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SCIENTIFIC NOTES

A VERY RAPIDLY CONVERGENT PRODUCT EXPANSION FOR π

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There has recently been considerable interest in an essentially quadratic method for computing π. The algorithm, first suggested by Salamin [6], is based upon an identity known to Gauss [3, p. 377], [4]. This iteration has been used by two Japanese researchers, Y. Tamura and Y. Kanada, to compute 2^{23} decimal digits of π in under 7 hours. They have now successfully computed 2^{24} digits (more than 16.7 million places). This is reported in the January 1983 Scientific American and the February 1983 Discover Magazine. In the process of surveying this and related fast methods of computing elementary functions ([2], [5]), the authors discovered a new quadratically convergent product expansion for π [1]. Our algorithm, like Salamin's, is intimately related to the Gaussian arithmetic-geometric mean iteration. However, it requires considerably less elliptic function theory to establish. The iteration is initialized by

$$\pi_0 := 2+\sqrt{2}, \qquad x_0 := \sqrt{2} \quad \text{and} \quad y_1 := 2^{\frac{1}{4}}.$$

Let

$$x_{n+1} := \tfrac{1}{2}(\sqrt{x_n}+1/\sqrt{x_n}) \qquad (n \geqq 0), \tag{1}$$

$$y_{n+1} := (y_n\sqrt{x_n}+1/\sqrt{x_n})/(y_n+1) \qquad (n \geqq 1), \quad \text{and} \tag{2}$$

$$\pi_n := [(x_n+1)/(y_n+1)]\pi_{n-1} \qquad (n \geqq 1). \tag{3}$$

Then π_n decreases monotonically to π and

$$|\pi_n - \pi| \leqq 10^{-2^{n+1}}$$

for $n \geqq 4$.

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We observe that x_n in (1) is actually Legendre's form of the AGM iteration for the ratio of the arithmetic to the geometric mean [4], while (2) represents a shifted "derivative" mean.

Thus we see that the number of correct digits essentially doubles each time. (Actually it does slightly better.) The algorithm can be derived from that given in [1, § 4] by the appropriate change of variables and manipulations. To 20 significant digits we have

$$\pi_1 = 3.1426067539416226007$$

$$\pi_2 = 3.1415926609660442304$$

and

$$\pi_3 = 3.1415926535897932386.$$

Since both x_n and y_n converge quadratically to 1 the algorithm is very stable and is easily scaled to work in fixed point arithmetic. Using an integer arithmetic language and carrying 1005 digits we verified that π_8 is correct to 694 digits, while π_9 yielded 1004 digits of π. Twenty-three iterations should produce over eight million digits.

Iteration	Digits correct	Iteration	Digits correct
1	3	5	83
2	8	6	170
3	19	7	345
4	40	8	694

The algorithm appears to be competitive, at least in principle, with Salamin's.

It is not too difficult to establish error bounds for (3) once one knows that the limit is π. Indeed one can show that

$$y_{n+1}-1 \leqq \pi_n - \pi \leqq 2(y_{n+1}-1) \tag{4}$$

while

$$\lim_{n\to\infty} \frac{\pi_n - \pi}{y_{n+1}} = \frac{\pi}{2}. \tag{5}$$

From (4) we have a most satisfactory stopping criterion. If we wish 2^m decimal digits of π we calculate all quantities to that precision and terminate as soon as $y_{n+1}-1$ is zero in 2^m+1 places. Unfortunately, in contrast to Newton's method, we must work to full precision throughout most of the calculation.

One can also show that actually

$$\pi_n - \pi \leqq 2\pi_n 10^{-2^{n+3}/3} \tag{6}$$

for $n \geqq 7$; and also that for n sufficiently large 4/3 can be replaced by any constant less than $\pi \log_{10} e \simeq 1.364$. This shows that our algorithm has much the same asymptotic behavior as Salamin's does [6]. It also follows from (4), and the fact that $y_{n+1} - 1 \leqq \frac{3}{8}(y_n - 1)^2$, that

$$\pi_{n+1} - \pi \leqq \tfrac{3}{4}(\pi_n - \pi)^2. \tag{7}$$

and the algorithm is truly quadratic.

Finally, we observe that we have the following explicit product formula for π:

$$\pi = (2+\sqrt{2}) \prod_{n=1}^{\infty} (1+x_n)/(1+y_n), \tag{8}$$

with x_n and y_n generated by (1) and (2).

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