THE EXPECTED L_p NORM OF RANDOM POLYNOMIALS.

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Abstract. The results of this paper concern the expected L_p norm of random polynomials on the boundary of the unit disc (equivalently of random trigonometric polynomials on the interval $[0, 2\pi]$). Specifically, for a random polynomial

$$q_n(\theta) = \sum_0^{n-1} X_k e^{ik\theta}$$

let

$$||q_n||_p^p = \int_0^{2\pi} |q_n(\theta)|^p \, d\theta/(2\pi) \, .$$

Assume the random variables $X_k; k \geq 0$ are independent and identically distributed, have mean 0, variance equal to 1 and, if $p > 2$ a finite p^{th} moment $E(|X_k|^p)$. Then

$$\frac{\mathrm{E}(||q_n||_p^p)}{n^{p/2}} \to \Gamma(1 + p/2)$$

and

$$\frac{\mathrm{E}(||q_n^{(r)}||_p^p)}{n^{(2r+1)p/2}} \to (2r+1)^{-p/2}\Gamma(1 + p/2)$$

as $n \to \infty$.

In particular if the polynomials in question have coefficients in the the set $\{+1, -1\}$ (a much studied class of polynomials) then we can compute the expected L_p norms of the polynomials and their derivatives

$$\frac{\mathrm{E}(||q_n||_p)}{n^{1/2}} \to (\Gamma(1 + p/2))^{1/p}$$

and

$$\frac{\mathrm{E}(||q_n^{(r)}||_p)}{n^{(2r+1)/2}} \to (2r+1)^{-1/2}(\Gamma(1 + p/2))^{1/p}.$$

This complements results of Fielding in the $p := 0$ case; Newman and Byrnes in the $p := 4$ case; and Littlewood et al in the $p = \infty$ case.

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1. INTRODUCTION

There are a number of difficult old conjectures that concern the possible rates of growth of polynomials with all coefficients in the set $\{+1,-1\}$. Since many of these were raised by Littlewood we denote the set of such polynomials by $\mathcal{L}_n$ and refer to them as Littlewood polynomials. Specifically

$$\mathcal{L}_n := \left\{ p : p(x) = \sum_{j=0}^{n} a_j x^j \,, \quad a_j \in \{-1,1\} \right\}.$$

In [Li-66] Littlewood conjectures that it is possible to find $p_n \in \mathcal{L}_n$ so that

$$C_1\sqrt{n+1} \le |p_n(z)| \le C_2\sqrt{n+1}$$

for all complex z of modulus 1. Such polynomials are often called "flat". Because the L_2 norm of a polynomial from $\mathcal{L}_n$ is exactly $\sqrt{n+1}$ the constants must satisfy $C_1 \le 1$ and $C_2 \ge 1$. This is discussed in some detail in problem 19 of Littlewood's delightful monograph [Li-68]. A sequence of polynomials that satisfies just the upper bound is given by the Rudin-Shapiro polynomials. No sequence is known that satisfies the lower bound.

This conjecture is complemented by a conjecture of Erdős [Er-62] that the constant C_2 is bounded away from 1 (independently of n). This is also still open. Though a remarkable result of Kahane's [Kah-80, Kah-85] shows that if the polynomials are allowed to have complex coefficients of modulus 1 then "flat" polynomials exist and indeed that it is possible to make C_1 and C_2 asymptotically arbitrarily close to 1. Equally remarkable is a result of Beck [Bec-91] who proves that " flat" polynomials exist from the class of polynomials of degree n whose coefficients are 400th roots of unity.

The relationship between these problems and Barker polynomials is discussed in [Sa-90]. The most famous problem concerning polynomials with coefficients in the set $\{0,-1,+1\}$ is the celebrated, now resolved, Littlewood conjecture (Ko-80). It asserts that the L_1 norm of a polynomial $\sum_{j=0}^{n} \pm x^{k_j}$ must grow at least like $\log(n)$. Here and in what follows the L_p norms are on the boundary of the unit disk in the complex plane.

In [Sa-54] it is shown that for all but $o(2^n)$ Littlewood polynomials the supremum on the unit disc lies between $c_1\sqrt{n\log n}$ and $c_2\sqrt{n\log n}$. In fact, Halász [Ha-73] shows that the $\lim ||q_n/\sqrt{n\log n}||_\infty = 1$ almost surely. See also [An-83].

The expected L_4 to the fourth power of a Littlewood polynomial of degree n is computed by Newman and Byrnes [Ne-90]. They show that

$$\mathrm{E}(||p||_4^4) = 2(n+1)^2 - (n+1)$$

where p is a random element of $\mathcal{L}_n$.

In the L_0 case Fielding [Fi-70] computes the expected norm (which in this case is the Mahler Measure) over the polynomials with *complex* coefficients of modulus 1. He proves that

$$\mathrm{E}(||p||_0) \ge \exp(-\gamma/2)\sqrt{n}(1 + O(n^{-1/2+\delta}))$$

where γ is Euler's constant. See also [Ul-88].

Our principal aim in this paper is to to compute the expected L_p norms of Random Littlewood polynomials. The complete results are stated in the next sections. For random Littlewood polynomials, $q_n \in \mathcal{L}_n$, we have

$$\frac{\mathrm{E}(||q_n||_p)}{n^{1/2}} \to (\Gamma(1+p/2))^{1/p}$$

and for their derivatives

$$\frac{\mathrm{E}(||q_n^{(r)}||_p)}{n^{(2r+1)/2}} \to (2r+1)^{-1/2}(\Gamma(1+p/2))^{1/p}.$$

From this and the inequality [Bor-95, p406]

$$\frac{||q_n'||_p}{n||q_n||_p} \le 1$$

we can deduce an expected Bernstein Inequality for Littlewood polynomials namely

$$\mathrm{E}\left(\frac{||q_n'||_p}{n||q_n||_p}\right) \to \frac{1}{\sqrt{3}}.$$

This should be compared to interesting results of Nazarov and Queffélec and Saffari [Qe-96] which says that

$$\max_{q_n \in \mathcal{L}_n} \frac{||q_n'||_p}{n||q_n||_p} \to 1$$

for all $p > 1$ except $p = 2$ where the $\limsup$ is $1/\sqrt{3}$.

Of course, because of the monotonicity ot the L_p norms it is relevant to rephrase Littlewood's conjecture in other norms. It has been conjectured that

$$||p||_4^4 \ge (7-\delta)n^2/6$$

for $p \in \mathcal{L}_n$ and n sufficiently large. This would be best possible and would imply Erdős' conjecture above. See [Bor-98] for a discussion of this.

Random polynomials have been much looked at, particularly the location of their roots. See for example [Bh-86], [Bri-95] and [Kac-48].

2. Results

Consider a random polynomial

$$q_n(\theta) = \sum_0^{n-1} X_k e^{ik\theta}$$

for $0 \le \theta \le 2\pi$. We study the p^{th} power of the L_p norm of q_n, that is:

$$||q_n||_p^p = \int_0^{2\pi} |q_n(\theta)|^p \, d\theta/2\pi$$

Theorem 1. *Fix $0 < p < \infty$. Assume that the random variables $X_k; k \geq 0$ are independent and identically distributed, have mean 0, variance equal to 1 and, if $p > 2$, a finite p^{th} moment $E(|X_k|^p)$. Then*

$$\frac{E(||q_n||_p^p)}{n^{p/2}} \to \Gamma(1+p/2)$$

as $n \to \infty$. If, in addition, $E(|X_k|^{2p}) < \infty$ then

$$\frac{||q_n||_p}{n^{1/2}} \to \Gamma(1+p/2)^{1/p}$$

in probability and

$$\frac{E(||q_n||_p)}{n^{1/2}} \to (\Gamma(1+p/2))^{1/p}.$$

For ± 1 coefficients randomly chosen the moment conditions are trivially satisfied for all finite p. Numerical confirmation of the principle results based on computations up to degree 24 is presented in the two figures accompanying this paper. The calculations are courtesy of Lesley Robinson [Ro-97]. The first figure shows the average L_3 and L_4 norms (normalized by division by $\sqrt{n+1}$) for the Littlewood polynomials up to degree 24. The second figure is a similar graph for L_3^3 and L_4^4 (normalized by division by $(n+1)^2$).

Proof. In what follows all unlabelled sums run from 0 through $n-1$. Define

$$\sigma_{n,c}^2(\theta) = \sum \cos^2(j\theta)$$

and

$$\sigma_{n,s}^2(\theta) = \sum \sin^2(j\theta)$$

and write

$$a_{k,n}(\theta) = \frac{\cos(k\theta)}{\sigma_{n,c}(\theta)}$$

and, for θ not a multiple of π,

$$b_{k,n}(\theta) = \frac{\sin(k\theta)}{\sigma_{n,s}(\theta)}.$$

Then

Lemma 2.1. *There is a constant M, free of n, k and θ such that*

$$|a_{k,n}(\theta)| + |b_{k,n}(\theta)| \leq \frac{M}{\sqrt{n}}.$$

Postpone the proof of the lemma and consider an arbitrary sequence θ_n. The lemma permits application of the Lindeberg central limit theorem to show that for an arbitrary sequence θ_n

$$C_n(\theta_n) \equiv \frac{\sum_0^{n-1} X_k \cos(k\theta_n)}{\sigma_{n,c}(\theta_n)} = \sum_0^{n-1} a_{k,n}(\theta_n) X_k$$

converges in distribution to standard normal and, provided no θ_n is an integer multiple of π,

$$S_n(\theta_n) \equiv \frac{\sum_0^{n-1} X_k \sin(k\theta_n)}{\sigma_{n,s}(\theta_n)}$$

converges in distribution to standard normal. Moreover, for any fixed θ not an integer multiple of π, the elementary convergences $\sigma_{n,c}^2/n \to 1/2$, $\sigma_{n,s}^2/n \to 1/2$ and $\sum \cos(k\theta)\sin(k\theta)/n \to 0$ show that the covariance matrix of $(C_n(\theta), S_n(\theta))$ converges to the 2 by 2 identity so that $(C_n(\theta), S_n(\theta))$ converges in distribution to (Z_1, Z_2) where the Z_i are independent standard normals. It follows for such θ that $|q_n(\theta)|^2/n$ converges in distribution to $(Z_1^2 + Z_2^2)/2$ which has a standard exponential distribution.

For $p > 2$, a theorem of Bernstein ([Be-39], [Br-70]) asserts that, in the central limit theorem, pth moments converge to the pth moment of the normal distribution provided an analogue of the Lindeberg condition holds. In particular,

$$E(|C_n(\theta_n)|^p) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |u|^p \exp(-u^2/2)\, du$$

provided

$$\sum \mathrm{E}(|a_{k,n}|^p |X_k|^p 1(a_{k,n}^2 X_k^2 > \eta)) \to 0$$

for each positive η (and analogously for S_n replacing $a_{k,n}$ by $b_{k,n}$). (For $p \le 2$ we could proceed straight from the central limit theorem.) The quantity in question is bounded by

$$\sum |a_{k,n}|^p \mathrm{E}(|X_0|^p 1(X_0^2 > n\eta/M^2)) \le \sum |a_{k,n}|^p \mathrm{E}(|X_0|^p 1(X_0^2 > n\eta/M^2))\,.$$

But $\sum |a_{k,n}|^p$ is, for $p \ge 2$, bounded by $\sum a_{k,n}^2 = 1$ and X_0 has a finite p^{th} moment so the bound converges to 0.

Now note

$$\begin{aligned} |q_n(\theta)|^p/n^{p/2} &= \left(\frac{\sigma_{n,c}^2(\theta)C_n^2(\theta) + \sigma_{n,s}^2(\theta)S_n^2(\theta)}{n}\right)^{p/2} \\ &\le 2^{p/2-1}\left[\left(\frac{\sigma_{n,c}^2(\theta)}{n}\right)^{p/2} |C_n(\theta)|^p + \left(\frac{\sigma_{n,s}^2(\theta)}{n}\right)^{p/2} |S_n(\theta)|^p\right] \\ &\le 2^{p/2-1}\left[|C_n(\theta)|^p + |S_n(\theta)|^p\right] \end{aligned}$$

Putting $g_n(\theta) = \mathrm{E}(|q_n(\theta)|^p/n^{p/2})$ and $h_n(\theta) = 2^{p/2-1}\mathrm{E}(|C_n(\theta)|^p + |S_n(\theta)|^p)$ we see that $0 \le g_n \le h_n$ almost everywhere and that h_n converges uniformly on $(0,\pi) \cup (\pi, 2\pi)$ to $2^{p/2}E(|Z|^p)$ where Z is standard normal. If we establish that g_n converges almost everywhere to $\Gamma(1 + p/2)$ then the theorem will follow by dominated convergence.

To do so we can apply the following result. Suppose U_n and V_n are random variables with U_n converging in distribution to some U, V_n converging in distribution to some

V and $0 \le U_n \le V_n$. If $\mathrm{E}(V_n) \to \mathrm{E}(V) < \infty$ then $\mathrm{E}(U_n) \to \mathrm{E}(U)$. (This is the Dominated Convergence Theorem for convergence in distribution.)

Use the result with $U_n = |q_n(\theta)|^p/n^{p/2}$, which converges in distribution, for θ not an integer multiple of π, to $U = ((Z_1^2 + Z_2^2)/2)^{p/2}$, and

$$V_n = 2^{p/2-1}\left[|C_n(\theta_n)|^p + |S_n(\theta_n)|^p\right]$$

which converges in distribution to $V = 2^{p/2-1}\left(|Z_1|^p + |Z_2|^p\right)$. Since $\mathrm{E}(U) = \Gamma(1 + p/2)$ we are done.

To establish the convergence in probability we compute the variance of $||q_n||_p^p/n^{p/2}$ and show that this converges to zero. It suffices to show that

$$\frac{\mathrm{E}(||q_n||_p^{2p})}{n^p} \to \Gamma^2(1+p/2)\,.$$

But

$$\frac{\mathrm{E}(||q_n||_p^{2p}))}{n^p} = \int_0^{2\pi}\int_0^{2\pi} g_n^{(*)}(\theta_1,\theta_2)\,d\theta_1\,d\theta_2\,.$$

where $g_n^{(*)}(\theta_1,\theta_2) = \mathrm{E}(|q_n(\theta_1)q_n(\theta_2)|^p)/n^p$. For fixed θ_1 and θ_2, neither an integer multiple of π, it is easily checked that the variance covariance matrix of $(C_n(\theta_1), S_n(\theta_1), C_n(\theta_2), S_n(\theta_2))$ converges to the 4 by 4 identity matrix. In view of the Lindeberg condition already checked the random vector converges in distribution to the standard normal and $|q_n(\theta_1)q_n(\theta_2)|^p)/n^p$ converges in distribution to $(Z_1^2+Z_2^2)^p(Z_3^2+Z_4^2)^p/2^{2p}$ where $Z_1,\ldots,Z_4$ are independent standard normal. Since

$$g_n^{(*)}(\theta_1,\theta_2) \le h_n^{(*)}(\theta_1,\theta_2) \equiv \left(\frac{\mathrm{E}(|q_n(\theta_1)|^{2p})}{n^p} + \frac{\mathrm{E}(|q_n(\theta_2)|^{2p})}{n^p}\right)/2$$

we may apply the first part of the theorem and the dominated convergence theorem to conclude that $g_n^{(*)}$ converges almost everywhere to $\Gamma^2(1+p/2)$. Moreover we have already checked that $h_n^{(2)}$ converges almost everywhere to $\Gamma(1+p)$ and that

$$\int_0^{2\pi}\int_0^{2\pi} h_n^{(*)}(\theta_1,\theta_2)\,d\theta_1\,d\theta_2/(4\pi^2) \to \Gamma(1+p)\,.$$

Hence

$$\int_0^{2\pi}\int_0^{2\pi} g_n^{(*)}(\theta_1,\theta_2)\,d\theta_1\,d\theta_2/(4\pi^2) \to \Gamma^2(1+p/2)$$

by dominated convergence. This establishes the convergence in probability.

The final statement of the theorem is simply dominated convergence applied via the elementary inequality $||q_n/n^{1/2}||_p \le 1 + ||q_n/n^{1/2}||_p^p$. □

Proof of Lemma. : For $\theta \le \pi/(2n)$ and $1 \le j \le n-1$ we have $2j\theta/\pi \le \sin(j\theta) \le j\theta$ whence

$$|b_{k,n}| \le \frac{\pi k\theta}{2\sqrt{\sum_{j=1}^{n-1} j^2\theta^2}} = \frac{\sqrt{6}\pi k}{2\sqrt{(n-1)n(2n-1)}} \le \frac{\sqrt{6}\pi/2}{\sqrt{n}}\,.$$

Moreover, on $[0,\pi/(2n)]$ the function $\cos(k\theta)$ is monotone decreasing for all $k \le n$. Hence

$$\begin{aligned} |a_{k,n}(\theta)| &\le \frac{1}{\sigma_{n,c}(\theta)} \\ &\le \frac{1}{\sigma_{n,c}(\pi/(2n))} \\ &= \sqrt{\frac{2}{n+1}} \\ &\le \frac{\sqrt{2}}{\sqrt{n}}. \end{aligned}$$

On the other hand if $\theta > \pi/(2n)$ then

$$\sum\{\cos^2(k\theta) - \sin^2(k\theta)\} = \sin^2(n\theta) + \sin(n\theta)\cos(\theta)\cos(n\theta)/\sin(\theta)$$

Since

$$\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta) \ge \theta$$

we see that for $\pi/(2n) \le \theta \le \pi/2$ we have

$$\left|\sum\{\cos^2(k\theta) - \sin^2(k\theta)\}\right| \le 1 + 1/\tan(\theta) \le 1 + 2n/\pi \le \delta_0 n$$

for a $\delta_0 < 1$ and all $n \ge 3$. It follows that

$$\sum \cos^2(k\theta) \ge \frac{1-\delta_0}{2} n$$

and

$$\sum \sin^2(k\theta) \ge \frac{1-\delta_0}{2} n$$

from which the Lemma follows for $0 \le \theta \le \pi/2$. Use easy symmetries of the weights $a_{k,n}$ and $b_{k,n}$ to get all values of θ. $\square$

Similar techniques permit the extension of our main result to the derivative $q_n^{(r)}$ of order r.

Theorem 2. *Fix $0 < p < \infty$. Assume that the random variables $X_k; k \ge 0$ are independent and identically distributed, have mean 0, variance equal to 1 and, if $p > 2$ a finite p^{th} moment $E(|X_k|^p)$. Then*

$$\frac{E(||q_n^{(r)}||_p^p)}{n^{(2r+1)p/2}} \to (2r+1)^{-p/2}\Gamma(1+p/2)$$

as $n \to \infty$. If, in addition, $E(|X_k|^{2p}) < \infty$ then

$$\frac{||q_n^{(r)}||_p}{n^{(2r+1)/2}} \to (2r+1)^{-1/2}(\Gamma(1+p/2))^{1/p}$$

in probability and

$$\frac{E(||q_n^{(r)}||_p)}{n^{(2r+1)/2}} \to (2r+1)^{-1/2}(\Gamma(1+p/2))^{1/p}.$$

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