GENERAL FORMS FOR MINIMAL SPECTRAL VALUES FOR A CLASS OF QUADRATIC PISOT NUMBERS.

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ABSTRACT. We study the spectrum that results when all height one polynomials are evaluated at a Pisot number. This continues the research theme initiated by Erdős, Joó and Komornik in 1990. We are particularly interested in the minimal non-zero value of this spectrum. Formally we denote this value as $l^1(q)$, and extend this definition to all height m polynomials as

$$l^m(q) := \inf(|y| : y = \epsilon_0 + \epsilon_1 q^1 + \cdots + \epsilon_n q^n, \epsilon_i \in \mathbb{Z}, |\epsilon_i| \leq m, y \neq 0).$$

A recent result in 2000, of Komornik, Loreti and Pedicini gives a complete description of $l^m(q)$ when q is the Golden ratio. This paper extends this result to include all unit quadratic Pisot numbers. A main theorem is

Theorem. *Let q be a quadratic Pisot number that satisfies a polynomial of the form $p(x) = x^2 - ax \pm 1$, with conjugate r. Let q have convergents $\left\{\frac{C_k}{D_k}\right\}$ and let k be the maximal integer such that*

$$|D_k r - C_k| \leq m \frac{1}{1 - |r|}$$

then

$$l^m(q) = |D_k q - C_k|.$$

A value related to $l(q)$ is $a(q)$, the minimal non-zero value when all ± 1 polynomials are evaluated at q . Formally this is

$$a(q) := \inf(|y| : y = \epsilon_0 + \epsilon_1 q^2 + \cdots + \epsilon_n q^n, \epsilon_i = \pm 1, y \neq 0).$$

An open question concerning how often $a(q) = l(q)$ is also answered here.

1. INTRODUCTION

Erdős, Joó and Komornik in 1990 [5] initiate the study of the spectra resulting from evaluating certain classes of polynomials at values $q > 1$. Recall:

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Definition 1. A *Pisot number* is a real algebraic integer, all of whose conjugates are of modulus strictly less than 1. A *Pisot polynomial* is the minimal polynomial of a Pisot number.

It is known that if we evaluate a height m polynomial at a Pisot number, then it is either zero, or bounded away from zero [3, 8, 11]. Thus we have that the infimums of these spectra being studied are greater than zero for all Pisot numbers. To this end Erdős, Joó and Joó [7] define these infimums as $l^m(q)$.

Definition 2. Define $l^m(q)$ as:

$$l^m(q) := \inf(|y| : y = \epsilon_0 + \epsilon_1 q^1 + \cdots + \epsilon_n q^n, \epsilon_i \in \mathbb{Z}, |\epsilon_i| \leq m, y \neq 0).$$

We typically denote $l(q) := l^1(q)$.

A related area of interest is the case where the class of polynomials is restricted to polynomials with ± 1 coefficients [2, 14]. The minimal value in this case is defined as $a(q)$.

Definition 3. Define $a(q)$ as:

$$a(q) := \inf(|y| : y = \epsilon_0 + \epsilon_1 q^1 + \cdots + \epsilon_n q^n, \epsilon_i = \pm 1, y \neq 0).$$

An open question concerning when $a(q) = l(q)$ is answered in Section 5. For additional history of problems relating to $l^m(q)$ and $a(q)$, see [2, 12].

Specific values of $l^m(q)$ are calculated for some Pisot numbers q. If q is the Pisot number that satisfies $q^3 - q^2 - 1$, then $l(q) = q^2 - 2$ [13]. If q is the Pisot number satisfying $q^n - q^{n-1} - \cdots - 1$ then $l(q) = \frac{1}{q}$ [7, 13]. If q is the Golden ratio, (the greater root of $x^2 - x - 1$) then $l^2(q) = q^3 - 2q^2 + 2q - 2 = 2q - 3$ (this corrects a misprint in [3], which used the notation $\liminf(u_n^{(2)})$ for $l^2(q)$). For general m, and q the Golden ratio, all $l^m(q)$ are known. If F_k is the kth Fibonacci number ($F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$), and $q^{k-2} < m \leq q^{k-1}$ then $l^m(q) = |F_k q - F_{k+1}|$ [13].

In [2] an algorithm is given to calculate $l^m(q)$ for any Pisot number q and any integer m, limited only by the memory of the computer. Although this method can make calculations for any given q and m, it obviously only solves the problem for specific examples. A tabulation of other $l^m(q)$ for various m and q, based on these methods of calculation, is found at [10]. Upon examination of these tables, we find another pattern, similar to that of $l^m(q)$ when q is the Golden ratio. This pattern concerns $l^m(q)$ for unit quadratic Pisot numbers. The class of Pisot numbers we consider is:

Definition 4. Let $\mathcal{P}$ be the set of unit quadratic Pisot numbers. This is easily seen to be the appropriate roots of polynomials of the form $x^2 - rx - 1$ for $r = 1, 2, 3, \cdots$, and $x^2 - rx + 1$ for $r = 3, 4, 5, \cdots$.

Let $q \in \mathcal{P}$. This paper shows that $l^m(q) = |Dq - C|$ where $\frac{C}{D}$ is a convergent of the continued fraction of q. A better description of which convergent $l^m(q)$ is equal to is given in Theorem 2.1, in Section 2.

2. A DESCRIPTION OF $l^m(q)$

In this section we give a description of $l^m(q)$ for all $q \in \mathcal{P}$. First though, we need a few lemmas and definitions. Here and throughout let a and b be two fixed integers.

Definition 5. We define the sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ as

1. $A_0 = 0,\ A_1 = 1,\ A_n = aA_{n-1} + bA_{n-2}$,
2. $B_0 = 1,\ B_1 = 0,\ B_n = aB_{n-1} + bB_{n-2}$.

Lemma 1. *Using the notation of Definition 5*

$$\det\left(\begin{bmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{bmatrix}\right) = (-b)^{n-1}$$

Proof:

$$\begin{aligned}
\det\left(\begin{bmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{bmatrix}\right) &= \det\left(\begin{bmatrix} aA_{n-1} + bA_{n-2} & A_{n-1} \\ aB_{n-1} + bB_{n-2} & B_{n-1} \end{bmatrix}\right) \\
&= -b\det\left(\begin{bmatrix} A_{n-1} & A_{n-2} \\ B_{n-1} & B_{n-2} \end{bmatrix}\right) \\
&\vdots \\
&= (-b)^{n-1}\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= (-b)^{n-1}
\end{aligned}$$

■

Lemma 2. *For $n \geq 0$ we have*

$$x^n \equiv A_n x + B_n \pmod{x^2 - ax - b}.$$

Proof: Simple induction argument.

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Lemma 3. *Let q and r be the two roots of $x^2 - ax - b$. Then*

$$A_n = \frac{1}{q-r} q^n + \frac{1}{r-q} r^n$$

and

$$B_n = \frac{r}{r-q} q^n + \frac{q}{q-r} r^n.$$

Proof: This is a standard result from recurrence relations, see for example [9].

■

By combining Lemmas 2 and 3 we get:

Lemma 4. *Let q and r be the two roots of $x^2 - ax - b$. Then*

$$x^n \equiv \frac{1}{q-r} q^n (x-r) + \frac{1}{r-q} r^n (x-q) \pmod{x^2 - ax - b}.$$

The next result is well known in the literature on continued fractions, see for example [4].

Lemma 5. *Let q be a real number, and m some integer, then the best approximation to q by $\frac{C}{D}$, where $0 < D \leq m$ is a convergent $\frac{C_n}{D_n}$ of the continued fraction of q, for n maximal such that $D_n \leq m$.*

One property of a continued fraction C/D is that $|D'q - C'| > |Dq - C|$ for all $0 < D' < D$.

The next lemma is reminiscent to those lemmas in Section 3 of [13], but the presentations is different. For this lemma we need the following definition:

Definition 6. Define $\widehat{\mathbb{R}}[x] = \{p \in \mathbb{R}[x] : H(p) \leq 1\}$. Here $H(\sum_{i=0}^n a_i x^i) = \max\{|a_i| : i = 0 \cdots n\}$.

Lemma 6. *Let $p(x) = x^2 - ax \pm 1$ be associated with some $q \in \mathcal{P}$. Let m be such that $|A_n| \leq 2m|B_n|$ for all $n \geq 2$. Let $y' \in m\widehat{\mathbb{R}}[x]$ such that $y' \equiv cx + d \pmod{p(x)}$ where $c, d \in \mathbb{Z}$. Then there exists a $y \in m\widehat{\mathbb{R}}[x] \cap \mathbb{Z}[x]$ such that $y \equiv cx + d \pmod{p(x)}$.*

Proof: First it is worth noting that we can always find an m such that $|A_n| \leq m|B_n|$ as $\lim_{n\to\infty} \left|\frac{A_n}{B_n}\right| = \left|\frac{1}{r}\right| < \infty$.

By assuming that $y \equiv cx + d \pmod{p(x)}$ with $c, d \in \mathbb{Z}$, we show here how to find y' such that $y' \equiv y \pmod{p(x)}$ and $y' \in \mathbb{Z}[x] \cap m\widehat{\mathbb{R}}[x]$. Let

$$y = a_n x^n + \cdots a_0.$$

If $a_n \in \mathbb{Z}$ then continue inductively on $a_{n-1}x^{n-1} + \cdots + a_0$. If $a_0 \in \mathbb{Z}$ then continue inductively on $a_n x^{n-1} + \cdots + a_1$. If neither a_0 nor a_n is an integer, then use the identity that $x^n - A_n x - B_n \equiv 0 \pmod{p(x)}$ along with the fact that $|A_n| \leq 2m|B_n|$ to solve for α where $a_n + \alpha \in \mathbb{Z}$ or $a_0 - \alpha B_n \in \mathbb{Z}$, and $|a_n + \alpha|, |a_1 - A_n\alpha|, |a_0 - B_n\alpha| \leq m$. By solving for α such that $a_0 - \alpha B_n = \lceil a_0 \rceil$ or $\lfloor a_0 \rfloor$ we get two possible values, one negative and one positive. The possible values have the property that they differ by less than $1/|B_n|$. Similarly by solving for α such that $a_n + \alpha = \lceil a_n \rceil$ or $\lfloor a_n \rfloor$ we again get two possible value, one negative and one positive. Let α_l be the maximal of the two negative values, and α_u be the minimal of the two positive values. So if α is equal to either α_l or α_u we have that one of $a_n + \alpha \in \mathbb{Z}$ of $a_0 - \alpha B_n \in \mathbb{Z}$ and further $|a_n + \alpha|, |a_0 - B_n\alpha| \leq m$. Lastly, by noticing that $\alpha_u - \alpha_l \leq 1/|B_n|$ we see that for at least one of these two values $|a_1 - \alpha_1 A_1| \leq m$. Continue inductively on $a_n x^n + \cdots + a_0 + \alpha(x^n - A_n x - B_n)$.

By repeated application of this we see that y' is such that all of the a_i are integers with the possible exception of two consecutive terms, a_j and a_{j-1}. Notice that $a_n x^n + \cdots + a_{j+1}x^{j+1} + a_{j-2}x^{j-2} + \cdots a_0 \equiv c'x + d'$ for some $d', c' \in \mathbb{Z}$. Thus we see that $a_j x^j + a_{j-1}x^{j-1} \equiv (c - c')x + (d - d')$, where $c - c', d - d' \in \mathbb{Z}$. By Lemma 2 we know that $a_j x^j + a_{j-1}x^{j-1} \equiv a_j A_j x + a_{j-1}A_{j-1}x + a_j B_j + a_{j-1}B_{j-1}$. Thus we have that

$$\begin{bmatrix} A_j & A_{j-1} \\ B_j & B_{j-1} \end{bmatrix} \begin{bmatrix} a_j \\ a_{j-1} \end{bmatrix} = \begin{bmatrix} c - c' \\ d - d' \end{bmatrix}$$

By noticing that the determinant of $\begin{bmatrix} A_j & A_{j-1} \\ B_j & B_{j-1} \end{bmatrix}$ is ± 1, we get that the inverse of this matrix is integral, and thus $a_j, a_{j-1} \in \mathbb{Z}$.

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What is interesting here is that this proof is constructive, and a computer algorithm can be designed from this. This is described in Section 3.

Theorem 2.1. *Let $q \in \mathcal{P}$ satisfy a polynomial of the form $p(x) = x^2 - ax \pm 1$, with conjugate r. Let q have convergents $\left\{ \frac{C_k}{D_k} \right\}$ and let k be the maximal integer such that*

$$|D_k r - C_k| \leq m \frac{1}{1 - |r|}$$

then

$$l^m(q) = |D_k q - C_k|.$$

It is worth noting, when q is the Golden ratio then the result is equivalent to Theorem 3.1 in [13].

Proof: The case $|A_n| > 2m|B_n|$ satisfies the theorem trivially with $l^m(q) = 1$.

Recall from Lemma 4 that

$$x^n \equiv \frac{1}{q-r} q^n (x-r) + \frac{1}{r-q} r^n (x-q).$$

It follows from this that if

$$\widehat{\mathbb{R}}[x] \equiv \sum \alpha_n x^n \pmod{p(x)}$$

for $|\alpha_n| \leq 1$ then

$$\begin{aligned} \widehat{\mathbb{R}}[x] &\equiv \sum \alpha_n \left(\frac{1}{q-r} q^n (x-r) + \frac{1}{r-q} r^n (x-q) \right) \pmod{p(x)} \\ &\equiv v(x-r) + w(x-q) \pmod{p(x)} \end{aligned}$$

where $v \in \mathbb{R}$ and $|w| \leq \frac{1}{(q-r)(1-|r|)}$.

Consider the continued fraction of q, $\left\{ \frac{C_k}{D_k} \right\}$. Lemma 5 indicates that the best linear terms are of the form $D_k q - C_k$. Lemma 6 indicates that if $D_k x - C_k \in m\widehat{\mathbb{R}}[x]$ then there exists a $y \in \mathbb{Z}[x] \cap \widehat{\mathbb{R}}[x] \pmod{p(x)}$ such that $y \equiv D_k x - C_k \pmod{p(x)}$. It follows that $l^m(q) = D_k q - C_k$ when $D_k x - C_k \in m\widehat{\mathbb{R}}[x] \pmod{p(x)}$ with k maximal. Write $D_k x - C_k$ as $v(x-r) + w(x-q)$. As $D_k x - C_k \in m\widehat{\mathbb{R}}[x] \pmod{p(x)}$ we have $|w| \leq \frac{m}{(q-r)(1-|r|)}$. Thus

$$\begin{aligned} |D_k r - C_k| &= |v(r-r) + w(r-q)| \\ &= |w(q-r)| \\ &\leq \frac{m}{(q-r)(1-|r|)}(q-r) \\ &\leq \frac{m}{1-|r|} \end{aligned}$$

This is the desired result.

■

Corollary 1. *Define $F_n = rF_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$ and q the Pisot root of $x^2 - rx - 1$. If $q^{k-1}(q-1) \leq m < q^k(q-1)$ then $l^m(q) = |F_k q - F_{k+1}|$.*

Proof: It is easy to verify that $\left\{ \frac{F_{k+1}}{F_k} \right\}$ are the continued fractions of q. A simple calculation shows that

$$F_k = \frac{1}{q-r}(q^k - r^k)$$

and that $r = \frac{-1}{q}$. This yields that $l^m(q) = |F_k q - F_{k+1}|$ when

$$\begin{aligned}
|F_k r - F_{k+1}| &\leq \frac{m}{1-|r|}, \\
\left|\frac{1}{q-r}((q^k - r^k)r - (q^{k+1} - r^{k+1}))\right| &\leq \frac{m}{1-|r|}, \\
\left|\frac{1}{q-r}(q^k r - q^{k+1})\right| &\leq \frac{m}{1-|r|}, \\
\left|\frac{r-q}{q-r}q^k\right| &\leq \frac{m}{1-|r|}, \\
q^k &\leq \frac{m}{1-|r|}, \\
(q-1)q^{k-1} &\leq m,
\end{aligned}$$

and the result follows.

■

Table 1 gives the ranges that m is in, for $l^m(q) = F_{k-1}q - F_k$.

$l^m(q)$	q^2-q-1	q^2-2q-1	q^2-3q-1	q^2-4q-1	q^2-5q-1
$\|F_0q - F_1\|$		[1,1]	[1,2]	[1,3]	[1,4]
$\|F_1q - F_2\|$	[1,1]	[2,3]	[3,7]	[4,13]	[5,21]
$\|F_2q - F_3\|$		[4,8]	[8,25]	[14,58]	[22,113]
$\|F_3q - F_4\|$	[2,2]	[9,19]	[26,82]	[59,245]	[114,586]
$\|F_4q - F_5\|$	[3,4]	[20,48]	[83,274]	[246,1042]	[587,3048]
$\|F_5q - F_6\|$	[5,6]	[49,115]	[275,904]	[1043,4413]	[3049,15826]
$\|F_6q - F_7\|$	[7,11]	[116,280]	[905,2989]	[4414,18698]	[15827,82183]
$\|F_7q - F_8\|$	[12,17]	[281,675]	[2990,9871]	[18699,79205]	[82184,426742]
$\|F_8q - F_9\|$	[18,29]	[676,1632]	[9872,32605]	[79206,335521]	[426743,2215893]

Table 1: Relation between $l^m(q)$ and $F_{k-1}q - F_k$.

With a proof similar to that of Corollary 1, we get

Corollary 2. *Define* $E_n = rE_{n-1} - E_{n-2}$ *with* $E_0 = 0$ *and* $E_1 = 1$. *Define* $G_n = rG_{n-1} - G_{n-2}$ *with* $G_0 = 1$ *and* $G_1 = 1$. *Let* q *be the Pisot root of* $x^2 - rx + 1$. *If* $q^{k-3}(q-1)^2 \leq m < q^{k-2}(q-1)$ *then* $l^m(q) = |G_{k-1}q - G_k|$ *and if* $q^{k-2}(q-1) \leq m < q^{k-2}(q-1)^2$ *then* $l^m(q) = |E_{k-1}q - E_k|$.

Table 2 gives the ranges that m is in, for $l^m(q) = |G_{k-1}q - G_k|$ and $l^m(q) = |E_{k-1}q - E_k|$

$l^m(q)$	q^2-3q+1	q^2-4q+1	q^2-5q+1	q^2-6q+1	q^2-7q+1
$\lvert G_0q-G_1\rvert$					
$\lvert E_0q-E_1\rvert$	[1,1]	[1,2]	[1,2]	[1,3]	[1,5]
$\lvert G_1q-G_2\rvert$			[3,3]	[4,4]	
$\lvert E_1q-E_2\rvert$	[2,2]	[3,7]	[4,14]	[5,23]	[6,34]
$\lvert G_2q-G_3\rvert$	[3,4]	[8,10]	[15,18]	[24,28]	[35,40]
$\lvert E_2q-E_3\rvert$	[5,6]	[11,27]	[19,68]	[29,135]	[41,234]
$\lvert G_3q-G_4\rvert$	[7,11]	[28,38]	[69,87]	[136,164]	[235,275]
$\lvert E_3q-E_4\rvert$	[12,17]	[39,103]	[88,329]	[165,791]	[276,1609]

Table 2: Relation between $l^m(q)$, $E_{k-1}q-E_k$ and $G_{k-1}q-G_k$.

3. Finding the height m polynomials

For $q \in \mathcal{P}$ with minimal polynomial $p(x)$, we have $l^m(q) = |Dq - C|$ for some integers C and D where $\frac{C}{D}$ is a convergent of q. What this section is interested in is finding the particular height m polynomial that $l^m(q)$ relates to. We notice that Lemma 6 can be implemented into an algorithm. Thus it is sufficient to find a $t(x) \in m\widehat{\mathbb{R}}[x]$ such that $t(x) \equiv Dx - C \pmod{p(x)}$. For this we can use the simplex method [15]. Write

$$Dx - C + (\sum_{k=0}^{n} a_k x^k)p(x) = \sum_{k}^{n+2} b_k x^k$$

for unknowns a_k. We wish $-m \leq b_k \leq m$ for all $k = 0, \cdots, (n+2)$. So for the correct value of n we minimize for h with

$$-h \leq b_k \leq h \tag{1}$$

and solve for the a_k. A careful calculation can yield the minimal value for n that works as

$$n = \left\lfloor \ln\left(\frac{m + |Dr - C|\,|r| - |Dr - C|}{m\,|r|}\right) (\ln(|r|))^{-1} \right\rfloor \tag{2}$$

Using the simplex method in this way works for any polynomial, though the value for n given in equation (2) is specifically for $q \in \mathcal{P}$. Thus if we wish to implement this algorithm for polynomials that come from some $q \notin \mathcal{P}$ we can simply take n increasing until we find one that works.

We now do an example

Example 1. Let q be the root of $q^2 - 3q + 1$. A simple calculation demonstrates that $l^7(q) = 5q - 13$. Using equation (2) we have the minimal value for n is 3.

So minimizing h with respect to the constraints in equation (1) as $n = 3$ gives $h = \frac{305}{44} < 7$. This gives a polynomial of:

$$\frac{17}{4}x^5 - \frac{305}{44}x^4 - \frac{305}{44}x^3 - \frac{305}{44}x^2 - \frac{305}{44}x - \frac{305}{44}$$

Here we use the techniques in Lemma 6 iteratively. Notice that at any step, only three coefficients are altered.

$$\begin{aligned}
\frac{17}{4}x^5 - \frac{305}{44}x^4 - \frac{305}{44}x^3 - \frac{305}{44}x^2 - \frac{305}{44}x - \frac{305}{44} &\equiv 5x - 13 \pmod{x^2 - 3x + 1}, \\
\frac{327}{77}x^5 - \frac{305}{44}x^4 - \frac{305}{44}x^3 - \frac{305}{44}x^2 - \frac{520}{77}x - 7 &\equiv 5x - 13 \pmod{x^2 - 3x + 1}, \\
\frac{371}{88}x^5 - \frac{305}{44}x^4 - \frac{305}{44}x^3 - \frac{553}{88}x^2 - 7x - 7 &\equiv 5x - 13 \pmod{x^2 - 3x + 1}, \\
4x^5 - \frac{305}{44}x^4 - \frac{229}{44}x^3 - \frac{305}{44}x^2 - 7x - 7 &\equiv 5x - 13 \pmod{x^2 - 3x + 1}, \\
4x^5 - 7x^4 - 5x^3 - 7x^2 - 7x - 7 &\equiv 5x - 13 \pmod{x^2 - 3x + 1}.
\end{aligned}$$

Thus we have found a height 7 integer polynomial $p(x)$ where $l^7(q) = 5q - 13 = p(q)$.

4. Non-unit quadratic Pisot numbers

It is worth noting that Theorem 2.1 does not work for all quadratic Pisot numbers. The problem is that Lemma 6 doesn't work for all Pisot polynomials $x^2 - ax - b$. For example if q is the Pisot root of $x^2 - 2x - 2$ (of approximately 2.732050808) then we see that

$$\frac{7}{16}x^8 - 2x^7 + 3x^6 - 3x^5 + 3x^4 - 3x^3 + 3x^2 - 3x + 3 \equiv 8 - 3x \pmod{x^2 - 2x - 2}$$

is in $3\widehat{\mathbb{R}}[x]$ but it is not in $\mathbb{Z}[x] \cap 3\widehat{\mathbb{R}}[x]$. Worse we have that $|8 - 3q| = 0.196152424 < 0.267949192 = l^3(q)$.

5. The existence of an infinite family of Pisot numbers where $l(q) = a(q)$

It is easy to give an infinite set of Pisot numbers q where $l(q) = a(q)$. We know from [7] that if $q^n - q^{n-1} - \cdots - 1 = 0$ then $l(q) = q^{n-1} - q^{n-2} - \cdots - 1$. It is clear in this case that $l(q) = a(q)$. This answers question 1 in [2] in the negative, as it gives an infinite family of Pisot numbers where $l(q) = a(q)$.

6. Further research

The counter example of Section 4 shows that Theorem 2.1 does not work for all quadratic Pisot numbers. Despite this, the spirit of Theorem 2.1 appears to be true. Let $\{\frac{C_k}{D_k}\}$ be the convergents of a quadratic Pisot number q, with conjugate r, not necessarily a unit. If Theorem 2.1 could be extended to this case, then $l^m(q) = |D_k q - C_k|$ where k is maximal such that $|D_k r - C_k| \leq \frac{m}{1-|r|}$. Computationally, it appears that $l^m(q) = |D_k q - C_k|$ or $|D_{k-1} q - C_{k-1}|$, but no proof of this is known. It would be interesting to know if this is indeed the case.

Secondly it would be of interest if a Lemma similar to Lemma 6 could be found that would work for all polynomials p where $p(0) = \pm 1$, regardless of the degree of $p(x)$. If something like this could be found then this could be used to prove, for $q \in (1, 2)$, that $l(q) > 0$ if and only if q is Pisot. This is believed to be true by a number of people, see for example [2, 12]. The second part of this Lemma easily extends to arbitrary degree, but it is not clear that there is an algorithm that forces all but d consecutive terms to be integers (where d is the degree of $p(x)$).

While searching for patterns among various Pisot numbers, it appears that a nice description exists for the Pisot roots of $x^3 - x - 1$ and $x^3 - x^2 - 1$. Some work is done on this [1] but this is not fully answered as of yet.

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