THE MULTIVARIATE INTEGER CHEBYSHEV PROBLEM

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ABSTRACT. The multivariate integer Chebyshev problem is to find polynomials with integer coefficients that minimize the supremum norm over a compact set in $\mathbb{C}^d$. We study this problem on general sets, but devote special attention to product sets such as cube and polydisk. We also establish a multivariate analog of the Hilbert-Fekete upper bound for the integer Chebyshev constant, which depends on the dimension of space. In the case of single variable polynomials in the complex plane, our estimate coincides with the Hilbert-Fekete result.

1. The integer Chebyshev problem and its multivariate counterpart

The supremum norm on a compact set $E \subset \mathbb{C}^d,\ d \in \mathbb{N}$, is defined by

$$\|f\|_E := \sup_{z \in E} |f(z)|.$$

We study polynomials with integer coefficients that minimize the sup norm on a set E, and investigate their asymptotic behavior. The univariate case ($d = 1$) has a long history, but the problem is virtually untouched for $d \geq 2$. Let $\mathcal{P}_n(\mathbb{C})$ and $\mathcal{P}_n(\mathbb{Z})$ be the classes of algebraic polynomials in one variable, of degree at most n, respectively with complex and with integer coefficients. The problem of minimizing the uniform norm on E by *monic* polynomials from $\mathcal{P}_n(\mathbb{C})$ is the classical *Chebyshev problem* (see [5], [22], [26], etc.) For $E = [-1, 1]$, the explicit solution of this problem is given by the monic Chebyshev polynomial of degree n:

$$T_n(x) := 2^{1-n} \cos(n \arccos x), \quad n \in \mathbb{N}.$$

By a linear change of variable, we immediately obtain that

$$t_n(x) := \left(\frac{b-a}{2}\right)^n T_n\left(\frac{2x-a-b}{b-a}\right)$$

is a monic polynomial with real coefficients and the smallest norm on $[a, b] \subset \mathbb{R}$ among all *monic* polynomials of degree n from $\mathcal{P}_n(\mathbb{C})$. In fact,

$$\|t_n\|_{[a,b]} = 2\left(\frac{b-a}{4}\right)^n, \quad n \in \mathbb{N}, \tag{1.1}$$

and the *Chebyshev constant* for $[a, b]$ is given by

$$t_{\mathbb{C}}([a, b]) := \lim_{n\to\infty} \|t_n\|_{[a,b]}^{1/n} = \frac{b-a}{4}. \tag{1.2}$$

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The Chebyshev constant of an arbitrary compact set $E \subset \mathbb{C}$ is defined similarly:

$$t_{\mathbb{C}}(E) := \lim_{n\to\infty} \|t_n\|_E^{1/n}, \tag{1.3}$$

where t_n is the Chebyshev polynomial of degree n on E. It is known that $t_{\mathbb{C}}(E)$ is equal to the transfinite diameter and the logarithmic capacity $\mathrm{cap}(E)$ of the set E (cf. [21] and [26, pp. 71-75] for definitions and background material).

We say that $Q_n \in \mathcal{P}_n(\mathbb{Z})$ is an *integer Chebyshev polynomial* for a compact set $E \subset \mathbb{C}$ if

$$\|Q_n\|_E = \inf_{0\not\equiv P_n\in\mathcal{P}_n(\mathbb{Z})} \|P_n\|_E, \tag{1.4}$$

where the inf is taken over all polynomials from $\mathcal{P}_n(\mathbb{Z})$ that are not identically zero. The *integer Chebyshev constant* (or integer transfinite diameter) for E is given by

$$t_{\mathbb{Z}}(E) := \lim_{n\to\infty} \|Q_n\|_E^{1/n}. \tag{1.5}$$

In general, $0 \le t_{\mathbb{Z}}(E) \le 1$, because $P_n(z) \equiv 1$ is one of the competing polynomials for the inf in (1.4). One may readily observe that if $E = [a, b]$ and $b - a \ge 4$, then $Q_n(x) \equiv 1,\ n \in \mathbb{N}$, by (1.1) and (1.5), so that

$$t_{\mathbb{Z}}([a, b]) = 1, \quad b - a \ge 4. \tag{1.6}$$

We also obtain directly from the definition and (1.2) that

$$\frac{b-a}{4} = t_{\mathbb{C}}([a, b]) \le t_{\mathbb{Z}}([a, b]), \quad b - a < 4. \tag{1.7}$$

The results of Hilbert [15] imply the important upper bound

$$t_{\mathbb{Z}}([a, b]) \le \sqrt{\frac{b-a}{4}}. \tag{1.8}$$

These results were generalized to the case of an arbitrary compact set $E \subset \mathbb{C}$ by Fekete [11], who developed a new analytic setting for the problem, by introducing the transfinite diameter of E and showing that it is equal to $t_{\mathbb{C}}(E)$. Both, the transfinite diameter and the Chebyshev constant, were later proved to be equal to the logarithmic capacity $\mathrm{cap}(E)$, by Szegő [24]. Therefore we state the result of Fekete as follows:

$$t_{\mathbb{Z}}(E) \le \sqrt{t_{\mathbb{C}}(E)} = \sqrt{\mathrm{cap}(E)}, \tag{1.9}$$

where E is $\mathbb{R}$-symmetric. It contains Hilbert's estimate (1.8) as a special case, since $t_{\mathbb{C}}([a, b]) = (b - a)/4$ by (1.2). The following useful observation on the asymptotic sharpness for the estimates (1.8) and (1.9) is due to Trigub [25]. For the sequence of the intervals $I_m := [1/(m + 4), 1/m]$, we have $t_{\mathbb{Z}}(I_m) \ge \frac{1}{m+2}$ and

$$\lim_{m\to\infty} \frac{t_{\mathbb{Z}}(I_m)}{\sqrt{|I_m|/4}} = 1.$$

Furthermore, it was shown in [19] that, for the circle $L_{1/n} = \{z : |nz - 1| = 1/n\},\ n \in \mathbb{N},\ n \ge 2$, we have $t_{\mathbb{Z}}(L_{1/n}) = 1/n$ and $t_{\mathbb{C}}(L_{1/n}) = 1/n^2$. Hence *equality* holds in (1.9) in this case.

The majority of lower bounds for the integer Chebyshev constant are obtained for intervals by using the resultant method, see [17, 6, 12]. They heavily depend on the arithmetic properties of endpoints of the interval. Different methods based on weighted potential theory are employed in [19]. We note that the exact values of

$t_{\mathbb{Z}}$ are not known for any segment of length less than 4. On the other hand, close upper and lower bounds are available for many intervals, with $[0,1]$ being the most thoroughly studied.

Even the classical Chebyshev problem for multivariate polynomials is considerably more complicated than its univariate version. Concerning the multivariate integer Chebyshev problem, very little is known at all. But some special cases of small integer polynomials in many variables were certainly studied before. For example, this problem received attention in light of the Gelfond-Schnirelman method in the distribution of prime numbers (see Gelfond's comments in [9, pp. 285–288], and see [18, 10, 20] for further developments).

By $\mathcal{P}_n^d(\mathbb{C})$ and $\mathcal{P}_n^d(\mathbb{Z})$, we denote the classes of algebraic polynomials in d variables, of degree at most $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, respectively with complex and with integer coefficients. The general form of such polynomials is as follows:

$$P_n(z) = \sum_{|k| \le n} a_k z^k, \quad z \in \mathbb{C}^d,$$

where $k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d,\ z^k = z_1^{k_1} \ldots z_d^{k_d}$, and $|k| = \sum_{i=1}^d k_i$.

Definition 1.1. *A multivariate integer Chebyshev polynomial $C_n \in \mathcal{P}_n^d(\mathbb{Z})$ for a compact set $E \subset \mathbb{C}^d$ is defined by*

$$\|C_n\|_E = \inf_{0 \not\equiv P_n \in \mathcal{P}_n^d(\mathbb{Z})} \|P_n\|_E$$

We write $t_{\mathbb{Z}}(n, E) := \|C_n\|_E$, and define the multivariate integer Chebyshev constant for E as

$$t_{\mathbb{Z}}(E) := \lim_{n \to \infty} \|C_n\|_E^{1/n}.$$

The multivariate integer Chebyshev constant is a monotone and continuous set function, which is consistent with the classical one-dimensional version.

Proposition 1.2. *Let $E \subset \mathbb{C}^d$ and $F \subset \mathbb{C}^d$ be compact sets.*
(i) *If $E \subset F$ then $t_{\mathbb{Z}}(n, E) \le t_{\mathbb{Z}}(n, F),\ n \in \mathbb{N}_0$, and $t_{\mathbb{Z}}(E) \le t_{\mathbb{Z}}(F)$.*
(ii) *Define $E_\delta := \bigcup_{w \in E} \{z : |z - w| \le \delta\}$, where $|z - w|$ is the Euclidean distance in $\mathbb{C}^d$ (as $\mathbb{R}^{2d}$). For any $\varepsilon > 0$ we can find $\delta > 0$ such that*

$$0 \le t_{\mathbb{Z}}(E_\delta) - t_{\mathbb{Z}}(E) \le \varepsilon.$$

Another property similar to the univariate case states that if the set is sufficiently large, then the integer Chebyshev polynomials are given by $C_n(z) \equiv 1,\ n \in \mathbb{N}_0$. We also estimate the multivariate integer Chebyshev constant of E by the integer Chebyshev constants of its coordinate projections.

Proposition 1.3. *Suppose that $E_j \subset \mathbb{C},\ j = 1, \ldots, d$, are compact sets, and define $E := E_1 \times \ldots \times E_d$. We have*

$$t_{\mathbb{Z}}(E) \le \min_{1 \le j \le d} t_{\mathbb{Z}}(E_j).$$

If $t_{\mathbb{C}}(E_j) \ge 1,\ j = 1, \ldots, d$, then $C_n(z) \equiv 1,\ n \in \mathbb{N}_0$, and $t_{\mathbb{Z}}(E) = 1$.

Note that if $E \subset \mathbb{C}^d$ is arbitrary, then we have $E \subset E_1 \times \ldots \times E_d$, where E_j is a projection of E onto the jth coordinate plane. Hence the estimate $t_{\mathbb{Z}}(E) \le \min_{1 \le j \le d} t_{\mathbb{Z}}(E_j)$ is valid in this case too.

We now state a result on vanishing of the multivariate integer Chebyshev polynomials on the product lattice of algebraic integers contained in the set.

Theorem 1.4. *Suppose that $\Lambda_j,\ j=1,\dots,d$, are complete sets of conjugate algebraic integers, and define $\Lambda := \Lambda_1\times\ldots\times\Lambda_d$. If $\Lambda\subset E$ for a compact set $E\subset\mathbb{C}^d$ with $t_{\mathbb{Z}}(E)<1$, then the integer Chebyshev polynomials for E satisfy $C_n(z)=0,\ z\in\Lambda$, for all large $n\in\mathbb{N}$.*

A very interesting problem is how one can find and describe factors of the integer Chebyshev polynomials. Alternatively, one may ask what are the manifolds of zeros in $\mathbb{C}^d$ connecting points of the lattice Λ in Theorem 1.4.

Following the univariate case [13, 17, 6, 12, 19], it is possible to give a lower bound for the multivariate integer Chebyshev constant by using the leading coefficients of minimal polynomials for algebraic numbers in the set.

Theorem 1.5. *Let $E\subset\mathbb{C}^d$ be a compact set. Suppose that $\Lambda_j,\ j=1,\dots,d$, are complete sets of conjugate algebraic numbers such that $\Lambda := \Lambda_1\times\ldots\times\Lambda_d\subset E$. Denote the leading coefficient of the minimal polynomial for Λ_j by $a_j\in\mathbb{Z}$. Assume that for each $j=1,\dots,d$ there are infinitely many sets $\Lambda_j(m_j)$, of cardinality $|\Lambda_j|=m_j\to\infty$, satisfying the above stated conditions, and set $s_j := \limsup_{m_j\to\infty}|a_j(m_j)|^{-1/m_j}$. Then*

$$t_{\mathbb{Z}}(E)\ge\prod_{j=1}^{d}s_j.$$

We consider examples of the problem on polydisks, rectangles and other special sets in the next section.

2. Special sets

2.1. **Polydisks.** For $r=(r_1,\dots,r_d)$, let $D_r := \{(z_1,\dots,z_d)\in\mathbb{C}^d : |z_j|\le r_j,\ r_j>0,\ j=1,\dots,d\}$ be a polydisk in $\mathbb{C}^d$, centered at the origin. This region is probably the simplest in terms of the integer Chebyshev problem, because the extremal polynomials are monomials, and the solution coincides with that of the regular Chebyshev problem for $\mathcal{P}_n^d(\mathbb{C})$.

Proposition 2.1. *Let $r_m := \min\limits_{1\le j\le d} r_j,\ 1\le m\le d$. If $r_m<1$ then an integer Chebyshev polynomial of degree $n\in\mathbb{N}_0$ on D_r is $C_n(z)=z_m^n$, with $t_{\mathbb{Z}}(n,D_r)=r_m^n$ and $t_{\mathbb{Z}}(D_r)=r_m$. If $r_m\ge 1$ then $C_n(z)\equiv 1,\ n\in\mathbb{N}_0$, so that $t_{\mathbb{Z}}(n,D_r)=t_{\mathbb{Z}}(D_r)=1$.*

2.2. **Rectangles.** Let $E=[a_1,b_1]\times\ldots\times[a_d,b_d]\subset\mathbb{R}^d$ be a (real) rectangle with faces parallel to the coordinate planes. Proposition 1.3 and Theorem 1.5 give the upper and the lower bounds for $t_{\mathbb{Z}}(E)$ in terms of one-dimensional bounds for the integer Chebyshev constant of the intervals $[a_j,b_j]$. However, it is of great interest to investigate the problem from a multivariate point of view, and determine the shape of the multivariate integer Chebyshev polynomials. We restrict our discussion to the case $d=2$.

Consider $E=[a,b]\times[c,d]$, where $a,b,c,d\in\mathbb{R}$. Suppose that $\ell:[a,b]\to[c,d]$ is a linear function with integer coefficients, i.e., $y=\ell(x)\in\mathcal{P}_1^1(\mathbb{Z})$. Set $F=\{(x,\ell(x)) : x\in[a,b]\}$. Let C_n^E and C_n^F be the integer Chebyshev polynomials of degree n for E and F. If $C_n^E|_F\not\equiv 0$ then

$$t_{\mathbb{Z}}(n,E)=\|C_n^E\|_E\ge\|C_n^E\|_F\ge\|C_n^F\|_F=t_{\mathbb{Z}}(n,F).$$

But $C_n^F|_F = C_n^F(x, \ell(x)) \in \mathcal{P}_n^1(\mathbb{Z})$ and

$$\|C_n^F\|_F = \|C_n^F \circ \ell\|_{[a,b]} = \|C_n^F \circ \ell\|_E \geq t_{\mathbb{Z}}(n, E).$$

Hence $t_{\mathbb{Z}}(n, E) = t_{\mathbb{Z}}(n, F)$, $C_n^E(x, y) = C_n^F(x, y)$ and $C_n^E(x, \ell(x)) = Q_n(x)$, where Q_n is a univariate integer Chebyshev polynomial for $[a, b]$.

For example, consider the square $E = [0, 1] \times [0, 1]$ and $y = \ell(x) = 1 - x$. Numerical computations suggested the polynomial $C_5(x, y) = xy(y-1)(x-1)(x-y)$ [**?**]. It does not vanish on $F = \{(x, \ell(x)) : x \in [0, 1]\}$, and $C_5(x, 1 - x) = x^2(1 - x)^2(2x - 1) = Q_5(x)$, where Q_5 is an integer Chebyshev polynomial for $[0, 1]$ (cf. [14]). Since $t_{\mathbb{Z}}(E) > t_{\mathbb{Z}}(F)$, we conclude that $C_n|_F \equiv 0$ for large n. In fact, the numerical computation of C_6 through C_9 show that they have the factor $1 - x - y$ [**?**]. As a consequence of Proposition 1.3 and Theorem 1.5, together with [19] and [10, 17], we state the bounds

$$(0.4207)^2 < t_{\mathbb{Z}}([0, 1] \times [0, 1]) \leq t_{\mathbb{Z}}([0, 1]) < 0.4232.$$

We plan a more detailed study of the integer Chebyshev problem for the square $[0, 1] \times [0, 1]$ in a forthcoming paper.

2.3. **Polylemniscates.** We consider polynomial mappings $q = (q_1, \ldots, q_d) : \mathbb{C}^d \to \mathbb{C}^d,\ d \geq 2$, such that $q_j \in \mathcal{P}_l^d(\mathbb{Z})$ with $\deg(q_j) = l,\ j = 1, \ldots, d$. Furthermore, we assume that the homogeneous parts $\hat{q}_j$ of degree l in q_j satisfy

$$\bigcap_{j=1}^{d} \hat{q}_j^{-1}(0) = 0.$$

The latter condition is equivalent to

$$\liminf_{|z| \to \infty} \frac{|q(z)|}{|z|^l} > 0,$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{C}^d$, see Theorem 5.3.1 of [16]. A polynomial mapping q of degree l is called simple if $\hat{q}_j(z) = z_j^l,\ j = 1, \ldots, d$.

Proposition 2.2. *Let $q = (q_1, \ldots, q_d) : \mathbb{C}^d \to \mathbb{C}^d,\ d \geq 2$, be a simple polynomial mapping such that $q_j \in \mathcal{P}_l^d(\mathbb{Z}),\ j = 1, \ldots, d$. For a polydisk $D_r,\ r = (r_1, \ldots, r_d)$, define the (filled-in) polylemniscate $L_r(q) := q^{-1}(D_r)$. If $r_m = \min\limits_{1 \leq j \leq d} r_j < 1$ then an integer Chebyshev polynomial of degree ln on $L_r(q)$ is $C_{ln}(z) = q_m^n$, with $t_{\mathbb{Z}}(ln, L_r(q)) = r_m^n$ and $t_{\mathbb{Z}}(L_r(q)) = r_m^{1/l}$.*

Note that the solutions of the integer and the regular Chebyshev problems coincide in Proposition 2.2. It would be of interest to find examples of sets in $\mathbb{C}^d,\ d > 1$, where the integer Chebyshev constants and polynomials are known explicitly, and they differ from the regular Chebyshev case. When $d = 1$, such examples of lemniscates for univariate polynomials were found in [19, Theorem 1.5].

3. A generalization of the Hilbert-Fekete upper bound

Recall that the space of polynomials $\mathcal{P}_n^d(\mathbb{C})$ of degree at most n in d complex variables has the dimension $h_n := \binom{d+n}{n}$, which corresponds to the count of monomials $z^k = z_1^{k_1} \ldots z_d^{k_d}$ or multi-indices $k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$ with $|k| = \sum_{j=1}^d k_j \leq n$. We arrange all multi-indices in the increasing sequence $\{k_i\}$, by following the standard lexicographic order. This order means that $k_i \prec k_{i+1}$ for the multi-indices k_i and

k_{i+1} if either $|k_i| \le |k_{i+1}|$ or $|k| = |k_{i+1}|$ and the first non-zero entry of $k_i - k_{i+1}$ is negative.

Given a set of points $z_i \in \mathbb{C}^d,\ i = 1, \dots, h_n$, we define the Vandermonde determinant by

$$V(z_1, \dots, z_{h_n}) := \det(z_i^{k_j})_{i,j=1}^{h_n}.$$

When $d = 1$ and $z_i \in \mathbb{C},\ i = 1, \dots, n+1$, it is well known that

$$V(z_1, \dots, z_{n+1}) = \prod_{1 \le i < j \le n+1} (z_i - z_j).$$

However, no simple factorization formula is available for $d \ge 2$.

For a compact set $E \subset \mathbb{C}^d$ and $n \in \mathbb{N}$, define an nth set of Fekete points $\{\zeta_i\}_{i=1}^{h_n} \subset E$ as maximizers for the Vandermonde determinant:

$$|V(\zeta_1, \dots, \zeta_{h_n})| = \max_{\{z_i\}_{i=1}^{h_n} \subset E} |V(z_1, \dots, z_{h_n})|.$$

All Fekete points change with n in general, but we avoid double indices to simplify the notation. Note that the degree of $V(z_1, \dots, z_{h_n})$ as a multivariate polynomial is equal to $l_n := d\binom{d+n}{d+1}$. Let

$$t_{\mathbb{C}}(E) := \lim_{n \to \infty} |V(\zeta_1, \dots, \zeta_{h_n})|^{1/l_n}$$

be the multivariate transfinite diameter of E in $\mathbb{C}^d$. In the univariate case, the sequence under the limit is increasing, which immediately implies existence of the limit. Furthermore, the transfinite diameter of $E \subset \mathbb{C}$ is equal to the Chebyshev constant of E and to the logarithmic capacity of E, as we already mentioned is Section 1 (cf. [21]). The multivariate case is much more delicate, and the existence of the defining limit for $t_{\mathbb{C}}(E)$ in the general setting was established much later, see [27] and [1].

We state the following generalization of the Hilbert-Fekete estimate (1.9).

Theorem 3.1. *For any compact set $E \subset \mathbb{C}^d$ that is invariant under complex conjugation in each coordinate variable, we have*

$$t_{\mathbb{Z}}(E) \le (t_{\mathbb{C}}(E))^{d/(d+1)}. \tag{3.1}$$

Clearly, if $d = 1$ then (3.1) yields $t_{\mathbb{Z}}(E) \le \sqrt{t_{\mathbb{C}}(E)}$, which is the Hilbert-Fekete inequality (1.9). We prove Theorem 3.1 for $E \subset \mathbb{R}^d$ here, to avoid a substantially more technical excursion into pluripotential theory.

4. Proofs

Proof of Proposition 1.2. (i) Suppose that C_n^E and C_n^F are arbitrary integer Chebyshev polynomials of degree n for E and F. It follows from Definition 1.1 that

$$\|C_n^E\|_E \le \|C_n^F\|_E \le \|C_n^F\|_F.$$

Hence part (i) is an immediate consequence of the same definition.

(ii) Consider $\varepsilon > 0$ and choose n such that $\|C_n\|_E^{1/n} \le t_{\mathbb{Z}}(E) + \varepsilon/2$. Since E is compact, there is a closed ball $B_R \subset \mathbb{C}^d$ of sufficiently large radius $R > 0$ that contains E strictly inside. Hence $E \subset H := \{z \in \mathbb{C}^d : |C_n(z)|^{1/n} \le t_{\mathbb{Z}}(E) + \varepsilon/2\} \bigcap B_R$. On the other hand, $H \subset W := \{z \in \mathbb{C}^d : |C_n(z)|^{1/n} \le t_{\mathbb{Z}}(E) + \varepsilon\} \bigcap B_{2R}$. Furthermore, the boundary of W is disjoint from H by the maximum modulus

principle applied to C_n, so that we can set $\delta := \operatorname{dist}(H, \partial W) = \min\limits_{z\in H, w\in\partial W} |z-w| > 0$. Hence $E_\delta \subset W$ and

$$t_{\mathbb{Z}}(E_\delta) \le t_{\mathbb{Z}}(W) \le t_{\mathbb{Z}}(E) + \varepsilon$$

by (i), where the last inequality follows by considering a sequence of polynomials $(C_n)^m,\ m \in \mathbb{N}$, on W. The lower bound in (ii) is an immediate consequence of (i). □

Proof of Proposition 1.3. If we consider a univariate integer Chebyshev polynomial Q_n for a set E_j, then

$$t_{\mathbb{Z}}(n, E) = \|C_n\|_E \le \|Q_n\|_E = \|Q_n\|_{E_j},$$

where C_n is a multivariate integer Chebyshev polynomial for E. After extracting the nth root and passing to the limit, we obtain that $t_{\mathbb{Z}}(E) \le t_{\mathbb{Z}}(E_j),\ j = 1, \dots, n$.

Suppose now that $t_{\mathbb{C}}(E_j) \ge 1,\ j = 1, \dots, d$. Since the Euclidean diameter $\operatorname{diam}(E_j) \ge 2\, t_{\mathbb{C}}(E_j) \ge 2$ (cf. [21, Theorem 5.3.4]), we can find a point $\zeta_j \in E_j$ such that $|\zeta_j| \ge 1,\ j = 1, \dots, d$. Substituting these values $z_j = \zeta_j,\ j = 2, \dots, d$, into a multivariate integer Chebyshev polynomial C_n, we obtain a polynomial in one variable z_1 with a leading coefficient of the form $a \prod_{j=2}^{d} \zeta_j^{n_j}$, where a is a nonzero integer. Note that this coefficient is at least one in absolute value. Recall that for any monic univariate polynomial $P_k(z_1)$ of degree k, one has $\|P_k\|_{E_1} \ge (t_{\mathbb{C}}(E_1))^k \ge 1$, see [21, Theorem 5.5.4]. It follows that $\|C_n\|_E \ge 1$. Since C_n is an arbitrary multivariate integer Chebyshev polynomial, we obtain that $t_{\mathbb{Z}}(n, E) \ge 1$ and $t_{\mathbb{Z}}(E) \ge 1$. Hence we can take $C_n(z) \equiv 1$, so that $t_{\mathbb{Z}}(n, E) = 1$ and $t_{\mathbb{Z}}(E) = 1$.

□

Proof of Theorem 1.4. Suppose that $\Lambda_j = \{\lambda_{j,k}\}_{k=1}^{m_j},\ j = 1, \dots, d$. Since non-real algebraic integers in Λ_j come in complex conjugate pairs, each Λ_j is invariant under complex conjugation, which works as a permutation of $\lambda_{j,k}$. For any integer Chebyshev polynomial C_n on E, consider

$$P(z_1) := \sum_{(\lambda_2,\dots,\lambda_d)\in\Lambda_2\times\dots\times\Lambda_d} |C_n(z_1, \lambda_2, \lambda_3, \dots, \lambda_d)|^{2n},$$

which is a polynomial in z_1 and $\bar{z}_1$ because $|C_n|^2 = C_n\overline{C}_n$. Since C_n has integer coefficients, the coefficient of $P(z_1)$ near each term $z_1^l \bar{z}_1^m$ is a symmetric function of $\lambda_{j,k} \in \Lambda_j$ with integer coefficients, for each $j = 2, \dots, d$. Therefore, these coefficients of $P(z_1)$ are integers. From $t_{\mathbb{Z}}(E) < 1$, we have that $\|C_n\|_E < 1$ for all sufficiently large $n \in \mathbb{N}$. It also follows that $\|P\|_{E_1} < 1$ for all sufficiently large $n \in \mathbb{N}$, where $E_1 := \{z_1 : (z_1, \dots, z_d) \in E\}$. Hence the we have for the product

$$\prod_{k=1}^{m_1} |P(\lambda_{1,k})| < 1.$$

But this product is an integer as a symmetric function in $\lambda_{1,k}$, and therefore must vanish. This means each $\lambda_{1,k}$ is a root of P for $k = 1, \dots, m_1$, so that all terms in the sum defining P must vanish on the lattice Λ.

Note that if the cardinality of Λ_k can be arbitrarily large for a certain k, then C_n must vanish for all values of the variable z_k when other variables z_j are assigned values from Λ_j. Indeed, in this case the univariate polynomial $C_n(\lambda_1, \dots, z_k, \dots, \lambda_d)$ in z_k vanishes on Λ_k, where sets Λ_k have an accumulation point as $|\Lambda_k| \to \infty$.

□

Proof of Theorem 1.5. For simplicity, we first assume that $E \subset \mathbb{C}^2$ and $C_n(z_1, z_2)$ is an integer Chebyshev polynomial for E. Consider the univariate polynomial

$$P(z_1) := a_2^n \prod_{\lambda_2 \in \Lambda_2} C_n(z_1, \lambda_2).$$

The coefficients of $P(z_1)$ are symmetric functions of algebraic numbers $\lambda_2 \in \Lambda_2$, and are integers because of the factor a_2^n. Furthermore, the number

$$N := a_1^{m_2 n} \prod_{\lambda_1 \in \Lambda_1} P(\lambda_1) = a_1^{m_2 n} a_2^{m_1 n} \prod_{\lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2} C_n(\lambda_1, \lambda_2)$$

is an integer by applying the fundamental theorem on symmetric forms in a similar way. This integer cannot be zero for $m_1 > m_2 n$, since P has fixed degree at most $m_2 n$. Thus $|N| \geq 1$, and

$$|a_1|^{m_2 n} |a_2|^{m_1 n} \|C_n\|_E^{m_1 m_2} \geq |N| \geq 1.$$

Thus the result follows after taking the power $1/(m_1 m_2 n)$ and passing to lim sup's.

In the general case $d \geq 2$, one observes that

$$N := \prod_{j=1}^{d} a_j^{n \prod_{k \neq j} m_k} \prod_{(\lambda_1, \ldots, \lambda_d) \in \Lambda_1 \times \ldots \times \Lambda_d} C_n(\lambda_1, \ldots, \lambda_d)$$

is a nonzero integer, so that

$$\prod_{j=1}^{d} |a_j|^{n \prod_{k \neq j} m_k} \|C_n\|_E^{\prod_{k=1}^{d} m_k} \geq |N| \geq 1.$$

□

Proof of Proposition 2.1. Let z^k, $|k| \leq n$, be the leading monomial of an integer polynomial $P_n \in \mathcal{P}_n^d(\mathbb{Z})$, with the corresponding leading coefficient $a_k \in \mathbb{Z}$. It follows from Proposition 4 of [2, p. 428] that

$$\|P_n\|_{D_r} \geq |a_k| \|z^k\|_{D_r} \geq r_m^{|k|}.$$

If $r_m < 1$ then the smallest possible value for the norm is clearly attained by the monomial $C_n(z) = z_m^n$, so that $t_{\mathbb{Z}}(n, D_r) = r_m^n$ and $t_{\mathbb{Z}}(D_r) = r_m$. If $r_m \geq 1$ then $C_n(z) \equiv 1$, $n \in \mathbb{N}_0$, because any other polynomial with integer coefficients has the norm at least equal to 1 by the above estimate. Hence $t_{\mathbb{Z}}(n, D_r) = t_{\mathbb{Z}}(D_r) = 1$.

□

Proof of Proposition 2.2. Note that $C_n(z) = z_m^n$ is both a Chebyshev and an integer Chebyshev polynomial for the polydisk D_r by Proposition 2.1. Applying Theorem 4 of [2], we conclude that $C_n \circ q = q_m^n$ is both a Chebyshev and an integer Chebyshev polynomial for the polylemniscate $L_r(q)$.

□

Proof of Theorem 3.1. We give a proof for $E \subset \mathbb{R}^d$ here. A proof of the general case involves more substantial machinery of pluripotential theory, and will be published separately.

Suppose first that E is not pluripolar in $\mathbb{C}^d$. Then the Vandermonde determinant of the Fekete points for E does not vanish, i.e., $V(\zeta_1, \ldots, \zeta_{h_n}) \neq 0$ for any $n \in \mathbb{N}$

(see [16, 3]). We define the fundamental Lagrange interpolation polynomials in Fekete points by

$$l_j^{(n)}(z) := \frac{V(\zeta_1, \ldots, z, \ldots, \zeta_{h_n})}{V(\zeta_1, \ldots, \zeta_j, \ldots, \zeta_{h_n})}, \quad j = 1, \ldots, h_n,$$

where the variable z replaces ζ_j in the numerator. It is clear from this construction that $l_j^{(n)}(\zeta_j) = 1$ and $l_j^{(n)}(\zeta_i) = 0$ for $i \neq j$. Next we express a polynomial $P_n \in \mathcal{P}_n^d(\mathbb{Z})$ as

$$P_n(z) = \sum_{j=1}^{h_n} P_n(\zeta_j) l_j^{(n)}(z),$$

by the Lagrange interpolation formula. Since

$$|V(\zeta_1, \ldots, z, \ldots, \zeta_{h_n})| \leq |V(\zeta_1, \ldots, \zeta_j, \ldots, \zeta_{h_n})|, \quad z \in E,$$

by the defining property of Fekete points, we obtain that

$$\|l_j^{(n)}\|_E \leq 1, \quad j = 1, \ldots, h_n.$$

It follows at once that

$$\|P_n\|_E \leq \sum_{j=1}^{h_n} |P_n(\zeta_j)| \leq h_n \max_{1 \leq j \leq h_n} |P_n(\zeta_j)|.$$

Observe that

$$f_j := P_n(\zeta_j) = \sum_{|k| \leq n} a_k \zeta_j^k, \quad j = 1, \ldots, h_n,$$

are linear forms in a_k's, with real coefficients. Applying Minkowski's theorem (see [8, p. 73]), we conclude that there exists a set of integers $\{a_k\}_{|k| \leq n}$, not all zero, such that

$$|f_j| \leq |\det(\zeta_i^{k_j})_{i,j=1}^{h_n}|^{1/h_n} = |V(\zeta_1, \ldots, \zeta_{h_n})|^{1/h_n}.$$

Thus we can find a sequence of polynomials $P_n(z) = \sum_{|k| \leq n}^{n} a_k z^k \not\equiv 0$ with integer coefficients, satisfying

$$\|P_n\|_E \leq h_n \, |V(\zeta_1, \ldots, \zeta_{h_n})|^{1/h_n}, \quad n \in \mathbb{N}.$$

Note that $\lim\limits_{n \to \infty} h_n^{1/n} = 1$ and that

$$\frac{l_n}{n \, h_n} = \frac{d}{n} \binom{d+n}{d+1} \binom{d+n}{n}^{-1} = \frac{d}{d+1}.$$

Hence we have that

$$\|P_n\|_E^{1/n} \leq h_n^{1/n} \left(|V(\zeta_1, \ldots, \zeta_{h_n})|^{1/l_n} \right)^{d/(d+1)},$$

and (3.1) follows by passing to the limit as $n \to \infty$.

If E is pluripolar in $\mathbb{C}^d$, then we consider the compact sets $E_m := E \bigcup \{z = (z_1, \ldots, z_d) : |z_j| \leq 1/m, \ j = 1, \ldots, d\}$. Clearly, each $E_m, \ m \in \mathbb{N}$, is not pluripolar [16], and $\lim_{m \to \infty} t_{\mathbb{C}}(E_m) = t_{\mathbb{C}}(E) = 0$, see [1, p. 287] and [16]. Hence the first part of the proof and Proposition 1.2 (i) give that

$$t_{\mathbb{Z}}(E) \leq t_{\mathbb{Z}}(E_m) \leq \left(t_{\mathbb{C}}(E_m)\right)^{d/(d+1)} \to 0, \quad \text{as } m \to \infty.$$

It follows that $t_{\mathbb{Z}}(E) = 0$, and (3.1) is trivially satisfied. Note also that E is pluripolar if and only if $t_{\mathbb{C}}(E) = 0$, cf. [1, p. 287]. Thus $t_{\mathbb{Z}}(E) = t_{\mathbb{C}}(E) = 0$ in this case.

□

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