)^{1/2})^{1/3} - 720\, (2)^{2/3}\, (2223011372695 + 100719\, (849)^{1/2})^{1/3} - 14916480$

$CA = 1280\, (15279215061012656364008 + 806969743425\, (849)^{1/2})^{1/3} - 1280\, (-15279215061012656364008 + 806969743425\, (849)^{1/2})^{1/3}$
$\quad + 31762181212$

$CB = 2304000\, (2)^{2/3}\, (283)^{1/3}\, (341638160183501 + 2907\, (849)^{1/2})^{1/3} - 2304000\, (2)^{2/3}\, (283)^{1/3}\, (-341638160183501 + 2907\, (849)^{1/2})^{1/3}$
$\quad + 1678623925032$

$d = -307$

$PJ = 20791296000 + 1649203200\, x + 93025440\, x^2 + x^3$

$PA = -2171781534677184 + 18423545571504\, x - 285596316996\, x^2 + x^3$

$PB = -8684908457046188544 + 7493376156813504\, x - 15720692138136\, x^2 + x^3$

$CJ = 1440\, (-9985126832108 + 221247\, (921)^{1/2})^{1/3} - 1440\, (9985126832108 + 221247\, (921)^{1/2})^{1/3} - 31008480$

$CA = 128\, (411399854822599371196241923 + 19982442754854000\, (921)^{1/2})^{1/3}$
$\quad - 128\, (-411399854822599371196241923 + 19982442754854000\, (921)^{1/2})^{1/3} + 95198772332$

$CB = 18432\, (307)^{1/3}\, (74850630640566730935101 + 573936129000\, (921)^{1/2})^{1/3}$
$\quad - 18432\, (307)^{1/3}\, (-74850630640566730935101 + 573936129000\, (921)^{1/2})^{1/3} + 5240230712712$

Table 2 (continued)

$d = -331$

$PJ = 382\,987\,173\,888 - 13\,756\,068\,864\ x + 188\,025\,696\ x^2 + x^3$

$PA = -193\,046\,772\,130\,776\,000 - 649\,374\,037\,160\,400\ x - 820\,684\,418\,100\ x^2 + x^3$

$PB = -1\,435\,632\,442\,845\,943\,491\,072 - 421\,424\,462\,984\,728\,896\ x - 46\,907\,259\,191\,352\ x^2 + x^3$

$CJ = 288\,(2)^{2/3}\,(-2\,576\,624\,645\,404\,847 + 59\,732\,127\,(993)^{1/2})^{1/3} - 288\,(2)^{2/3}\,(2\,576\,624\,645\,404\,847 + 59\,732\,127\,(993)^{1/2})^{1/3} - 62\,675\,232$

$CA = 1280\,(9\,761\,912\,085\,273\,963\,152\,194\,414 + 550\,254\,791\,639\,871\,(993)^{1/2})^{1/3}$
$\quad - 1280\,(-9\,761\,912\,085\,273\,963\,152\,194\,414 + 550\,254\,791\,639\,871\,(993)^{1/2})^{1/3} + 273\,561\,472\,700$

$CB = 1\,677\,312\,(2)^{2/3}\,(331)^{1/3}\,(611\,824\,497\,271\,160\,651 + 6\,944\,949\,(993)^{1/2})^{1/3}$
$\quad - 1\,677\,312\,(2)^{2/3}\,(331)^{1/3}\,(-611\,824\,497\,271\,160\,651 + 6\,944\,949\,(993)^{1/2})^{1/3} + 15\,635\,753\,063\,784$

$d = -379$

$PJ = 2\,490\,287\,652\,864 + 26\,122\,411\,008\ x + 714\,262\,176\ x^2 + x^3$

$PA = -3\,129\,133\,923\,190\,296\,000 + 2\,723\,906\,951\,146\,800\ x - 6\,076\,263\,133\,140\ x^2 + x^3$

$PB = -28\,798\,814\,700\,595\,775\,937\,024 + 2\,030\,652\,587\,246\,222\,016\ x - 371\,625\,965\,400\,312\ x^2 + x^3$

$CJ = 432\,(2)^{2/3}\,(-41\,850\,231\,197\,998\,291 + 506\,925\,237\,(1137)^{1/2})^{1/3} - 432\,(2)^{2/3}\,(41\,850\,231\,197\,998\,291 + 506\,925\,237\,(1137)^{1/2})^{1/3} - 238\,087\,392$

$CA = 640\,(31\,696\,114\,862\,878\,504\,387\,860\,728\,537 + 547\,992\,352\,039\,286\,646\,(1137)^{1/2})^{1/3}$
$\quad - 640\,(-31\,696\,114\,862\,878\,504\,387\,860\,728\,537 + 547\,992\,352\,039\,286\,646\,(1137)^{1/2})^{1/3} + 2\,025\,421\,044\,380$

$CB = 129\,024\,(379)^{1/3}\,(2\,335\,090\,257\,174\,672\,939\,642\,164 + 8\,102\,893\,579\,209\,(1137)^{1/2})^{1/3}$
$\quad - 129\,024\,(379)^{1/3}\,(-2\,335\,090\,257\,174\,672\,939\,642\,164 + 8\,102\,893\,579\,209\,(1137)^{1/2})^{1/3} + 123\,875\,321\,800\,104$

$d = -499$

$PJ = 167\,163\,228\,389\,376 + 960\,245\,369\,856\ x + 14\,430\,665\,664\ x^2 + x^3$

$PA = -1\,738\,536\,908\,186\,085\,624\,000 + 606\,592\,414\,710\,759\,600\ x - 551\,797\,427\,717\,700\ x^2 + x^3$

$PB = -24\,064\,382\,283\,671\,706\,510\,352\,896 + 597\,035\,021\,372\,739\,778\,752\ x - 38\,723\,964\,839\,988\,120\ x^2 + x^3$

$CJ = 34\,848\,(-2\,630\,036\,327\,482\,738 + 2\,505\,321\,(1497)^{1/2})^{1/3} - 34\,848\,(2\,630\,036\,327\,482\,738 + 2\,505\,321\,(1497)^{1/2})^{1/3} - 4\,810\,221\,888$

$CA = 640\,(23\,737\,519\,185\,228\,735\,469\,859\,210\,208\,233\,993 + 9\,750\,784\,388\,851\,369\,755\,414\,(1497)^{1/2})^{1/3}$
$\quad - 640\,(-23\,737\,519\,185\,228\,735\,469\,859\,210\,208\,233\,993 + 9\,750\,784\,388\,851\,369\,755\,414\,(1497)^{1/2})^{1/3} + 183\,932\,475\,905\,900$

$CB = 18\,450\,432\,(499)^{1/3}\,(686\,207\,600\,339\,595\,812\,898\,244 + 56\,418\,000\,759\,(1497)^{1/2})^{1/3}$
$\quad - 18\,450\,432\,(499)^{1/3}\,(-686\,207\,600\,339\,595\,812\,898\,244 + 56\,418\,000\,759\,(1497)^{1/2})^{1/3} + 12\,907\,988\,279\,996\,040$

$d = -547$

$PJ = 4\,367\,388\,672\,000 + 566\,564\,198\,400\ x - 43\,320\,375\,360\ x^2 + x^3$

$PA = -37\,641\,143\,160\,190\,708\,416 + 641\,821\,608\,961\,316\,784\ x - 2\,870\,043\,265\,025\,124\ x^2 + x^3$

$PB = -117\,693\,586\,653\,896\,166\,068\,736 + 303\,153\,044\,219\,253\,998\,784\ x - 210\,878\,342\,558\,402\,904\ x^2 + x^3$

$CJ = 720\,(2)^{2/3}\,(-2\,016\,765\,426\,909\,483\,927\,605 + 45\,499\,699\,161\,(1641)^{1/2})^{1/3}$
$\quad - 720\,(2)^{2/3}\,(2\,016\,765\,426\,909\,483\,927\,605 + 45\,499\,699\,161\,(1641)^{1/2})^{1/3} - 14\,440\,125\,120$

$CA = 512\,(2)^{2/3}\,(1\,630\,916\,372\,297\,873\,832\,089\,266\,529\,811\,485\,107 + 1\,804\,432\,179\,625\,101\,129\,375\,(1641)^{1/2})^{1/3}$
$\quad - 512\,(2)^{2/3}\,(-1\,630\,916\,372\,297\,873\,832\,089\,266\,529\,811\,485\,107 + 1\,804\,432\,179\,625\,101\,129\,375\,(1641)^{1/2})^{1/3} + 956\,681\,088\,341\,708$

$CB = 258\,048\,(547)^{1/3}\,(36\,952\,460\,016\,205\,849\,031\,198\,818\,739\,056 + 4\,576\,555\,564\,014\,365\,(1641)^{1/2})^{1/3}$
$\quad - 258\,048\,(547)^{1/3}\,(-36\,952\,460\,016\,205\,849\,031\,198\,818\,739\,056 + 4\,576\,555\,564\,014\,375\,(1641)^{1/2})^{1/3} + 70\,292\,780\,852\,800\,968$

Table 2 (continued)

$d = -643$

$PJ = 675\,369\,750\,528\,000 + 18\,886\,206\,412\,800\ x + 340\,695\,161\,280\ x^2 + x^3$

$PA = -26\,899\,324\,789\,544\,591\,249\,088 + 18\,813\,175\,513\,500\,994\,992\ x - 63\,299\,324\,509\,069\,476\ x^2 + x^3$

$PB = -281\,126\,588\,715\,231\,815\,026\,487\,808 + 16\,936\,292\,323\,622\,083\,845\,312\ x - 5\,042\,599\,015\,352\,394\,456\ x^2 + x^3$

$CJ = 159\,120\ (2)^{2/3}\ (-90\,886\,594\,432\,580\,273 + 1\,099\,701\ (1929)^{1/2})^{1/3}$
$\quad - 159\,120\ (2)^{2/3}\ (90\,886\,594\,432\,580\,273 + 1\,099\,701\ (1929)^{1/2})^{1/3} - 113\,565\,053\,760$

$CA = 2560\ (559\,903\,995\,269\,050\,425\,173\,551\,506\,376\,836\,363\,133 + 664\,221\,982\,539\,568\,499\,996\,850\ (1929)^{1/2})^{1/3}$
$\quad - 2560\ (-559\,903\,995\,269\,050\,425\,173\,551\,506\,376\,836\,363\,133 + 664\,221\,982\,539\,568\,499\,996\,850\ (1929)^{1/2})^{1/3} + 21\,099\,774\,836\,356\,492$

$CB = 59\,904\,000\ (643)^{1/3}\ (34\,357\,463\,895\,158\,903\,488\,208\,076\,439 + 5\,864\,544\,131\,094\ (1929)^{1/2})^{1/3}$
$\quad - 59\,904\,000\ (643)^{1/3}\ (-34\,357\,463\,895\,158\,903\,488\,208\,076\,439 + 5\,864\,544\,131\,094\ (1929)^{1/2})^{1/3} + 1\,680\,866\,338\,450\,798\,152$

$d = -883$

$PJ = 5\,517\,741\,993\,984\,000 + 454\,817\,006\,899\,200\ x + 32\,680\,708\,332\,480\ x^2 + x^3$

$PA = -406\,793\,552\,432\,356\,723\,163\,328 + 3\,225\,578\,464\,047\,253\,550\,256\ x - 59\,468\,545\,813\,780\,904\,388\ x^2 + x^3$

$PB = -2\,574\,302\,250\,130\,549\,079\,820\,231\,168 + 2\,038\,049\,808\,718\,242\,855\,082\,176\ x - 5\,551\,592\,025\,996\,996\,495\,768\ x^2 + x^3$

$CJ = 720\ (2)^{2/3}\ (-865\,870\,888\,072\,767\,908\,118\,672\,822\,131 + 29\,352\,625\,301\,131\,209\ (2649)^{1/2})^{1/3}$
$\quad - 720\ (2)^{2/3}\ (865\,870\,888\,072\,767\,908\,118\,672\,822\,131 + 29\,352\,625\,301\,131\,209\ (2649)^{1/2})^{1/3} - 108\,930\,569\,444\,160$

$CA = 256\ (464\,278\,204\,622\,963\,232\,635\,339\,264\,335\,727\,336\,976\,993\,486\,769\,232 + 61\,584\,897\,224\,359\,967\,096\,742\,677\,701\,875\ (2649)^{1/2})^{1/3}$
$\quad - 256\ (-464\,278\,204\,622\,963\,232\,635\,339\,264\,335\,727\,336\,976\,993\,486\,769\,232 + 61\,584\,897\,224\,359\,967\,096\,742\,677\,701\,875\ (2649)^{1/2})^{1/3}$
$\quad + 19\,822\,848\,604\,593\,634\,796$

$CB = 129\,024\ (2)^{2/3}\ (883)^{1/3}\ (835\,327\,281\,508\,916\,099\,554\,861\,447\,582\,907\,918\,375\,537\,729 + 9\,961\,405\,766\,482\,632\,088\,708\,125\ (2649)^{1/2})^{1/3}$
$\quad - 129\,024\ (2)^{2/3}\ (883)^{1/3}\ (-835\,327\,281\,508\,916\,099\,554\,861\,447\,582\,907\,918\,375\,537\,729 + 9\,961\,405\,766\,482\,632\,088\,708\,125\ (2649)^{1/2})^{1/3}$
$\quad + 1\,850\,530\,675\,332\,332\,165\,256$

$d = -907$

$PJ = 5\,303\,371\,235\,328\,000 - 863\,731\,153\,612\,800\ x + 49\,741\,611\,099\,840\ x^2 + x^3$

$PA = -4\,634\,375\,653\,281\,672\,773\,198\,784 + 44\,456\,371\,015\,450\,646\,847\,024\ x - 111\,668\,298\,509\,456\,843\,316\ x^2 + x^3$

$PB = -20\,880\,964\,330\,119\,359\,918\,068\,345\,344 + 28\,694\,333\,242\,035\,085\,634\,944\,704\ x - 10\,565\,338\,480\,812\,006\,196\,536\ x^2 + x^3$

$CJ = 4320\ (2)^{2/3}\ (3)^{1/3}\ (-4\,711\,544\,446\,661\,617\,873\,062\,970\,863 + 52\,735\,595\,419\,633\ (2721)^{1/2})^{1/3}$
$\quad - 4320\ (2)^{2/3}\ (3)^{1/3}\ (4\,711\,544\,446\,661\,617\,873\,062\,970\,863 + 52\,735\,595\,419\,633\ (2721)^{1/2})^{1/3} - 16\,580\,537\,033\,280$

$CA = 27\,136\ (2\,581\,002\,591\,670\,714\,650\,084\,289\,323\,501\,202\,067\,163\,298\,721 + 99\,780\,432\,501\,542\,041\,707\,016\,500\ (2721)^{1/2})^{1/3}$
$\quad - 27\,136\ (-2\,581\,002\,591\,670\,714\,650\,084\,289\,323\,501\,202\,067\,163\,298\,721 + 99\,780\,432\,501\,542\,041\,707\,016\,500\ (2721)^{1/2})^{1/3}$
$\quad + 37\,222\,766\,169\,818\,947\,772$

$CB = 193\,019\,904\ (907)^{1/3}\ (6\,696\,886\,031\,513\,505\,648\,275\,135\,384\,091\,973\,612 + 22\,970\,050\,316\,722\,125\ (2721)^{1/2})^{1/3})$
$\quad - 193\,019\,904\ (907)^{1/3}\ (-6\,696\,886\,031\,513\,505\,648\,275\,135\,384\,091\,973\,612 + 22\,970\,050\,316\,722\,125\ (2721)^{1/2})^{1/3}$
$\quad + 3\,521\,779\,493\,604\,002\,065\,512$

Identity $(*)$ holds for all $t > 0$. It is of particular interest when $t$ is a positive integer ($\equiv 3$ mod 4), in which case

$$\exp\left\{\tfrac{1}{2}(3+\sqrt{-t})2\pi i\right\} - \exp(-\sqrt{t}\,\pi)$$

and $j(t)$ is the singular $j$-invariant corresponding to the imaginary quadratic fields with discriminant $t$. When the class number is $p$, then $(j(t))^{1/3}$ and also $a(t)$ and $b(t)$ are algebraic integers of degree $p$. There are ten of class number one fields

$$[1, 2, 3, 7, 11, 19, 27, 43, 67, 163].$$

The class number two fields are

$$[5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427].$$

For class number three we have

$$[23, 31, 59, 83, 107, 139, 211, 283, 307, 331, 379, 499, 547, 643, 883, 907],$$

all of which are 3 mod 4. In Table 2,

$$PJ[t], \quad PA[t], \quad PB[t]$$

will be the minimal polynomials for

$$(j(t))^{1/3}, \quad a(t), \quad b(t),$$

respectively, while

$$CJ[t], \quad CA[t], \quad CB[t]$$

will be the appropriate values of

$$(j(t))^{1/3}, \quad a(t), \quad b(t),$$

which in each case is the unique real root of the minimal polynomial. The computations were all carried out in Maple. The minimal polynomials were computed using the expansions above and then a lattice basis reduction algorithm. (The precision required is quite high, though 300 digits suffices in all cases.) It is possible of course to compute $CJ[t] = j(t)^{1/3}$ by other methods, see [1] where implicit forms of these cubic invariants are given. The number of correct digits per term in the associated series is easy to estimate (it behaves like $1728/j(t)$) (see Table 1). In Table 2 the series are all in the form $(t = -d)$

$$\frac{\sqrt{-CJ[t]^3}}{\pi} = \sum_{n=0}^{\infty} \frac{CA[t] + nCB[t]}{(CJ[t])^{3n}} \frac{(6n)!}{(3n)!\,(n!)^3}.$$

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