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The Desmic Conjecture

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The twelve point Desmic configuration in Euclidean three space is composed of three finite sets with the property that any line intersecting points of two of the sets also intersects the remaining set. The Desmic conjecture asserts that this is the only such configuration. In this paper the Desmic conjecture is proven.

1. INTRODUCTION

The Desmic configuration is a three-dimensional configuration consisting of three sets each containing four points with the property that any line that intersects two of these sets also intersects the third set. (See Fig. 1.) Edelstein and Kelly in 1963 asked whether any other such confugurations exist. This problem soldified into the Desmic conjecture.

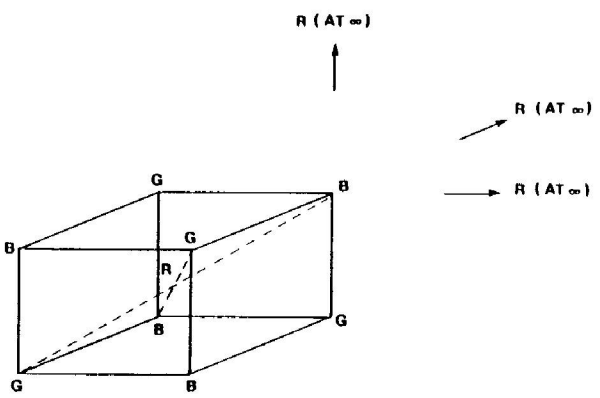

FIGURE 1

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The Desmic Conjecture. Let $\mathscr{R}$(red), $\mathscr{B}$(blue), and $\mathscr{G}$(green) be three finite disjoint sets whose union spans E^3 (Euclidean three space). If every line through any two of the sets intersects the remaining set, then the configuration is (projectively equivalent to) the Desmic configuration.

Edelstein and Kelly pose this problem in [1] and prove a number of related results. They show that many configurations of the above type exist in the plane and that none exist in four or more dimensions. We shall call any configuration satisfying the conditions of the conjecture "Desmic-like." Nwankpa, in his 1970 doctoral thesis under the supervision of Kelly, showed exhaustively that the only Desmic-like configuration with fewer than twenty-seven points is, in fact, the Desmic configuration. The remainder of this paper is concerned with deriving a series of propositions that combine to establish the Desmic conjecture.

2. Mixed Lines

We shall refer to any line (or plane) intersection two or more of the sets under consideration as "mixed". A line (or plane) intersecting only one of the sets will be termed "monochrome". A line through exactly two points is called a "normal line". The triangle defined by three points, P_1, P_2, and P_3, will be denoted by $\Delta(P_1, P_2, P_3)$ and the segment defined by P_1 and P_2 will be denoted by $S(P_1, P_2)$. The conjecture is valid in real projective three space though some of the arguments are more conveniently phrased in Euclidean space.

The first proposition concerns mixed lines and is due to Edelstein. It is reproduced in Nwankpa's thesis and represents the only substantial progress towards the conjecture. Since it is vital to what follows and has not appeared in any conventional form elsewhere, we present it now.

Proposition 1 (Edelstein). *Every mixed line in a Desmic-like configuration contains exactly three points.*

Proof. Suppose there exists a mixed line l that intersects one of the sets (say $\mathscr{R}$) in at least two points P_1 and P_2. We show that this is impossible. Project $\mathscr{B} \cup \mathscr{G}$ from P_1 into a plane π that is not parallel to any line of the configuration. If S is the intersection of l with π, then all lines through S and any point of the projection contain at least one additional point. Motzkin [2] show that this implies that every line bounding the "residence" of S must be normal. (The lines of the configuration that do not pass through S divide the plane into components. The component that contains S is its residence.) Let l' be such a bounding line and let Q_1 and Q_2 be two of the points that define it. We note that there must be a point P_3 in $\mathscr{R}$ so that its image Q_3 under the

same projection lies on l' and is distinct from Q_1 and Q_2. Since every mixed line contains at least three points it follows that the set of all connecting lines of the projection of $\mathscr{B} \cup \mathscr{G}$ is identical to the set of all connecting lines of the projection of $\mathscr{B} \cup \mathscr{G} \cup \{P_3\}$. Furthermore, the line joining S and Q_3 contains an additional point of the projection. Thus, l' is a nonnormal bounding line of S in the projection of $\mathscr{B} \cup \mathscr{G} \cup \{P_3\}$ which contradicts the aforementioned result of Motzkin. ∎

An immediate consequence of the previous proposition is that $\text{card}(\mathscr{R}) = \text{card}(\mathscr{B}) = \text{card}(\mathscr{G})$ in any Desmic-like configuration. Also, if π is any mixed plane, then $\text{card}(\mathscr{R} \cap \pi) = \text{card}(\mathscr{B} \cap \pi) = \text{card}(\mathscr{G} \cap \pi)$.

3. Monochrome Lines

The next step is to examine lines defined by points of one set.

Proposition 2. *Every monochrome line in a Desmic-like configuration contains exactly two points.*

Suppose there exists a Desmic-like configuration with three collinear points of the same colour (say red). If we project this configuration from one of these points to a plane we get a configuration of $\mathscr{P}$ and $\mathscr{L}$ points (corresponding to the projection of the red points and the other points, respectively) that satisfies the following three conditions:

(a) Every line through two $\mathscr{L}$ points contains a $\mathscr{P}$ point.

(b) Every line through a $\mathscr{P}$ point and a $\mathscr{L}$ point contains an additional $\mathscr{L}$ point.

(c) There is a special $\mathscr{P}$ point P (corresponding to the projection of the three collinear red points) with the property that every line through P and a $\mathscr{L}$ point contains at least two additional $\mathscr{L}$ points.

These conditions follow from the observations that in a Desmic-like configuration every mixed line contains exactly one point of each colour and that every mixed plane contains exactly the same number of points of each colour. Thus, a plane through the point of projection that contains exactly $k \geqslant 2$ points of each colour projects to a line that contains exactly k $\mathscr{L}$ points and at least one $\mathscr{P}$ point.

In Lemma 1 we prove, in dual formulation, that such a configuration cannot exist. This, of course, establishes Proposition 2.

Lemma 1. *There does not exist (a noncurrent) finite configuration of $\mathscr{X}$ and $\mathscr{Y}$ lines in the plane that satisfies the following three conditions:*

(a) *Every vertex that contains two $\mathscr{Y}$ lines contains an $\mathscr{X}$ line.*

(b) *Every vertex that contains an $\mathscr{X}$ line and a $\mathscr{Y}$ line contains an additional $\mathscr{Y}$ line.*

(c) *There is a special $\mathscr{X}$ line l_0 with the property that every vertex on l_0 that contains a $\mathscr{Y}$ line contains at least two additional $\mathscr{Y}$ lines.*

Proof. We represent $\mathscr{X}$ lines by solid lines and $\mathscr{Y}$ lines by broken lines. Consider the following minimum configuration: Let $\Delta(P_1, P_2, P_3)$ be a triangle with edges composed of $\mathscr{X}$ lines and with base $S(P_1, P_2)$ on l_0 with the additional requirements:

(1) there is a mixed vertex V between P_1 and P_2;

(2) $\Delta(P_1, P_2, P_3)$ has minimum altitude (from $S(P_1, P_2)$);

(3) if more than one such triangle exists, then $\Delta(P_1, P_2, P_3)$ is also assumed to have minimum area.

It can be seen that l_0 must be cut by at least two $\mathscr{X}$ lines and at least one $\mathscr{Y}$ line. Thus, by performing an initial collineation, if necessary, we can ensure that such a configuration exists.

Since vertex V lies on l_0 at least three $\mathscr{Y}$ lines pass through V and at least two of the, say l_1 and l_2, cut one side of $\Delta(P_1, P_2, P_3)$. (See Fig. 2.)

Let l_3 be the $\mathscr{X}$ line that cuts l_1 at a point A in such a way that no other $\mathscr{X}$ line cuts l_1 between A and V and let l_4 be the $\mathscr{X}$ line that cuts l_2 at a point B so that no other $\mathscr{X}$ line cuts l_2 between B and V. (A or A and B may possibly lie on $S(P_1, P_3)$.) Note that, by the minimality assumptions, neither l_3 nor l_4 cuts $S(P_3, P_2)$ or $S(V, P_2)$. Now, include $\mathscr{X}$ lines so that the convex region $(V, B, Q_2, \ldots, Q_n)$ has no $\mathscr{X}$ lines passing through its interior. (See Fig. 3.)

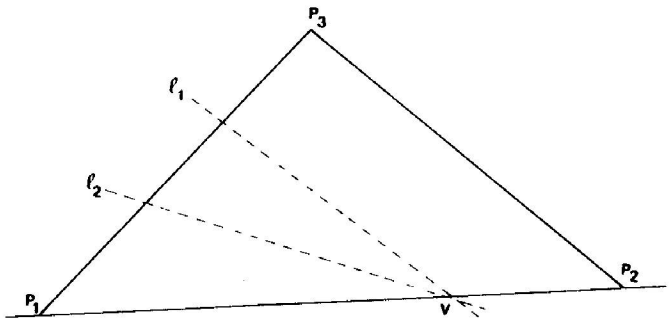

FIGURE 2

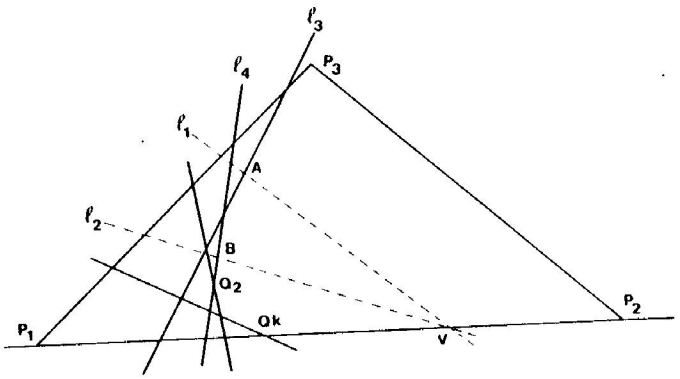

FIGURE 3

The vertex at B contains an additional $\mathscr{Y}$ line l_5. This line cannot cut $S(A, V)$ since this would generate a prohibited $\mathscr{K}$ line. Nor can it cut the interior of the segment $S(P_1, Q_k)$ without violating the minimality assumptions. Thus, it must cut $S(Q_k, V)$. Call this point W. (See Fig. 4.) Note that $W \neq V$. There are two additional $\mathscr{Y}$ lines through W that cannot intersect $S(B, V)$. No two $\mathscr{Y}$ lines meet in $(V, B, Q_2, \ldots, Q_k)$ and we find it is now impossible to place the requisite additional $\mathscr{Y}$ line through C (the point where one of these lines intersects the boundary of $(V, B, Q_2, \ldots, Q_k)$) without contradicting the minimality assumptions. (The case where $W = Q_k$ needs a small additional argument of the same variety.) ∎

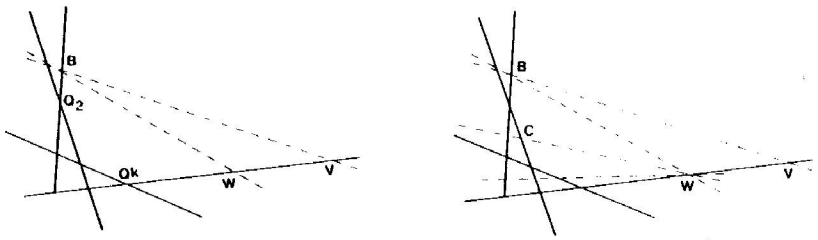

FIGURE 4

4. Mixed Planes

If we project a Desmic-like configuration from any point of the configuration to a plane, in the same fashion as in the comments following the statement of Proposition 2, we find that we have a configuration of $\mathscr{P}$ and $\mathscr{Q}$ points which satisfy:

(a) Every line through a $\mathscr{Q}$ point and a $\mathscr{P}$ point contains an additional $\mathscr{Q}$ point.

(b) Every line through exactly k $\mathscr{Q}$ points contains exactly $k-1$ $\mathscr{P}$ points.

Condition (b) is a consequence of Propositions 1 and 2.

We now show, one again in dual formulation, that if such a planar configuration exists, then every mixed line contains exactly three points.

Lemma 2. *There does not exist a (nonconcurrent) finite configuration of $\mathscr{X}$ and $\mathscr{Y}$ lines in the plane that satisfies these three conditions*:

(a) *Every vertex formed by the intersection of an $\mathscr{X}$ line and a $\mathscr{Y}$ line contains an additional $\mathscr{Y}$ line.*

(b) *Every vertex that contains exactly k $\mathscr{Y}$ lines contains exactly $k-1$ $\mathscr{X}$ lines.*

(c) *At least one vertex V contains three or more $\mathscr{Y}$ lines.*

Proof. Once again $\mathscr{X}$ lines are represented by solid lines and $\mathscr{Y}$ lines by broken lines. We make the following observation: any triangle with edges in $\mathscr{Y}$ has an even number of lines of the configuration cutting into it at the vertices. (By cutting we mean passing through the interior.) This follows since any line cutting a triangle cuts it twice and at any vertex on the interior of an edge of the triangle there are an even number of lines (excluding the edge itself) cutting the triangle. Thus, to preserve parity, an even number must cut at vertices.

We project the vertex V to infinity in such a way that no line of the configuration goes to the line at infinity and so that the lines through V are vertical. Let $l_1,\ldots,l_k$ be the $k \geqslant 3$ $\mathscr{Y}$ lines passing through V ordered as in Fig. 5. Let l^* be a $\mathscr{Y}$ line chosen so that no $\mathscr{Y}$ line intersects l_1 above the vertex formed by l_1 and l^*. Let $A_i = l_i \cap l^*$ for $i = 1, 2,\ldots, k$. The vertex A_2 must be cut by an $\mathscr{X}$ line and since $S(V, A_1)$ is uncut it must intersect $S(V, A_3)$. (See Fig. 5.) There is an odd number of lines through every mixed vertex by condition (b). There is an even number of lines cutting triangle $\Delta(A_2, A_3, V)$ at its vertices. No line cuts into $\Delta(A_2, A_3, V)$ at A_3 since such a line would cut l_1 above A_1. Thus, there exists a line through V that cuts $\Delta(A_2, A_3, V)$. Since l_2 and l_3 are adjacent $\mathscr{Y}$ lines through V this cutting line

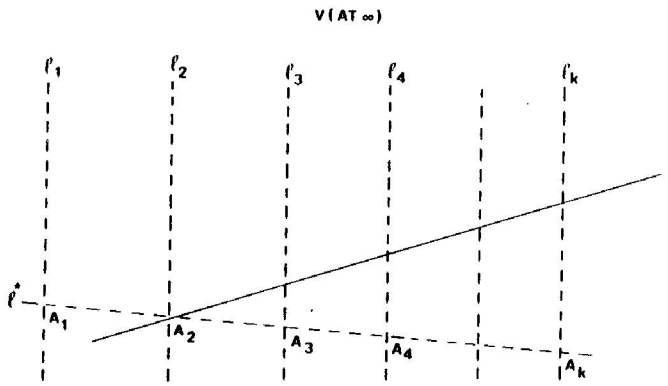

FIGURE 5

must be in $\mathscr{Z}$. Repeating the argument verbatim for the triangles $\Delta(A_i, A_{i+1}, V)$, $i = 3,\ldots, k-1$, we deduce that there is an $\mathscr{Z}$ line cutting each of these triangles at V. If we reverse the procedure (that is, consider the line that cuts l_k at the "highest" point) we see that there must also be an $\mathscr{Z}$ line cutting $\Delta(A_1, A_2, V)$ at V. This shown that each $\Delta(A_i A_{i+1}, V)$ has precisely one $\mathscr{Z}$ line cutting it at V. Now, consider a $\mathscr{Y}$ line l^{**} that cuts l_2 in such a way that no line cuts l_2 above the vertex C formed by the intersection of l^{**} and l_2. Let B and D denote the intersections of l_1 and l_3 with l^{**} (see Fig. 6). The triangle $\Delta(B, D, V)$ is uncut at vertices B and D and has

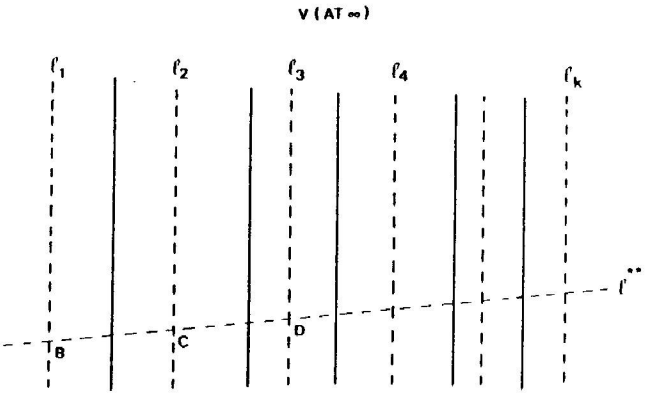

FIGURE 6

exactly three lines cutting it at V. This, however, is contrary to the previous observation that the number of vertex cuts must be even. ∎

Propositions 1 and 2 and Lemma 2 combine to yield

PROPOSITION 3. *Every mixed plane in a Desmic-like configuration contains exactly six points.*

5. COMPLETION OF PROOF

We now know enough about the structure of Desmic-like configurations to finish the proof of the conjecture. The next proposition, in conjunction with the result of Nwankpa mentioned in the introduction, shows that every Desmic-like configuration consists of exactly twelve points.

PROPOSITION 4. *If every mixed plane in a Desmic-like configuration contains exactly six points, then the total number of points in the configuration is less than twenty seven.*

Proof. If we project from an $\mathscr{R}$ point R (as in the comments following the statement of Proposition 2), we arrive at a planar configuration of $\mathscr{P}$ and $\mathscr{Q}$ points.

Any three $\mathscr{Q}$ points come from the projection of three $\mathscr{E}$ and three $\mathscr{B}$ points. These six points in conjunction with R complete to a twelve point Desmic configuration. (One can deduce from the previous results that the remaining five points are uniquely determined.) There are only two possible projections of a twelve point Desmic configuration. (See Fig. 7.)

Thus we observe that if we completely triangulate the $\mathscr{Q}$ points, then every triangle in the triangulation has exactly one $\mathscr{P}$ point on its edges.

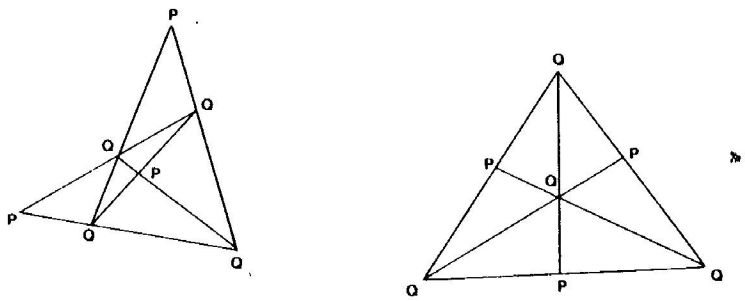

FIGURE 7

If we assume the existence of at least twenty-seven points in total, we guarantee the existence of a least nine $\mathscr{S}$ points in general position. This in turn ensures the existence of a convex pentagon with vertices in $\mathscr{S}$ and with, at most, one $\mathscr{S}$ point in its interior. An exhaustive argument shows that it is not possible that every triangulation of this pentagon has $\mathscr{S}$ points in the required locations. ∎

Acknowledgments

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