

Multiple Roots of $[-1,1]$ Power Series

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Abstract

We are interested in how small a root of multiplicity k can be for a power series of the form $f(z) := 1 + \sum_{n=1}^{\infty} a_n z^n$ with coefficients a_i in $[-1,1]$. Let $r(k)$ denote the size of the smallest root of multiplicity k possible for such a power series. We show that

$$1 - \frac{\log(e\sqrt{k})}{k+1} \leq r(k) \leq 1 - \frac{1}{k+1}.$$

We describe the form that the extremal power series must take and develop an algorithm that lets us compute the optimal root (which proves to be an algebraic number). The computations, for $k \leq 27$, suggest that the upper bound is close to optimal and that $r(k) \sim 1 - c/(k+1)$ where $c = 1.230\dots$

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1 Introduction

Let \mathcal{F} be the set of $[-1, 1]$ power series

$$\mathcal{F} = \left\{ f(z) = 1 + \sum_{i=1}^{\infty} a_i z^i : a_i \in [-1, 1] \right\}.$$

The set \mathcal{F} is uniformly bounded on compact subsets of the open unit disc and so the maximum number of roots of an element of \mathcal{F} on such a subset is also uniformly bounded.

For positive integers $k \geq 1$ let $\mathcal{P}(k)$ denote the set of real numbers $\alpha > 0$ that are k th order roots of some series f_α in \mathcal{F} , and define $r(k)$ to be the infimum of $\mathcal{P}(k)$. The question we examine is how small $r(k)$ can be. We solve this problem exactly for $k \leq 27$, give general bounds for arbitrary k , and offer a speculation on what the exact bound should be.

Notice that the problem is equivalent to asking for the k th order root of smallest absolute value amongst the power series with complex coefficients satisfying $|a_i| \leq 1$ (since if $\alpha_k = r e^{i\theta}$ is a k th order root then making the substitution $f(z e^{i\theta})$ and taking the real part of the resulting power series we obtain an element of \mathcal{F} with k th order real root r). Polynomials with similarly restricted coefficients can be found, with applications, in the work of Byrnes & Newman [7, 8].

It is easy to see that $r(k) \geq r(1) = 1/2$ – if α is a root of a power series in \mathcal{F} and $|\alpha| < 1$, then plainly

$$1 = \left| \sum_{i=1}^{\infty} a_i \alpha^i \right| \leq \frac{|\alpha|}{1 - |\alpha|},$$

with $1/2$ achieved for the series $f(x) = 1 - \sum_{i=1}^{\infty} x^i$.

It is not a priori obvious that \mathcal{F} contains any elements with multiple roots strictly inside the unit disc. We first show (by explicit construction) that $r(k)$ is indeed always strictly less than 1.

Theorem 1 *For all positive integers $k \geq 1$*

$$r(k) \leq 1 - \frac{1}{(k+1)}.$$

As we elaborate later we believe that the correct bound is of the form

$$r(k) \approx 1 - \frac{1.230\dots}{(k+1)}$$

and so the above result is likely giving the correct rate of growth.

It is interesting to note that as $k \rightarrow \infty$ the power series that give us this bound tend coefficient-wise to $\theta_4(x)/(1-x)$ where [2, p.64]

$$\theta_4(x) := 1 + 2 \sum_{i=1}^{\infty} (-1)^i x^{i^2} = \prod_{n=1}^{\infty} (1 - x^{2n-1})^2 (1 - x^{2n})$$

is the Jacobi theta function (a function with no zeros inside the unit circle but a “zero of infinite order” at $x = 1$ in the sense that $\theta_4(x) \sim \exp(-\pi^2/4(x-1))$ as $x \rightarrow 1$). The proof also emphasizes the relationship between our problem and an old question of E. Schmidt and Schur [11] concerning bounds on the maximum order zero at 1 possible for a polynomial (with bounded coefficients) in terms of its degree (see [4] for the best current bounds).

A straightforward application of Jensen’s Theorem yields a lower bound for $r(k)$:

Theorem 2 *For all $k \geq 1$*

$$r(k) \geq k^{-1/2k} \left(1 + \frac{1}{k}\right)^{-\frac{1}{2}(1+1/k)} \geq 1 - \frac{\log(e\sqrt{k})}{k+1}.$$

We should remark that this lower bound is really a bound on the radius of a disc containing k roots of a $[-1,1]$ power series (not specifically multiple roots).

Notice that the infimum $r(k)$ is actually achieved on \mathcal{F} . Indeed, the set \mathcal{F} is bounded (and closed) and thus compact for the topology of uniform convergence on every compact set of the open unit disc. So there exists a sequence $r_n \rightarrow r(k)$ and a convergent sequence f_n in \mathcal{F} such that each series f_n has a k th root at r_n . Calling f the limit of f_n in \mathcal{F} , we know that the derivatives $f'_n, \dots, f_n^{(k-1)}$ converge to $f', \dots, f^{(k-1)}$ uniformly on every compact set of the open unit disc, which ensures that f has a root of order k at $r(k)$.

We next observe that the power series for which the infimum $r(k)$ is achieved must take a very particular form. This is critical in allowing us to construct an algorithm to solve our problem for fixed k .

Theorem 3 *For each k there exists a unique β in $\mathcal{P}(k)$ for which there is a set of $(k-1)$ exponents $1 \leq m_1 < \dots < m_{k-1}$ such that the coefficients of the corresponding power series*

$$f_\beta(x) = 1 + \sum_{i=1}^{\infty} b_i x^i, \quad b_i \in [-1, 1]$$

satisfy

$$b_j = \begin{cases} -1 & \text{if } j \in (0, m_1) \cup \dots \cup (m_{2i}, m_{2i+1}) \cup \dots \\ +1 & \text{if } j \in (m_1, m_2) \cup \dots \cup (m_{2i+1}, m_{2i+2}) \cup \dots \end{cases}$$

Moreover

$$\beta = r(k),$$

and the series f_β is unique.

Consequently we can, by multiplying the extremal power series by $(1 - x)$, equivalently study the k th order roots of polynomials of the form

$$q(x; \mathbf{m}) = 1 - 2x - \sum_{i=1}^{k-1} (-1)^i ((1 + \alpha_i) + (1 - \alpha_i)x) x^{m_i}, \quad \alpha_i := (-1)^{i-1} a_{m_i} \in [-1, 1]. \quad (1)$$

The proofs of the Theorem's are given in §5. In §2 we discuss the computations.

A natural analogous question concerns the location of k th order zeros of power series with coefficients from the set $\{0, -1, 1\}$. Obviously this class is in \mathcal{F} so our lower bounds apply but the upper bounds are probably different. This is briefly discussed in §3 where it is shown that the nonexistence of arbitrarily high multiplicity repeat roots inside the unit disc would answer an old and well known problem of Lehmer.

2 Computations

Theorem 3 in the form of (1) allows us to construct an algorithm for computing values of $r(k)$. This we now describe:

For a trial set of exponents $\mathbf{m} = (m_1, \dots, m_{k-1})$ one can use the first $(k - 1)$ derivatives (linear equations in the α_i) to eliminate the α_i from $q(x; \mathbf{m})$;

For example, if we set

$$Q_0(x) := 1 - 2x - (1 + x) \sum_{i=1}^{k-1} (-1)^i x^{m_i},$$

iteratively generate polynomials

$$\begin{aligned} Q_i(x) &:= x(Q'_{i-1}(x)(1 - x) + iQ_{i-1}(x)), \quad i = 1, \dots, k - 1, \\ G_i(x) &:= -(1 - x)^{k-i-1} Q_i(x), \quad i = 1, \dots, k - 1, \end{aligned}$$

and define polynomials $A_i(x)$ by

$$\begin{pmatrix} A_1(x) \\ \vdots \\ A_{k-1}(x) \end{pmatrix} := \begin{pmatrix} m_1 & \cdots & m_{k-1} \\ \vdots & & \vdots \\ m_1^{k-1} & \cdots & m_{k-1}^{k-1} \end{pmatrix}^{-1} \begin{pmatrix} G_1(x) \\ \vdots \\ G_{k-1}(x) \end{pmatrix},$$

then the real root $0 < \beta < 1$ of the polynomial

$$F(x; \mathbf{m}) := (1-x)^{k-1} Q_0(x) + \sum_{i=1}^{k-1} A_i(x) \quad (2)$$

produces a candidate for $r(k)$; whether this is the correct set of exponents \mathbf{m} (and hence $r(k) = \beta$) is then determined by checking if the value β yields all

$$\alpha_i = (-1)^{i-1} \frac{A_i(\beta)}{\beta^{m_i} (1-\beta)^k}$$

in $[-1,1]$.

We computed $r(k)$ for all $k \leq 27$; The values $r(k)$ are listed at the end together with their successful \mathbf{m} .

The calculations were done in Maple. For $k = 27$ the polynomial (2) had degree in excess of 1000. It is presumably the minimum polynomial though we did not check it for irreducibility. We implemented our own Newton's method for finding the roots in (2) because we had to. We used 50 digit precision in order to compute the α_i accurately. The 27 case took several hours on a Silicon Graphics R3000 Indigo workstation.

It is noticeable that the exponents in \mathbf{m} seem to settle down to the squares (the exponents that appear in the polynomials giving the upper bound of Theorem 1), the α_i appearing to converge very slowly to 1. Plotting $1/(1-r(k))$ against k also suggests a linear growth rate resembling the upper rather than the lower bound; least squares analysis giving the approximation

$$r(k) \approx 1 - \frac{1}{1.23909318 + .81255949k}$$

for small k (see Figure 1), with .99999808 as a coefficient of correlation.

Notice that the $r(k)$ (and hence the coefficients in the corresponding power series) are necessarily roots of polynomials with integer coefficients. For example the corresponding polynomials for $k = 2, 3$ and 4 are;

$$k = 2 \quad 2x^5 - 8x^2 + 11x - 4$$

$$\begin{aligned}
k = 3 & \quad 10x^{12} - 14x^{11} + 14x^6 - 10x^5 - 80x^3 + 185x^2 - 147x + 40 \\
k = 4 & \quad 126x^{22} - 296x^{21} + 176x^{20} + 44x^{12} - 104x^{11} + 54x^{10} + 96x^7 \\
& \quad - 146x^6 + 56x^5 - 684x^4 + 2236x^3 - 2797x^2 + 1584x - 342
\end{aligned}$$

3 Integer polynomials, an open question

It is not surprising that, although all but $(k-1)$ of the coefficients of the optimal k th power series must be ± 1 , the remaining coefficients seem to be non-integer. It is interesting to ask whether there do exist power series (or polynomials) with all $\{0, \pm 1\}$ coefficients and a k th order root strictly inside the unit circle. This question also seems to be intriguingly related to the long-standing Lehmer Problem of Diophantine approximation; defining the Mahler measure $M(p)$ of an integer polynomial

$$p(x) := a \prod_{i=1}^n (x - \gamma_i)$$

by

$$M(p) := |a| \prod_{i=1}^n \max\{1, |\gamma_i|\},$$

Lehmer [9] asked whether there is an absolute constant $c > 1$ such that any non-cyclotomic integer polynomial p (that is an integer polynomial with at least one root off the unit circle) has Mahler measure $M(p) \geq c$.

Double roots are readily found;

$$\begin{aligned}
(x^3 + x^2 - 1)^2(x^2 - x + 1) &= x^8 + x^7 - x^5 + x^4 - x^2 - x + 1 \\
(x+1)(x^2+1)(x^4-x^3+1)^2 &= x^{11} - x^{10} + x^7 + x^6 - x^3 + x + 1,
\end{aligned}$$

these being the only examples with degree ≤ 12 (up to the substitutions $x \mapsto \pm x, \pm x^{-1}$).

It can be shown from the Bombieri-Vaaler version of Siegel's Lemma [1, Theorem 1] that any polynomial $p(x)$ whose Mahler measure $M(p) < 2^{1/k}$ must be a k 'th order factor of some $\{0, \pm 1\}$ polynomial. In particular there must certainly be $\{0, \pm 1\}$ polynomials with a fourth order factor $p_1(x)$ or $p_2(x)$

$$p_1(x) := x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1,$$

$$p_2(x) := x^{18} - x^{17} + x^{16} - x^{15} - x^{12} + x^{11} - x^{10} + x^9 - x^8 + x^7 - x^6 - x^3 + x^2 - x + 1,$$

where (see Boyd [6]) $p_1(x)$ and $p_2(x)$ are (apart from the trivial substitutions $x \mapsto \pm x^n$) the only known polynomials with integer coefficients and $1 < M(p) <$

$2^{1/4}$. Each has a single root, $1/M(p_1) = .850137\dots$ or $1/M(p_2) = .841490\dots$ respectively, in $|x| < 1$. Similarly, as the smallest known “limit point” of Mahler measures (see Boyd [5])

$$\lim_{n \rightarrow \infty} M(x^{2n} - x^{2n-1} - x^{n+1} + x^n - x^{n-1} - x + 1) = 1.25542\dots$$

is less than $2^{1/3}$, there must be infinitely many genuinely distinct polynomials that appear as triple factors of $\{0, \pm 1\}$ polynomials.

However the situation for $k \geq 5$ remains unresolved. Clearly if the conjectured lower bound of the Lehmer Problem were to prove to be false then we would certainly have roots with arbitrarily high multiplicity. There is no good reason to believe that the converse should be true but it does make the question of existence or nonexistence of arbitrary multiplicity roots of $\{0, \pm 1\}$ polynomials an interesting one.

Notice, since (as is clear from our bounds) $r(k) \rightarrow 1$ as $k \rightarrow \infty$, the maximum multiplicity of a given polynomial factor is certainly bounded; so for example (from the exact values of our table) there can be no $\{0, \pm 1\}$ polynomials with factor $p_1(x)^7$ or $p_2(x)^7$.

Related questions about zeros of $\{0, 1\}$ polynomials can be found in Odlyzko & Poonen [10].

4 A slight generalization

Although we have stated Theorem 3 for power series with coefficients in the interval $[-1, 1]$, a similar result holds if we vary the interval, or if we consider polynomials constructed from a finite set of monomials, (for example zeros of $[0, 1]$ power series have been extensively studied by Solomyak [12] in connection with beta-numbers). Hence we actually prove a slightly more general form of the result:

Given a set S of exponents $\mathcal{E}_S = \{n_1 < n_2 < \dots\}$ and non-trivial bounded intervals $I_i = [u_i, v_i]$ each containing 0, we set

$$\mathcal{F}_S = \left\{ 1 + \sum_{n_i \in \mathcal{E}_S} a_i z^{n_i} : a_i \in [u_i, v_i] \right\},$$

and let $\mathcal{P}_S(k)$ denote the set of k th order real roots $\alpha > 0$ of a series f_α in \mathcal{F}_S . We similarly define $r_S(k)$ to be the infimum $\mathcal{P}_S(k)$ (whenever this set is non-empty).

Theorem 4 *If $\mathcal{P}_S(k) \neq \emptyset$ then there exists a unique β in $\mathcal{P}_S(k)$ for which there is a set of $(k-1)$ exponents $1 \leq m_1 < \dots < m_{k-1}$ in S such that the coefficients of the corresponding power series*

$$f_\beta(x) = 1 + \sum_{n_i \in S} b_i x^{n_i}, \quad b_i \in [u_i, v_i]$$

satisfy

$$b_j = \begin{cases} u_i & \text{if } n_j \in (0, m_1) \cup \dots \cup (m_{2i}, m_{2i+1}) \cup \dots \\ v_i & \text{if } n_j \in (m_1, m_2) \cup \dots \cup (m_{2i+1}, m_{2i+2}) \cup \dots \end{cases}$$

Moreover

$$\beta = r_S(k),$$

and the series f_β is unique.

Of course in general we cannot ensure that $\mathcal{P}_S(k)$ is non-empty. However if we fix the intervals $I_i := [-g, g]$ for some $g > 0$ then in the polynomial case, (i.e. $|\mathcal{E}_S| < \infty$), we have $\mathcal{P}_S(k) \neq \emptyset$ iff $|\mathcal{E}_S| \geq k$ (see for example Lemma 1) and in the power series case (i.e. $\mathcal{E}_S = \mathbf{N}$) we have $r_S(k) < 1$ for all k . Indeed in this latter case,

$$\mathcal{F}_{S_g} := \{1 + \sum_{j=1}^{\infty} a_j x^j : a_i \in [-g, g]\}$$

we show the following analogues of the upper bound of Theorem 1

$$r_{S_g}(k) \leq \left(1 - \frac{1}{k+1}\right)^{\min\{g/9, 1\}} \quad (3)$$

and the lower bound of Theorem 3

$$r_{S_g}(k) \geq \left(1 + \frac{1}{k}\right)^{-1/2} (g^2 k + 1)^{-1/2k} \quad (4)$$

– where again this latter bound is a bound on the radius of a disc containing k roots rather than specifically multiple roots.

5 The Proofs

At several points in the proofs we shall need to construct polynomials with a given k th order root:

Lemma 1 *For any set of $(k+1)$ distinct integers $m_1 < m_2 < \dots < m_{k+1}$ and non-zero ξ there is a polynomial*

$$p(x; \mathbf{m}, \xi) = x^{m_1} - c_2 x^{m_2} + c_3 x^{m_3} - \dots + (-1)^k c_{k+1} x^{m_{k+1}},$$

given by

$$c_i = c_i(\mathbf{m}, \xi) := \left(\prod_{\substack{j=2 \\ j \neq i}}^{k+1} \frac{(m_j - m_1)}{|m_i - m_j|} \right) \xi^{m_1 - m_i}, \quad i = 2, 3, \dots, k+1$$

with a k th order zero at ξ .

Proof: The proof may be regarded as a straightforward exercise in Vandermonde determinants with the c_i 's the solution of

$$\begin{pmatrix} 1 & \dots & 1 \\ (m_2 - m_1) & \dots & (m_{k+1} - m_1) \\ \vdots & & \vdots \\ (m_2 - m_1)^{k-1} & \dots & (m_{k+1} - m_1)^{k-1} \end{pmatrix} \begin{pmatrix} c_2 \xi^{m_2 - m_1} \\ \vdots \\ (-1)^{k+1} c_{k+1} \xi^{m_{k+1} - m_1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

or as an exercise on the Residue Theorem on observing that

$$(\xi x)^{-m_1} p(\xi x; \mathbf{m}, \xi) = \prod_{j=2}^{k+1} (m_1 - m_j) \frac{1}{2\pi i} \int_{\Gamma} \frac{x^t}{\prod_{j=1}^{k+1} (t - (m_j - m_1))} dt,$$

where Γ is a simple closed contour surrounding the zeros of the denominator (the k th order zero of this integral at $x = 1$ becomes apparent by repeated differentiation and evaluation by expanding the contour to infinity).

Notice that we cannot construct a polynomial with a k th root at $\xi \neq 0$ and fewer than $(k+1)$ terms (since

$$\begin{pmatrix} 1 & \dots & 1 \\ m_1 & \dots & m_k \\ \vdots & & \vdots \\ m_1^{k-1} & \dots & m_k^{k-1} \end{pmatrix} \begin{pmatrix} c_1 \xi^{m_1} \\ \vdots \\ c_k \xi^{m_k} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

implies that $c_i \xi^{m_i} = 0$ for all $i = 1, \dots, k$, by the non-vanishing of the Vandermonde determinant). ■

Proof of Theorem 1: The proof resembles the construction of $[-1, 1]$ -polynomials with a high multiplicity root at 1 recently considered by Borwein-Erdélyi-Kós [4] (see [3, §3.4, Ex.9] for details).

For positive integers $k \geq 1$ we obtain from Lemma 1 a polynomial

$$P_k(x) := 1 + \sum_{i=1}^k (-1)^i d_{i,k} x^{i^2},$$

$$d_{i,k} := \frac{2(k!)^2}{(k-i)!(k+i)!} \left(1 + \frac{1}{k}\right)^{i^2}$$

with a k -fold root at $k/(k+1)$ whose coefficients alternate in sign, satisfy $d_{1,k} = 2$, and for $k \geq i > 1$ decrease in magnitude;

$$\frac{d_{i,k}}{d_{i-1,k}} = \left(\frac{k+1}{k}\right)^{2i-1} \left(\frac{k-i+1}{k+i}\right) < 1,$$

(To see this set

$$w_k(x) := \left(\frac{k+1}{k}\right)^{2x-1} \left(\frac{k-x+1}{k+x}\right)$$

and observe that for $x > 1$ and $k \geq 2$

$$\begin{aligned} \frac{w'_k(x)}{w_k(x)} &= 2 \log \left(1 + \frac{1}{k}\right) - \frac{(2k+1)}{k(k+1) - x(x-1)} \\ &< 2 \left(\frac{1}{k} - \frac{1}{2k^2} + \frac{1}{3k^3}\right) - \frac{(2k+1)}{k(k+1)} = \frac{(2-k)}{3k^3(k+1)} \leq 0. \end{aligned}$$

Finally we observe that

$$\begin{aligned} f_k(x) &:= \frac{P_k(x)}{(1-x)} \\ &= 1 + \sum_{i=1}^{k-1} b_{i,k} (x^{i^2} + x^{i^2+1} + \dots + x^{(i+1)^2-1}) + b_{k,k} (x^{k^2} + x^{k^2+1} + \dots), \end{aligned}$$

is a power series with a root of multiplicity k at $k/(k+1)$ and coefficients

$$b_{i,k} = 1 + \sum_{j=1}^i (-1)^j d_{j,k}$$

satisfying $|b_{i,k}| \leq 1$. Note that, for fixed i , the $d_{i,k} \rightarrow 2$ and hence the $b_{i,k} \rightarrow (-1)^i$ as $k \rightarrow \infty$. ■

Proof of the related bound (3): When $0 < g \leq 1$ setting $M := [9/g]$ a similar appeal to Lemma 1 yields a polynomial

$$\tilde{P}_{k,g}(x) := 1 - \tilde{d}_{0,k} x - \sum_{i=1}^k (-1)^i \tilde{d}_{i,k} x^{Mi^2},$$

with a $(k+1)$ th order root at $x = (1 + 1/k)^{-1/M}$ and (in the notation of the above proof)

$$0 < \tilde{d}_{i,k} := \frac{1}{Mi^2 - 1} d_{i,k} \leq \frac{2}{M-1} < g$$

for $i \geq 1$ and

$$1 < \tilde{d}_{0,k} := \left(1 + \frac{1}{k}\right)^{1/M} \prod_{j=1}^k \left(1 - \frac{1}{Mi^2}\right)^{-1}$$

so that

$$\tilde{d}_{0,k} < \exp\left(\frac{1}{M} \log 2 + 2 \sum_{i=1}^{\infty} \frac{1}{Mi^2}\right) < \exp(4/M) < \exp(g/2) < 1 + g.$$

Dividing by $(1-x)$ we similarly obtain a power series

$$\tilde{f}_{k,g}(x) := \frac{\tilde{P}_{k,g}(x)}{1-x} = 1 + \tilde{b}_{0,k} \frac{(x-x^M)}{(1-x)} + \sum_{i=1}^{k-1} \tilde{b}_{i,k} \frac{(x^{Mi^2} - x^{M(i+1)^2})}{(1-x)} + \tilde{b}_{k,k} \frac{x^{Mk^2}}{(1-x)}$$

where (since the $\tilde{d}_{i,k}$ decrease in magnitude for $i \geq 1$) the coefficients

$$\tilde{b}_{i,k} := 1 - \tilde{d}_{0,k} + \sum_{j=1}^i (-1)^{j+1} \tilde{d}_{j,k}$$

satisfy

$$-g < 1 - \tilde{d}_{0,k} < \tilde{b}_{i,k} < 1 - \tilde{d}_{0,k} + \tilde{d}_{i,k} < \tilde{d}_{1,k} < g. \quad \blacksquare$$

Proof of Theorem 2 and bound (4): We prove the more general bound (4). Suppose that $f(z) = 1 + \sum_{i=1}^{\infty} a_i z^i$ has its coefficients a_i in $[-g, g]$ and k roots inside or on the disc

$$|z| = r < R = \left(\frac{k}{k+1}\right)^{1/2}.$$

Then

$$\|f(Rz)\|_2 \leq \left(1 + \sum_{j=1}^{\infty} g^2 R^{2j}\right)^{1/2} = (1 + g^2 k)^{1/2}.$$

Hence by Jensen's Theorem, the concavity of the logarithm and Parseval's Formula we have

$$\begin{aligned} k \log(R/r) &\leq \sum_{\substack{f(\gamma_i) = 0 \\ |\gamma_i| \leq R}} \log |R/\gamma_i| = \frac{1}{2\pi} \int_{|z|=1} \log |f(Rz)| dz \\ &\leq \log \left(\frac{1}{2\pi} \int_{|z|=1} |f(Rz)|^2 dz \right)^{1/2} \leq \log(g^2 k + 1)^{1/2}, \end{aligned}$$

and (4) and the first inequality in Theorem 2 are plain.

The second inequality in Theorem 2 follows from the rough bounds

$$k^{-1/2k} = \exp\left(-\frac{\log \sqrt{k}}{k}\right) \geq 1 - \frac{\log(\sqrt{k})}{k}, \quad \left(1 + \frac{1}{k}\right)^{-\frac{1}{2}(1+1/k)} \geq \frac{k}{k+1}. \quad \blacksquare$$

For Theorems 3 and 4 we shall need the following Lemma:

Lemma 2 *If γ is the smallest element of $\mathcal{P}_S(k)$ then any corresponding series $f_\gamma(x) = 1 + \sum_{n_i \in S} a_{n_i} x^{n_i}$ in \mathcal{F}_S must have at least one non-zero coefficient at an end point $a_{n_J} = u_J$ or v_J .*

Proof: If $f_\gamma(x)$ is a polynomial then this is almost trivial:

If

$$f_\gamma(x) = 1 + \sum_{i=1}^M a_i x^i,$$

with

$$B := \max\left(\left\{(a_i/v_i)^{1/i} : a_i > 0\right\} \cup \left\{(a_i/u_i)^{1/i} : a_i < 0\right\}\right) < 1$$

then we could construct a polynomial $\tilde{f}_\gamma(x) := f_\gamma(z/B)$ still in \mathcal{F}_S with a root of multiplicity k at γB ; contradicting the minimality of γ .

If $f_\gamma(x)$ is not a polynomial but has all its non-zero coefficients strictly inside the appropriate interval then we show that we can replace $f_\gamma(x)$ by a polynomial with a k th order root at γ and this same property:

Let n_1, \dots, n_k denote the first k elements of S with $a_{n_i} \neq 0$. For any $\mathbf{w} = (w_1, \dots, w_k)$ in \mathbf{R}^k there is certainly an $\alpha(\mathbf{w}) = (\alpha_1, \dots, \alpha_k)$ in \mathbf{R}^k such that

$$\begin{pmatrix} 1 & \cdots & 1 \\ n_1 & \cdots & n_k \\ \vdots & & \vdots \\ n_1^{k-1} & \cdots & n_k^{k-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \gamma^{n_1} \\ \vdots \\ \alpha_k \gamma^{n_k} \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix}$$

and moreover for a sufficiently small constant $\delta = \delta(\gamma, a_{n_1}, \dots, a_{n_k})$ we have

$$\max_{1 \leq i \leq k} |w_i| < \delta \Rightarrow \max_{1 \leq i \leq k} |\alpha_i| < \min_{1 \leq i \leq k} \{v_i - a_{n_i}, a_{n_i} - u_i\}.$$

Now for a sufficiently large N we must be able to set

$$w_i := \sum_{j > N} a_{n_j} n_j^{i-1} \gamma^{n_j}, \quad i = 1, \dots, k,$$

with all the $|w_i| < \delta$. Hence we can truncate $f_\gamma(x)$ at the N th term and replace the remaining series by a polynomial with exponents n_1, \dots, n_k whose $0, 1, \dots, (k-1)$ th derivatives take the same values at γ and whose coefficients are small enough that the resulting polynomial

$$\tilde{f}_\gamma(z) := 1 + \sum_{i=1}^k (a_{n_i} + \alpha_i) x^{n_i} + \sum_{i=k+1}^N a_{n_i} x^{n_i}$$

still has all its non-zero coefficients \tilde{a}_{n_i} strictly inside the given interval and a root of multiplicity k at γ . ■

Proof of Theorems 3 and 4: Suppose that α is a k -fold root of a power series

$$f_\alpha(x) = 1 + \sum_{n_i \in S} a_{n_i} x^{n_i},$$

then, setting

$$F_i(x) := x \frac{d}{dx} F_{i-1}(x), \quad F_0(x) := f_\gamma(x),$$

observe that

$$F_i(\alpha) = 0, \quad i = 0, 1, \dots, k-1. \quad (5)$$

For a subset of $(k-1)$ exponents $S_1 = \{m_1, \dots, m_{k-1}\} \subseteq S$ we define the matrix

$$A_{S_1} := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & m_1 & \cdots & m_{k-1} \\ \vdots & \vdots & & \vdots \\ 0 & m_1^{k-1} & \cdots & m_{k-1}^{k-1} \end{pmatrix}$$

and use D_{ij} to denote the various minors (that is the determinant of the sub-matrix obtained by deleting the i th row and j th column from A_{S_1}).

Using the equations (5) for $i = 1, \dots, k-1$ to eliminate $a_{m_1}, \dots, a_{m_{k-1}}$ from the $i = 0$ equation we obtain

$$1 + \sum_{n_i \in S \setminus S_1} a_{n_i} P_{S_1}(n_i) \alpha^{n_i} = 0 \quad (6)$$

where $P_{S_1}(x)$ is the polynomial

$$P_{S_1}(x) := 1 + \sum_{i=1}^{k-1} (-1)^i \frac{D_{i+1,1}}{D_{11}} x^i.$$

(consider for example the relation

$$\begin{pmatrix} 1 \\ a_{n_1} \alpha^{n_1} \\ \vdots \\ a_{n_{k-1}} \alpha^{n_{k-1}} \end{pmatrix} = A^{-1} \begin{pmatrix} - \sum_{n_j \in S \setminus S_1} a_{n_j} \beta^{n_j} \\ \vdots \\ - \sum_{n_j \in S \setminus S_1} a_{n_j} n_j^{k-1} \beta^{n_j} \end{pmatrix}$$

and expand the expression on the right-hand side for 1).

One readily sees that

$$P_{S_1}(x) = \begin{cases} 0 & \text{if } x \in S_1 \\ > 0 & \text{if } x \in (0, m_1) \cup \cdots \cup (m_{2i}, m_{2i+1}) \cup \cdots =: T_1 \\ < 0 & \text{if } x \in (m_1, m_2) \cup \cdots \cup (m_{2i+1}, m_{2i+2}) \cup \cdots =: T_2. \end{cases}$$

(the vanishing of $P_{S_1}(x)$ at $x = m_i$ is easily seen by replacing the first column of A by the $(i+1)$ th and expanding the zero determinant).

Hence any such α certainly cannot be smaller than the (one) positive real root of

$$1 - \sum_{n_i \in T_1} |u_i| |P_{S_1}(n_i)| x^{n_i} - \sum_{n_i \in T_2} |v_i| |P_{S_1}(n_i)| x^{n_i} = 0. \quad (7)$$

Clearly if β is an element of $\mathcal{P}_S(k)$ whose power series $f_\beta(x)$'s coefficients satisfy

$$a_{n_j} = \begin{cases} u_j & \text{if } n_j \in (0, m_1) \cup \cdots \\ v_j & \text{if } n_j \in (m_1, m_2) \cup \cdots \end{cases}$$

then we attain (7) and β must indeed be the minimal element of $\mathcal{P}_S(k)$. Notice that this β cannot be a root of any other series in \mathcal{F}_S ; since any difference in the a_{n_i} for an n_i in $S \setminus S_1$ and we could not recover a root as small as β from the resulting (7), and the remaining a_{n_i} , n_i in S_1 are fully determined by the a_{n_i} , n_i in $S \setminus S_1$ (given β and \mathbf{m});

$$\begin{pmatrix} a_{m_1} \beta^{m_1} \\ \vdots \\ a_{m_{k-1}} \beta^{m_{k-1}} \end{pmatrix} = \begin{pmatrix} m_1 & \cdots & m_{k-1} \\ \vdots & & \vdots \\ m_1^{k-1} & \cdots & m_{k-1}^{k-1} \end{pmatrix}^{-1} \begin{pmatrix} - \sum_{n_j \in S \setminus S_1} a_{n_j} n_j \beta^{n_j} \\ \vdots \\ - \sum_{n_j \in S \setminus S_1} a_{n_j} n_j^{k-1} \beta^{n_j} \end{pmatrix} \quad (8)$$

Thus there remains to show that if $\gamma := r_S(k)$ then $f_\gamma(x)$ must be of this form.

From Lemma 1, for any vector \mathbf{m} of $(k+1)$ distinct integers $m_1 < m_2 < \cdots < m_{k+1}$ there is a polynomial

$$p(x; \mathbf{m}, \gamma) = x^{m_1} - c_2 x^{m_2} + c_3 x^{m_3} - \cdots + (-1)^{k+1} c_{k+1} x^{m_{k+1}},$$

with a k th order root at γ , where the $c_i = c_i(\mathbf{m}, \gamma)$ are explicit and more importantly positive. We set $C = C(\mathbf{m}, \gamma) := \max_i c_i$.

We first observe that $f_\gamma(x)$ can have at most $(k-1)$ of its a_{n_i} not at the endpoints; since if $a_{n_{r_1}}, \dots, a_{n_{r_k}}$ have $a_{n_{r_i}}$ in (u_{r_i}, v_{r_i}) we could take $\mathbf{m} = (0, n_{r_1}, \dots, n_{r_k})$ and, for a suitably small δ ,

$$0 < \delta < \min_{1 \leq i \leq k} \{v_{r_i} - a_{n_{r_i}}, a_{n_{r_i}} - u_{r_i}\} / C,$$

form a new $\tilde{f}_\gamma(x)$

$$\begin{aligned} \tilde{f}_\gamma(x) &:= (1 - \delta)f_\gamma(x) + \delta p(x; \mathbf{m}, \gamma) \\ &= 1 + \sum_{n_i \neq r_j} (1 - \delta)a_{n_i}x^{n_i} + \sum_{j=1}^k (a_{r_i} - \delta a_{r_i} + (-1)^i \delta c_{i+1})x^{r_i} \end{aligned}$$

with a root of multiplicity k at γ and all its non-zero coefficients \tilde{a}_{n_i} in (u_i, v_i) (in contradiction to the above lemma).

Further we cannot have a sequence of $(k+1)$ exponents

$$n_{l_1} < n_{l_2} < \dots < n_{l_{k+1}}$$

with

$$a_{n_{l_1}} \neq v_{l_1}, a_{n_{l_2}} \neq u_{l_2}, a_{n_{l_3}} \neq v_{l_3}, \dots$$

(or similarly

$$a_{n_{l_1}} \neq u_{l_1}, a_{n_{l_2}} \neq v_{l_2}, a_{n_{l_3}} \neq u_{l_3}, \dots).$$

If such a sequence existed we could take $\mathbf{m} = (n_{l_1}, \dots, n_{l_{k+1}})$ and for any small positive δ

$$0 < \delta < \min\{v_{l_1} - a_{n_{l_1}}, a_{n_{l_2}} - u_{l_2}, v_{l_3} - a_{n_{l_3}}, \dots\} / C$$

the series

$$\tilde{f}_\gamma(x) := f_\gamma(x) + \delta p(x; \mathbf{m}, \gamma)$$

would be in \mathcal{F}_S , have a k th order root at γ , and coefficients $\tilde{a}_{n_{l_i}}, i = 1, \dots, k+1$ in (u_{l_i}, v_{l_i}) ; contradicting our above assertion that all but $(k-1)$ coefficients take the value of an end-point.

Hence we can have at most k blocks of u_i or v_i (with at most $(k-1)$ remaining values not equal to u_i or v_i). But by Descartes Rule of Signs we must have at least k sign changes. Hence, for some set of $(k-1)$ exponents $m_1 < \dots < m_{k-1}$,

$$a_{n_i} = \begin{cases} v_i & \text{if } n_i \in (m_1, m_2) \cup \dots \cup (m_{2i+1}, m_{2i+2}) \cup \dots \\ u_i & \text{if } n_i \in (0, m_1) \cup \dots \cup (m_{2i}, m_{2i+1}) \cup \dots \end{cases}$$

as desired. ■

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k	$r(k)$	\mathbf{m}
1	.5	$\{\}$
2	.64913786	$\{4\}$
3	.72788323	$\{4,10\}$
4	.77732954	$\{4,9,19\}$
5	.81159422	$\{4,9,17,30\}$
6	.83672063	$\{4,9,17,28,45\}$
7	.85581352	$\{4,9,16,27,42,62\}$
8	.87095801	$\{4,9,16,26,40,58,83\}$
9	.88322694	$\{4,9,16,26,39,55,77,106\}$
10	.89334152	$\{4,9,16,26,38,54,73,98,132\}$
11	.90182864	$\{4,9,16,26,38,53,71,94,123,161\}$
12	.90908099	$\{4,9,16,25,37,52,70,91,117,150,194\}$
13	.91533200	$\{4,9,16,25,37,51,69,89,114,143,180,229\}$
14	.92077995	$\{4,9,16,25,37,51,68,88,111,139,172,213,267\}$
15	.92556147	$\{4,9,16,25,37,51,67,87,110,136,167,204,249,308\}$
16	.92979823	$\{4,9,16,25,37,50,67,86,108,134,163,198,239,288,353\}$
17	.93358190	$\{4,9,16,25,36,50,66,85,107,132,161,193,231,276,330,400\}$
18	.93698047	$\{4,9,16,25,36,50,66,85,106,131,158,190,226,268,316,375,451\}$
19	.94004491	$\{4,9,16,25,36,50,66,84,105,129,157,187,222,261,306,359,422,504\}$
20	.94282535	$\{4,9,16,25,36,50,66,84,105,129,155,185,219,257,299,348,405,473,561\}$
21	.94535801	$\{4,9,16,25,36,50,65,84,104,128,154,183,216,253,294,340,393,454,527,620\}$
22	.94767713	$\{4,9,16,25,36,50,65,83,104,127,153,182,214,250,289,334,384,440,505,583,683\}$
23	.94980718	$\{4,9,16,25,36,49,65,83,103,126,152,181,212,247,286,329,376,430,490,560,643,749\}$
24	.95177133	$\{4,9,16,25,36,49,65,83,103,126,151,179,211,245,283,324,371,422,479,543,617,706,818\}$
25	.95358636	$\{4,9,16,25,36,49,65,83,103,125,151,179,209,243,280,321,366,415,470,531,599,678,771,890\}$
26	.95526965	$\{4,9,16,25,36,49,65,83,103,125,150,178,208,241,278,318,362,410,462,521,585,658,741,840,965\}$
27	.95683539	$\{4,9,16,25,36,49,65,82,102,125,150,177,207,240,276,315,358,405,456,512,574,643,720,808,911\}$

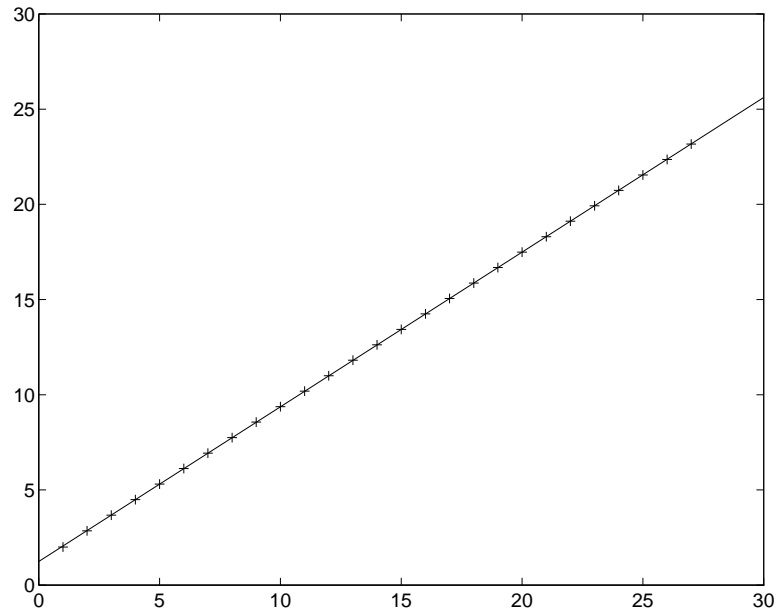


Figure 1: $(1-r(k))^{-1}$ against k with least squares line $1.23909318 + .81255949k$.

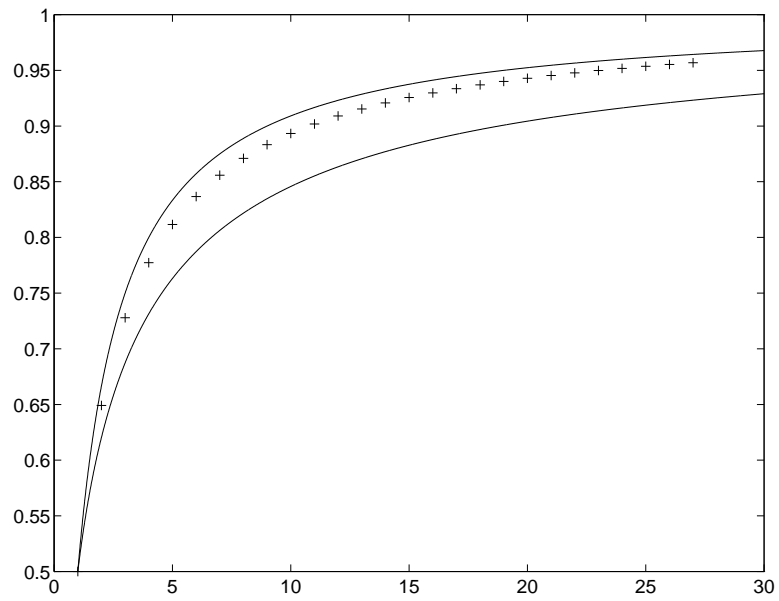


Figure 2: $r(k)$ against k with the upper and lower bounds.