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RATIONAL APPROXIMATIONS TO STIELTJES TRANSFORMS

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1.

Rational sums of the form $\sum a_i/(x+b_i)$ where a_i and b_i are positive can be expressed as Stieltjes transforms of discrete positive measures. The Stieltjes transforms of the measure $\alpha(t)$ is the function

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t}.$$

Rational approximations that interpolate such functions on positive intervals are particularly amenable to analysis because all of the poles of these approximations lie on the negative real axis [1]. Furthermore, if $g(x)$ is the Stieltjes transform of $\beta(t)$, $f(x)$ is the transform of $\alpha(t)$ and if α, β and $\alpha-\beta$ are all positive measures then g can be approximated more closely than f by rational functions on any positive interval (see Theorem 1). We will exploit these two observations to analyse the rate of rational approximation to certain functions of the form $\sum a_i/(x+b_i)$.

Let Π_n denote the real algebraic polynomials of degree at most n. Let $R_{n,m}$ denote the rational functions with numerators in Π_n and denominators in Π_m. Let

$$r_{n,m}(f\colon [a,b]) = \inf_{r \in R_{n,m}} \|f(x)-r(x)\|_{[a,b]}$$

where $\|\cdot\|_{[a,b]}$ denotes the supremum norm on $[a,b]$.

In a seminal paper ([6], see also [7]) Gončar shows that if f is the Stieltjes transform of a positive measure α with support in the interval $[a,b]$, if $\alpha'>0$ almost everywhere on $[a,b]$ and if $c>-a$ then

$$\lim_{n\to\infty} r_{n-1,n}(f\colon [c,d])^{1/n} = \frac{1}{\varrho^2} < 1$$

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where ϱ depends only on a, b, c, and d. These results have been extended by Ganelius [5] who shows that under slightly more restrictive conditions

$$k_1 \leqq r_{n-1,n}(f\colon [c,d])\varrho^{2n} \leqq k_2 .$$

Ganelius [4] also shows that for non-integral positive δ,

$$b_\delta |\sin \pi\delta| \leqq r_{n-1,n}(x^\delta\colon [0,1])e^{2\pi\sqrt{\delta n}} \leqq C_\delta e^{c_\delta n^{1/4}}$$

where b_δ, c_δ, and C_δ depend only on δ. This settles the conjecture of Gončar that

$$\lim n^{-1/2} \ln r_{n-1,n}(x^\delta\colon [0,1]) = -2\pi\sqrt{\delta} . \tag{1}$$

As a corollary to Theorem 1 we deduce that

$$\lim_{n\to\infty} n^{-1/2} \ln r_{n-1,n}(x \ln x\colon [0,1]) = -2\pi .$$

This amounts in some sense to the $\delta = 1$ case of Gončar's conjecture.

In contrast to the above situation we will also consider functions which arise as transforms of discrete measures, that is, functions of the form $\sum_{i=1}^{\infty} a_i/(x+b_i)$, $a_i, b_i \geqq 0$. We will, for example, obtain results of the following nature:

(a) $$r_{n-1,n}\left(\sum_{i=1}^{n+1} \frac{1}{x+i}\colon [0,1]\right) = \frac{a_n}{16^n(n!)^2} \quad \text{where } \lim_{n\to\infty} a_n^{1/2n} = .278\ldots$$

and

(b) $$r_{n-1,n}\left(\sum_{i=1}^{n+1} \frac{1}{x+i^2}\colon [0,1]\right) = \frac{b_n}{16^n(n!)^4} \quad \text{where } \lim_{n\to\infty} b_n^{1/2n} = .439\ldots .$$

The convergence problem for Padé approximants to functions of the form $\sum a_i/(x+b_i)$ is treated by Franzen in [3].

2. A comparison theorem.

A particularly useful theorem in polynomial approximation theory due to Bernstein states that if $|g^{(n+1)}(x)| \leqq f^{(n+1)}(x)$ on $[a,b]$, then the error in best uniform polynomial approximation of degree n to g is no greater than the corresponding error in approximating f. Our first result is a modest extension of this to the case of rational approximations to Stieltjes transforms.

THEOREM 1. *Let*

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t} \quad \textit{and} \quad g(x) = \int_0^\infty \frac{d\beta(t)}{x+t} .$$

Suppose that α, β and $\alpha-\beta$ are all non-negative measures. Suppose that for $k \geqq 0$

$$\frac{q_{n+k-1}(\zeta_i)}{p_n(\zeta_i)} - f(\zeta_i) = 0 = \frac{q^*_{n+k-1}(\zeta_i)}{p^*_n(\zeta_i)} - g(\zeta_i)$$

at $2n+k$ points $0 \leqq \zeta_1 \leqq \zeta_2 \ldots \leqq \zeta_{2n+k}$, where

$$q_{n+k-1}/p_n,\ q^*_{n+k-1}/p^*_n \in R_{n+k-1,n}\,.$$

Then, for $x>0$,

$$\left| g(x) - \frac{q^*_{n+k-1}(x)}{p^*_n(x)} \right| \leqq \left| f(x) - \frac{q_{n+k-1}(x)}{p_n(x)} \right| .$$

PROOF. If k of the ζ_i coincide, then we are assuming that $f-q_{n+k-1}/p_n$ and $g-q^*_{n+k-1}/p^*_n$ have zeros of multiplicity k at those ζ_i.

We may suppose that the ζ_i are distinct and that

$$f(x) = \sum_{i=1}^{\infty} \frac{\gamma_i}{x+\alpha_i} \quad \text{and} \quad g(x) = \sum_{i=1}^{\infty} \frac{\delta_i}{x+\alpha_i}$$

where for all i,

$$0 \leqq \alpha_i < \alpha_{i+1} \quad \text{and} \quad 0 \leqq \delta_i \leqq \gamma_i\,.$$

(The general argument is completed by taking limits.) Let I_f be the index of the first non-zero γ_i, and let I_g be the index of the first non-zero δ_i. Then, if $\beta = \alpha_{I_f}$ and $\beta^* = \alpha_{I_g}$, it follows from results in [1] that

$$\frac{q_{n+k-1}}{p_n} = \bar{q}_{k-1} + \sum_{k=1}^{n} \frac{c_i}{x+d_i} \qquad d_i > \beta,\ c_i > 0$$

and

$$\frac{q^*_{n+k-1}}{p^*_n} = \bar{q}^*_{k-1} + \sum_{i=1}^{n} \frac{e_i}{x+h_i} \qquad h_i > \beta^*,\ e_i > 0$$

where $\bar{q}_{k-1}, \bar{q}^*_{k-1} \in \pi_{k-1}$. $(\pi_{-1} \equiv 0.)$

Furthermore,

$$F(x) := \frac{q_{n+k-1}}{p_n} - f \quad \text{and} \quad G(x) := \frac{q^*_{n+k-1}}{q^*_n} - g$$

have exactly $2n+k$ simple zeros on

$$[-\beta, \infty) \quad \text{and} \quad [-\beta^*, \infty)$$

respectively. Also,

$$\lim_{x \to -\beta^+} F(x) = \lim_{x \to (-\beta^*)^+} G(x) = -\infty\,.$$

It follows that

$$\operatorname{sgn} F(x) = \operatorname{sgn} G(x) \quad \text{for } x \in [0, \infty)\,.$$

If there exists $x_0 > 0$, $x_0 \notin \{\zeta_1, \ldots, \zeta_{2n+k}\}$ so that

$$|F(x_0)| \leqq |G(x_0)| \tag{2}$$

then there exists $c > 1$ so that on $[0, \infty)$

$$cF(x) - G(x) \quad \text{has } 2n+k+1 \text{ zeros}\,.$$

Thus,

$$\frac{cq_{n+k-1}}{p_n} = cf - g + \frac{q^*_{n+k-1}}{p^*_n}$$

has $2n+k+1$ non-negative solutions.

If we differentiate the above k times we see that

$$c\sum_{i=1}^{n} \frac{c_i}{(x+d_i)^{k+1}} = \sum_{i=1}^{\infty} \frac{c\gamma_i - \delta_i}{(x+\alpha_i)^{k+1}} + \sum_{i=1}^{n} \frac{e_i}{(x+h_i)^{k+1}}$$

has $2n+1$ negative solutions. Since $c\gamma_i - \delta_i \geqq 0$, this violates Descartes rule of signs (see [1] for further details). Thus, assumption (2) is not possible and the proof is complete.

The interpolation condition in Theorem 1 is satisfiable for all choices of non-negative ζ_i (see [1]). This observation yields the following corollaries.

COROLLARY 1. *Let*

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t} \quad \text{and} \quad g(x) = \int_0^\infty \frac{d\beta(t)}{x+t}\,.$$

Suppose that α, β and $\alpha-\beta$ are all non-negative measures. Then, for any $n, k, a, b \geqq 0$,

$$r_{n+k-1,n}(g\colon [a,b]) \leqq r_{n+k-1,n}(f\colon [a,b])\,.$$

COROLLARY 2. *Let f and g be as above. Let $p_{n+k-1,n}(f; x)$ be the $(n+k-1, n)$ Padé approximant to f concentrated at the point $a \geqq 0$. Then, for $x \geqq 0$*

$$|g(x) - p_{n+k-1,n}(g; x)| \leqq |f(x) - p_{n+k-1,n}(f; x)|\,.$$

As an application of Theorem 1 we have

COROLLARY 3.

$$\lim_{n\to\infty} n^{-1/2}\ln r_{n,n}(x\ln x\colon [0,1]) = -2\pi .$$

PROOF. For $\delta \in (0,1)$, $x \geqq 0$

$$x^{\delta-1} = \frac{\sin(\delta\pi)}{\pi}\int_0^\infty \frac{t^{\delta-1}dt}{t+x} .$$

Let $s_{n-1,n} \in R_{n-1,n}$ interpolate $x^{\delta-1}$ at any $2n$ points in $(0,1]$ and let $t_{n-1,n} \in R_{n-1,n}$ interpolate

$$\frac{\sin(\delta\pi)}{\pi}(\ln(x+1)-\ln x) = \frac{\sin(\delta\pi)}{\pi}\int_0^1 \frac{dt}{t+x}$$

at the same points. By Theorem 1, for $x>0$,

$$\left|t_{n-1,n}(x)-\frac{\sin(\delta\pi)}{\pi}(\ln(x+1)-\ln x)\right| \leqq |s_{n-1,n}(x)-x^{\delta-1}|$$

and for suitably chosen interpolation points,

$$\begin{aligned}\left|xt_{n-1,n}(x)-\frac{\sin(\delta\pi)}{\pi}(x\ln(x+1)-x\ln x)\right| &\leqq |xs_{n-1,n}(x)-x^{\delta}| \\ &\leqq 2r_{n,n}(x^{\delta}\colon [0,1]) \\ &\leqq C_\delta e^{c_\delta n^{1/4}}/e^{2\pi\sqrt{\delta n}}\end{aligned}$$

where the latter inequality, due to Ganelius, was mentioned in the introduction. Since $x\ln(x+1)$ is analytic in a region containing $[0,1]$ there exists $\varrho<1$ so that

$$r_{n,n}(x\ln(x+1)\colon [0,1]) < \varrho^n$$

and hence

$$r_{n,n}(x\ln x\colon [0,1]) \leqq \left(\frac{C_\delta e^{c_\delta(n-n^{2/3})^{1/4}}}{e^{2\pi\sqrt{\delta(n-n^{2/3})}}}+\varrho^{n^{2/3}}\right)\left(\frac{\pi}{\sin(\delta\pi)}\right).$$

Taking logarithms and letting δ tend to 1 yields

$$\overline{\lim}\, n^{-1/2}\ln r_{n,n}(x\ln x\colon [0,1]) \leqq -2\pi .$$

The lower bound is achieved by observing that

$$\int_0^1 \frac{t^\delta dt}{t+x} = k_\delta x^\delta + f_\delta(x) \qquad \delta \in (0,1)$$

where $f_\delta(x)$ is analytic in $\{|z-1/2|<1\}$. We observe that by Theorem 1 (applied to the above and $\ln(x+1)-\ln x$) we have

$$2r_{n,n}(x\ln(x+1)-x\ln(x)\colon [0,1]) \geqq r_{n,n}(k_\delta x^{\delta+1}+xf_\delta(x)\colon [0,1])$$

and the lower bound is now completed in a similar fashion to the upper bound.

If f and g are Stieltjes transforms of non-negative measures then an immediate consequence of Corollary 1 is that for $a,b\geqq 0$,

$$r_{n+k-1,n}(f+g\colon [a,b]) \geqq \max\left(r_{n+k-1,n}(f\colon [a,b]), r_{n+k-1,n}(g\colon [a,b])\right).$$

Another application of Theorem 1 is

COROLLARY 4. *Suppose that* $0<\gamma_1<\gamma_2<\ldots<\gamma_m<1$ *and suppose that* $c_1,c_2,\ldots,c_m>0$. *Then*

$$\lim_{n\to\infty} n^{-1/2}\ln\left(r_{n,n}\left(\sum_{i=1}^m c_i x^{\gamma_i}\colon [0,1]\right)\right) = -2\pi\sqrt{\gamma_1}\,.$$

PROOF. That the limit exceeds $-2\pi\sqrt{\gamma_1}$ is apparent from the comment preceding the Corollary and (1). To derive an upper bound on the limit we observe once again that

$$c_i x^{\gamma_i} = d_i\int_0^1 \frac{t^{\gamma_i}}{x+t}+h_{\gamma_i}(x)$$

where h_{γ_i} is analytic on $\{|z-1/2|<1\}$. Thus,

$$\sum_{i=1}^m c_i x^{\gamma_i} = \int_0^1 \frac{\sum_{i=1}^n d_i t^{\gamma_i}}{x+t}+h^*(x)$$

where h^* is analytic on $\{|z-1/2|<1\}$. We note that for $t\in[0,1]$,

$$0 < \sum_{i=1}^n d_i t^{\gamma_i} \leqq \left(\sum_{i=1}^n d_i\right)t^{\gamma_1}.$$

We may now compare, as in the proof of Corollary 3, the rational approximation to $\sum_{i=1}^n c_i x^{\gamma_i}$, and the known rational approximation to x^{γ_1}.

3. Approximating rational sums.

We begin by examining rational approximations with n poles to certain rational sums with $n+1$ poles.

THEOREM 2. *Fix* $k\geqq 0$. *Suppose that* $\gamma_i>0$ *and* $\beta_{i+1}>\beta_i\geqq 0$. *Let*

$$f(x) = \sum_{i=1}^{n+1} \frac{\gamma_i}{x+\beta_i}.$$

If $\zeta_1,\ldots,\zeta_{2n+k}$ are $2n+k$ (not necessarily distinct) non-negative points then there exists $p_{n+k-1}/q_n \in R_{n+k,n}$ that interpolates f at each of the ζ_i. Furthermore,

a) $$\frac{p_{n+k-1}(x)}{q_n(x)} = p_{k-1}(x) + \sum_{k=1}^{n} \frac{\delta_i}{x+\alpha_i}$$

where $p_{k-1} \in \pi_{k-1}$, $\delta_i > 0$ and $\beta_i < \alpha_i < \beta_{i+1}$ for $i = 1,\ldots,n$.
Also,

b) $$f(x) - \frac{p_{n+k-1}(x)}{q_n(x)} = \frac{a_n \prod_{i=1}^{2n+k} (x-\zeta_i)}{\prod_{i=1}^{n} (x+\alpha_i) \prod_{i=1}^{n+1} (x+\beta_i)}$$

where, for all j,

$$|a_n| = \frac{\left| \prod_{\substack{i=1 \\ i \neq j}}^{n+1} (\beta_i - \beta_j) \prod_{i=1}^{n} (a_i - \beta_j) \right| \gamma_j}{\left| \prod_{i=1}^{2n+k} (\beta_j + \zeta_i) \right|}.$$

Furthermore, if $k=0$, then $|a_n| \leqq \gamma_{n+1}$.

Proof. Part a) can be found in [1]. Part b) is straightforward since $f - p_{n+k-1}/q_n$ is an element of $R_{2n+k,2n+1}$ with $2n+k$ zeros at the ζ_i and $2n+1$ poles at $-\alpha_i$ and $-\beta_i$. The bound on a_n is obtained by observing that

$$\lim_{x \to -\beta_j} (x+\beta_j)\left(f - \frac{p_{n+k-1}}{q_n}\right) = \gamma_j = \frac{a_n \prod (\beta_j + \zeta_i)(-1)^k}{\prod_{i \neq j} (\beta_i - \beta_j) \prod (\alpha_i - \beta_j)}.$$

When $k=0$ the right hand side of the above equation has absolute value greater than a_n for $j = n+1$.

For certain choices of β_i we can be more precise.

Example 1. Fix $c, k \geqq 0$. Let $\sum_{i=1}^{n} \delta_i/(x+\alpha_i) + p_{k-1}(x)$, $p_{k-1} \in \Pi_{k-1}$, interpolate $\sum_{i=1}^{n+1} 1/(x+i)$ at $2n+k$ points $\zeta_1,\ldots,\zeta_{2n+k} \in [0,c]$.
Then,

$$\left|\sum_{i=1}^{n+1} \frac{1}{x+i} - \left(\sum_{i=1}^{n} \frac{\delta_i}{x+\alpha_i} + p_{k-1}(x)\right)\right| = \frac{a_n \left|\prod_{i=1}^{2n+k} (x-\zeta_i)\right|}{\left|\prod_{i=1}^{n+1} (x+i) \prod_{i=1}^{n} (x+\alpha_i)\right|}$$

and independent of the choice of the ξ_i,

$$\lim_{n\to\infty} |a_n|^{1/2n} = .27846\ldots = v$$

where v is the solution of $ve^{1+v}=1$.

EXAMPLE 2. Fix $c, k \geqq 0$. Let $\zeta_1, \ldots, \zeta_{2n+k}$ be $2n+k$ points in $[0, c]$. Then there exists $\delta_i, \alpha_i > 0$, p_{k-1}, $i^2 < \alpha_i < (i+1)^2$ so that

$$\left|\sum_{i=1}^{n+1} \frac{1}{x+i^2} - \left(\sum_{i=1}^{n} \frac{\delta_i}{x+\alpha_i} + p_{k-1}(x)\right)\right| = \frac{a_n \left|\prod_{i=1}^{2n+k} (x-\zeta_k)\right|}{\left|\prod_{i=1}^{n+1} (x+i^2) \prod_{i=1}^{n} (x+\alpha_i)\right|}$$

where, independent of the choice of the ζ_i,

$$\lim_{n\to\infty} |a_n|^{1/2n} = .439\ldots = 1/\eta^2 - 1$$

where $\eta = .8335\ldots$ is the solution of $(1+\eta)/(1-\eta) = e^{2/\eta}$.

Both examples are consequences of Theorem 2. It is essentially just a calculus exercise to estimate the size of a_n. The two results (a) and (b) of the introduction follow from these examples by chosing the ζ_i to be the roots of the Čebyšev polynomial of degree $2n$ shifted to the interval $[0, 1]$.

COMMENT. For the circle $C = \{|z| = 1\}$ we observe the following. Suppose that

$$f(z) = \sum_{i=1}^{n+1} \frac{1}{z+\alpha_i} \qquad \alpha_{i+1} \geqq \alpha_i + 1 \geqq 2$$

and suppose that $p_{n+k-1}/q_n \in R_{n+k-1,n}$ interpolates $f(z)$ at $2n+k$ points $\zeta_1, \ldots, \zeta_{2n+k} \in [-1, 1]$. Then, by Theorem 2,

$$\frac{\min_{z\in C} |f(z) - p_{n+k-1}(z)/q_n(z)|}{\max_{z\in C} |f(z) - p_{n+k-1}(z)/q_n(z)|} \geqq \frac{\min_{z\in C} \left|\prod_{i=1}^{2n+k} (z-\zeta_i)\right|}{(\alpha_{n+1})^4 \max_{z\in C} \left|\prod_{i=1}^{2n+k} (z-\zeta_i)\right|}.$$

It follows from Rouche's theorem (see [2] for details) that if we choose

p^*_{n+k-1}/q^*_n to be the $(n+k-1, n)$ Padé approximant (i.e. $\zeta_i \equiv 0$) then p^*_{n+k-1}/q^*_n is, up to a multiple of $1/(\alpha_{n+1})^4$, as efficient as a best rational approximation of corresponding degree in the sense that

$$\|f - p^*_{n+k-1}/q^*_n\|_C \geqq r_{n+k-1,n}(f\colon C) \geqq \frac{1}{(\alpha_{n+1})^4}\|f - p^*_{n+k-1}/q^*_n\|_C\,.$$

We can use the previous results to get upper estimates for approximations to $\sum_{i=1}^{\infty} \delta_i/(x+\alpha_i)$.

THEOREM 3. *If*

$$f(x) = \sum_{i=1}^{\infty} \frac{\delta_i}{x+\alpha_i} \qquad 0 \leqq \delta_i \leqq 1,\ \ 1 \leqq c \leqq \alpha_i < \alpha_{i+1}$$

then

$$r_{n-1,n}(f\colon [0,1]) \leqq \left(\frac{c^2}{c^2-1}\right)\frac{2f(0)}{4^{2n-1}\left(\prod_{i=1}^{n} \alpha_i\right)^2}\,.$$

PROOF. Let $s_m \in R_{n+m-1,n+m}$ interpolate f at the $2n-1$ zeros of the $(2n-1)$th Čebyšev polynomial T_{2n-1} shifted to $[0,1]$ and also $2m+1$ times at zero. Note that, as in the proof of Theorem 1,

$$s_m = \sum_{i=1}^{n+m} \frac{\gamma_i^m}{x+\beta_i^m}$$

where $\beta_i^m > \alpha_i$ and

$$\sum_{i=1}^{n+m} \frac{\gamma_i^m}{\beta_i^m} = f(0)\,.$$

In particular, since each $\gamma_i^m, \beta_i^m \geqq 0$, we have for each m

$$\frac{\gamma_{n+m}^m}{\beta_{n+m}^m} \leqq f(0)\,.$$

From Theorem 2 we deduce that for $x \in [0,1]$

$$|s_{m+1}(x) - s_m(x)| \leqq \frac{\gamma_{n+m+1}^{m+1}|T_{2n-1}(x)|}{\left(\prod_{i=1}^{n+m} (\beta_i^{m+1})\right)^2 \beta_{n+m+1}^{m+1}}$$

$$\leqq \frac{f(0)|T_{2n-1}(x)|}{\prod_{i=1}^{n+m} \alpha_i^2}$$

$$\leqq \frac{2f(0)}{4^{2n-1}\prod_{i=1}^{n+m}\alpha_i^2}.$$

We finish by observing that

$$\|f-s_0\|_{[0,1]} \leqq \sum_{m=0}^{\infty} \|s_{i+1}-s_i\|_{[0,1]}.$$

We note that

$$e^z = 1+\frac{2z}{2-z+2z^2\sum_{n=1}^{\infty}\frac{1}{z^2+(2\pi n)^2}}.$$

By Theorem 3, there exists C_1 so that

$$r_{2n-2,2n}\left(\sum_{n=1}^{\infty}\frac{1}{z^2+(2\pi n)^2}:[-1,1]\right) \leqq \frac{C_1}{4^{2n}(2\pi)^{4n}(n!)^4}$$

and hence, there exists C_2 so that

$$r_{2n+1,2n+1}(e^z:[-1,1]) \leqq \frac{C_2}{4^{2n}(2\pi)^{4n}(n!)^4}.$$

This implies that

$$\overline{\lim}\,(n!\,n!\,r_{n,n}(e^z:[-1,1]))^{1/n} \leqq \frac{1}{(2\pi)^2} \leqq \frac{1}{39.4}.$$

This should be compared to the "correct" result due to Németh [8]

$$\lim_{n\to\infty}(n!\,n!\,r_{n,n}(e^z:[-1,1])^{1/n} = \tfrac{1}{64}.$$

Thus, our method yields good but inexact upper bounds for $r_{n,n}$.

It is apparent from Theorem 1 that if $0<c_1\leqq\gamma_i\leqq c_2$ and $\alpha_i\geqq 0$ then, on positive intervals,

$$c_1 r_{n+k,n}\left(\sum_{i=1}^{\infty}\frac{1}{x+\alpha_i}\right) \leqq r_{n+k,n}\left(\sum_{i=1}^{\infty}\frac{\gamma_i}{x+\alpha_i}\right) \leqq c_2 r_{n+k,n}\left(\sum_{i=1}^{\infty}\frac{1}{x+\alpha_i}\right)$$

and that

$$r_{n+k,n}\left(\sum_{i=1}^{n+1}\frac{1}{x+\alpha_i}\right) \leqq r_{n+k,n}\left(\sum_{i=1}^{\infty}\frac{1}{x+\alpha_i}\right).$$

Lower bounds for rational approximation to $\sum_{i=1}^{n+1} 1/(x+\alpha_i)$ will depend critically on the spacing of the α_i. However, the technique presented in this section can be extended to many more special cases.

ADDED IN PROOF. It has come to the author's attention that a version of Theorem 1 is derived by D. Braess in Numer. Math. 22 (1974), 219–232.

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