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On the Irrationality of $\sum (1/(q^n + r))$

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We prove that if q is an integer greater than one and r is a non-zero rational ($r \neq -q^m$) then $\sum_{n=1}^{\infty} (1/(q^n + r))$ is irrational and is not a Liouville number.

The focus of this paper is a proof that

$$\sum_{n=1}^{\infty} \frac{1}{q^n + r}$$

is irrational if q is an integer ($\geqslant 2$) and r is a non-zero rational. This is a known result if $q := 2$ and $r := -1$ since

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1} = \sum_{n=1}^{\infty} \frac{d(n)}{2^n},$$

where $d(n)$ is the divisor function, and this was proven irrational by Erdős in 1948 [7]. This is discussed in [8, p. 62] where Erdős and Graham claim that the irrationality of

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 3}$$

is unresolved. This of course is a special case of our result.

The technique we employ is to examine the Padé approximants to a particular function, and to show that, with some modification, they provide a rational approximation that is too rapid to be consistent with

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rationality. This general approach has been explored by Mahler [11], the Chudnovskys [6], and most recently by Wallisser [12]. Wallisser shows that the irrationality of the products

$$\prod_{n=1}^{\infty}\left(1+\frac{r}{q^n}\right),$$

where q is an integer ($\geqslant 2$) and r is a non-zero rational ($\neq -q^m$) can be deduced from an examination of the Padé approximant to the q-exponential (essentially the above infinity product as a function of r). There is no simple functional equation linking the q-exponentials to the series we consider that allows us to deduce our results from those of Wallisser.

One can also deduce, in similar fashion, the irrationality of the partial theta function

$$\sum_{n=0}^{\infty}\frac{r^n}{q^{n^2}}$$

when q is an integer ($\geqslant 2$) and r is a non-zero rational (see [5]).

Our approach is to analyse the Padé approximants to a q analogue of log. The function we wish to examine in detail is $L_q^*(x)$ defined by

$$L_q^*(x) := \sum_{n=1}^{\infty}\frac{x}{q^n-x}=\sum_{n=1}^{\infty}\frac{x^n}{q^n-1}, \qquad |q|>1. \tag{1}$$

(The first identity requires $x \neq q^m$ while the second requires $|x|<|q|$.) In particular we wish to construct the main diagonal Padé approximants to L_q^*. The (n, n) Pade approximant to L_q^* is the unique rational function $P_n(x)/Q_n(x)$ with numerator and denominator each of degree n which satisfies

$$Q_n(x)\,L_q^*(x)-P_n(x)=O(x^{2n+1}). \tag{2}$$

It is not often the case that one can explicitly construct or completely analyse the Padé approximants. The case we are in is one of the exceptions. The function $L_q^*(x)$, as a function of x, is a Stieltjes series. It also obeys a particularly simple functional relation namely

$$L_q^*(qx)=L_q^*(x)+\frac{x}{1-x}. \tag{3}$$

These two pieces of information allow for an easier analysis of the Padé approximation to L_q^*. See [5, 13] and, for the general Padé theory, [2].

The function L_q^* can be thought of as a q analogue of log in the sense that

$$\lim_{q \downarrow 1} (q-1)\, L_q^*(x) = \lim_{q \downarrow 1} \sum_{n=1}^{\infty} \frac{x^n}{1+q+\cdots+q^{n-1}}$$
$$= \sum_{n=1}^{\infty} \frac{x^n}{n}$$
$$= -\log(1-x). \tag{4}$$

We wish to introduce some pieces of notation that describe q analogues of factorials and binomial coefficients. More on q-functions can be accessed in [1, 5, 9, 10].

The *q-factorial* is

$$[n]_q! := [n]! := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q)}{(1-q)(1-q)\cdots(1-q)}, \tag{5}$$

where $[0]_q! := 1$. Since $(1-q^n)/(1-q) = 1 + \cdots + q^{n-1}$ it is clear that

$$\lim_{q \to 1} [n]_q! = n!. \tag{6}$$

The *q-binomial coefficient* (or Gaussian binomial coefficient) is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[n-k]!\,[k]!} \tag{7}$$

and as above

$$\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

We also have the q-binomial theorem (or Cauchy binomial theorem):

$$\sum_{k=0}^{n} x^k q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} = \prod_{k=1}^{n} (1+xq^k).$$

The facts we need about the Padé approximants are collected in the following results.

THEOREM 1. (a) *Let* $|q| > 1$ *and let*

$$Q_n(x) := q^{n^2} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}^2 \frac{\prod_{k=0}^{i-1} (1-xq^k)}{q^{i(2n-i)}}. \tag{8}$$

Then $Q_n(x)$ is the denominator of the (n, n) Padé approximant to

$$\sum_{n=1}^{\infty} \frac{x}{q^n - x}.$$

Q_n is polynomial of degree n in x and degree n^2 in q with integer coefficients.

(b) *We denote by P_n the (n, n) Padé numerator (normalized so that Q_n is as above). $P_n(x)$ is a polynomial of degree n in x. Furthermore*

$$\left\{ \prod_{k=[n/2]}^{n} (1 + q^k) \right\} P_n(x)$$

is also polynomial in q and it has integer coefficients.

THEOREM 2. *Let*

$$\bar{Q}_n(x) := \frac{Q_n(x)}{q^{n^2}} = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}^2 \frac{\prod_{k=0}^{i-1} (1 - xq^k)}{q^{i(2n-i)}}. \tag{9}$$

Then the three-term recursion that $\bar{Q}_n$ satisfies is given by

$$\bar{Q}_{n+1} = (a_n + b_n x)\, \bar{Q}_n + c_n x^2 \bar{Q}_{n-1},$$

where

$$a_n := \frac{(q^{2n+1} - 1)(q^{n+1} + 1)}{(q^{n+1} - 1)\, q^{2n+1}}$$

$$b_n := \frac{-2(q^{2n+1} - 1)}{(q^n + 1)(q^{n+1} - 1)\, q^{n+1}}$$

and

$$c_n := \frac{-(q^{n+1} + 1)(q^n - 1)}{(q^n + 1)(q^{n+1} - 1)\, q^{2n+1}}.$$

These results are discussed and derived in [5], except for the proof of part (b) of Theorem 1. This part is easily derived from considering $Q_n(x)\, L_q^*(x)$, with L_q^* written in the second form in (1). Note that the first n terms (in x) of $Q_n(x)\, L_q^*(x)$ are just $P_n(x)$.

One can deduce, from Theorem 2, that for $x \in [-1, 1]$,

$$|\bar{Q}_n(x)| \leqslant C_q,$$

where C_q is a constant depending only on q and hence, for $x \in [-1, 1]$,

$$|Q_n(x)| \leqslant q^{n^2} \cdot C_q. \tag{10}$$

THEOREM 3. *If P_n/Q_n is the (n, n) Pade approcimant to L_q^*, $q>1$, then for $x \in [-1, 0) \cup (0, 1]$*

$$0 < \left| L_q^*(x) - \frac{P_n(x)}{Q_n(x)} \right| < \frac{d_q\,|x|^{2n}}{q^{n(n+1)}},$$

where d_q is a constant depending only on q.

Theorem 3 is proved in [4].

We can now establish the main result.

THEOREM 4. *If q is an integer greater than 1 and r is a non-zero rational ($r \neq q^n$ for any $n \geqslant 1$) then*

$$\sum_{n=1}^{\infty} \frac{1}{q^n - r}$$

is irrational.

Proof. It clearly suffices to show that $L_q^*(r)$ is irrational. Now fix N a positive integer. Then

$$\begin{aligned} L_q^*(r/q^N) &= \sum_{n=1}^{\infty} \frac{r/q^N}{q^n - r/q^N} \\ &= \sum_{n=N+1}^{\infty} \frac{r}{q^n - r} \\ &= L_q^*(r) - \sum_{n=1}^{N} \frac{r}{q^n - r}. \end{aligned}$$

Also, by Theorem 3 with $x := r/q^N$ and N chosen so that $|r/q^N| < 1$,

$$\begin{aligned} 0 &< \left| L_q^*(r) - \sum_{n=1}^{N} \frac{r}{q^n - r} - \frac{P_N(r/q^N)}{Q_N(r/q^N)} \right| \\ &= \left| L_q^*(r/q^N) - \frac{P_N(r/q^N)}{Q_N(r/q^N)} \right| \\ &\leqslant d_q \frac{|r|^{2N}}{q^{2N^2} \cdot q^{N(N+1)}}. \end{aligned}$$

Let

$$T_N := \prod_{n=1}^{N} (q^n - r) \prod_{n=[N/2)}^{N} (1 - q^n).$$

Then

$$0 < |T_N| \leqslant e_{r,q} q^{7N(N+1)/8},$$

where $e_{r,q}$ is constant depending only on r and q. Thus,

$$0 < \left| T_N Q_N\left(\frac{r}{q^N}\right) L_q^*(r) - T_N Q_N\left(\frac{r}{q^N}\right) \sum_{n=1}^{N} \frac{r}{q^n - r} - T_N P_N\left(\frac{r}{q^N}\right)\right|$$
$$\leqslant f_{r,q} \frac{|Q_N(r/q^N)|\,|r|^{2N}}{q^{2N^2+N(N+1)/8}}$$

where $f_{r,q}$ depends only on r, q. If we multiply through by q^{N^2} we have with (10),

$$0 < |S_N(r)\, L_q^*(r) - U_N(r)| \leqslant \frac{g_{r,q}\,|r|^{2N}}{q^{N(N+1)/8}},$$

where $g_{r,q}$ depends only on r and q and where

$$S_N(r) := q^{N^2} T_N Q_N\left(\frac{r}{q^N}\right)$$

and

$$U_N(r) := q^{N^2} T_N \left(Q_N\left(\frac{r}{q^N}\right) \sum_{n=1}^{N} \frac{r}{q^n - r} - P_N\left(\frac{r}{q^N}\right)\right)$$

are both polynomials in r and q (of degree at most $2N$ in r) with integer coefficients. Finally, if $r := h/j$ with h and j integers then $S := j^{2N} S_N(r)$ and $U := j^{2N} U_N(r)$ are integers and

$$0 < |SL_q^*(r) - U| \leqslant \frac{g_{r,q}\,|h|^{2N}}{q^{N(N+1)/8}}$$

which tends to zero as N tends to infinity. This shows that $L_q^*(r)$ is irrational. ▮

The estimates of Theorem 4 are in fact sharp enough to prove, under the assumptions of the theorem, that

$$\left| L_q^*(r) - \frac{s}{t} \right| > \frac{1}{t^{\alpha}}$$

for some constant α and all integers s and t. In particular r is not a Liouville number. For t sufficiently large $\alpha = 26/3$ works. This is a standard argument of the type detailed in Section 11.3 of [3].

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