

2. If $x = 2 \sin \theta$, $dx = 2 \cos \theta d\theta$ and $\theta = \sin^{-1} \frac{x}{2}$.

Note $\sin \theta = \frac{x}{2}$, so $\cos \theta = \frac{\sqrt{4-x^2}}{2}$ and $\tan \theta = \frac{x}{\sqrt{4-x^2}}$.

$$\begin{aligned} \int_0^2 x^3 \sqrt{4-x^2} dx &= \int_0^{\pi/2} (2 \sin \theta)^3 (2 \cos \theta) (2 \cos \theta d\theta) = \int_0^{\pi/2} 32 \sin^3 \theta \cos^2 \theta d\theta = \\ &= \int_0^{\pi/2} 32 \sin \theta (1 - \cos^2 \theta) \cos^2 \theta d\theta = \int_0^{\pi/2} 32 (\cos^2 \theta - \cos^4 \theta) \sin \theta d\theta = \\ &= 32 \left(-\frac{1}{3} \cos^3 \theta + \frac{1}{5} \cos^5 \theta \right) \Big|_0^{\pi/2} = \frac{64}{15}. \end{aligned}$$

Alternatively if $x = 2 \tanh u$, $dx = 2 \operatorname{sech}^2 u du$, and $u = \tanh^{-1} \frac{x}{2} = \frac{1}{2} \ln \frac{1 + \frac{x}{2}}{1 - \frac{x}{2}} =$

$$= \frac{1}{2} \ln \frac{2+x}{2-x}. \text{ Note } \operatorname{sech} u = \frac{\sqrt{4-x^2}}{2}, \cosh u = \frac{2}{\sqrt{4-x^2}}, \text{ and } \sinh u = \frac{x}{\sqrt{4-x^2}}.$$

When $x = 0$, $u = 0$. As $x \rightarrow 2^-$, $u \rightarrow +\infty$.

$$\begin{aligned} \int_0^2 x^3 \sqrt{4-x^2} dx &= \int_0^{+\infty} (2 \tanh u)^3 (2 \operatorname{sech} u) (2 \operatorname{sech}^2 u du) = \int_0^{+\infty} 32 \tanh^3 u \operatorname{sech}^3 u du = \\ &= \int_0^{+\infty} 32 \tanh^2 u \operatorname{sech}^2 u (\operatorname{sech} u \tanh u du) = \int_0^{+\infty} 32 (1 - \operatorname{sech}^2 u) (\operatorname{sech}^2 u) (\operatorname{sech} u \tanh u du) = \\ &= \int_0^{+\infty} 32 (\operatorname{sech}^2 u - \operatorname{sech}^4 u) (\operatorname{sech} u \tanh u du) = -32 \left(\frac{1}{3} \operatorname{sech}^3 u - \frac{1}{5} \operatorname{sech}^5 u \right) \Big|_0^{+\infty} = \frac{64}{15}. \end{aligned}$$

8. If $x = \tan \theta$, $dx = \sec^2 \theta d\theta$ and $\theta = \tan^{-1} x$.

When $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \frac{\pi}{4}$.

Note $\cos \theta = \frac{1}{\sqrt{x^2+1}}$ and $\sin \theta = \frac{x}{\sqrt{x^2+1}}$, while $\tan \theta = x$.

$$\begin{aligned} \int_0^1 \sqrt{x^2+1} dx &= \int_0^{\pi/4} \sec \theta (\sec^2 \theta d\theta) = \int_0^{\pi/4} \sec^3 \theta d\theta = \\ &= \frac{1}{2} \left(\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right) \Big|_0^{\pi/4} = \frac{1}{2} \left(\sqrt{2} + \ln(\sqrt{2}+1) \right). \end{aligned}$$

Alternatively if $x = \sinh u$, $dx = \cosh u du$ and $u = \sinh^{-1} x = \ln(x + \sqrt{x^2+1})$.

Note $\cosh u = \sqrt{x^2+1}$. When $x = 0$, $u = 0$ and $\cosh u = 1$.

When $x = 1$, $u = \sinh^{-1} 1 = \ln(1 + \sqrt{1^2+1}) = \ln(\sqrt{2}+1)$ and $\cosh u = \sqrt{2}$.

$$\int_0^1 \sqrt{x^2+1} dx = \int_0^{\sinh^{-1} 1} \cosh u (\cosh u du) = \frac{1}{2} (\cosh u \sinh u + u) \Big|_0^{\sinh^{-1} 1} = \frac{1}{2} \left(\sqrt{2} + \ln(\sqrt{2}+1) \right).$$

12. If $x = \frac{3}{4} \sec \theta$, $dx = \frac{3}{4} \sec \theta \tan \theta d\theta$ and $\theta = \sec^{-1} \frac{4}{3} x$.

Note $\sec \theta = \frac{4x}{3}$, $\cos \theta = \frac{3}{4x}$, $\sin \theta = \frac{\sqrt{16x^2 - 9}}{4x}$, and thus $\tan \theta = \frac{\sqrt{16x^2 - 9}}{3}$.

$$\int \frac{dx}{x^2 \sqrt{16x^2 - 9}} = \int \frac{\frac{3}{4} \sec \theta \tan \theta d\theta}{\left(\frac{3}{4} \sec \theta\right)^2 \cdot 3 \tan \theta} = \int \frac{4}{9} \cos \theta d\theta = \frac{4}{9} \sin \theta + C = \frac{\sqrt{16x^2 - 9}}{9x} + C.$$

Alternatively if $x = \frac{3}{4} \cosh u$ then $dx = \frac{3}{4} \sinh u du$.

Note $\cosh u = \frac{4x}{3}$, so $u = \cosh^{-1} \frac{4x}{3} = \ln \left(\frac{4x}{3} + \sqrt{\left(\frac{4x}{3}\right)^2 - 1} \right) = \ln \frac{4x + \sqrt{16x^2 - 9}}{3}$.

Since $\sinh u = \frac{\sqrt{16x^2 - 9}}{3}$, $\tanh u = \frac{\sqrt{16x^2 - 9}}{4x}$.

$$\int \frac{dx}{x^2 \sqrt{16x^2 - 9}} = \int \frac{\frac{3}{4} \sinh u du}{\left(\frac{3}{4} \cosh u\right)^2 \cdot 3 \sinh u} = \int \frac{4}{9} \operatorname{sech}^2 u du = \frac{4}{9} \tanh u + C = \frac{\sqrt{16x^2 - 9}}{9x} + C.$$

16. If $x = 2 \tan \theta$, $dx = 2 \sec^2 \theta d\theta$ and $\theta = \tan^{-1} \frac{x}{2}$.

Note $\tan \theta = \frac{x}{2}$, $\sin \theta = \frac{x}{\sqrt{x^2 + 4}}$, and $\cos \theta = \frac{2}{\sqrt{x^2 + 4}}$.

$$\begin{aligned} \int \frac{x}{(x^2 + 4)^{5/2}} dx &= \int \frac{(2 \tan \theta)(2 \sec^2 \theta d\theta)}{32 \sec^5 \theta} = \frac{1}{8} \int \cos^2 \theta \sin \theta d\theta = -\frac{1}{24} \cos^3 \theta + C = \\ &= -\frac{1}{24} \left(\frac{2}{\sqrt{x^2 + 4}} \right)^3 + C = -\frac{1}{3(x^2 + 4)^{3/2}} + C. \end{aligned}$$

Alternatively if $x = 2 \sinh u$, then $dx = 2 \cosh u du$, $\sinh u = \frac{x}{2}$,

$u = \sinh^{-1} \frac{x}{2} = \ln \left(\frac{x}{2} + \sqrt{\left(\frac{x}{2}\right)^2 + 1} \right) = \ln \frac{x + \sqrt{x^2 + 4}}{2}$, and $\cosh u = \frac{\sqrt{x^2 + 4}}{2}$.

$$\begin{aligned} \int \frac{x}{(x^2 + 4)^{5/2}} dx &= \int \frac{(2 \sinh u)(2 \cosh u du)}{(2 \cosh u)^5} = \int \frac{1}{8} \frac{\sinh u}{(\cosh u)^4} = -\frac{1}{24} (\cosh u)^{-3} + C = \\ &= -\frac{1}{3(x^2 + 4)^{3/2}} + C. \end{aligned}$$

Alternatively just make the substitution $u = x^2 + 4$, $du = 2x dx$, and integrate directly.

26. Notice that $5 - 4x - x^2 = 9 - (x + 2)^2$.

If $x + 2 = 3 \sin \theta$, $dx = 3 \cos \theta d\theta$ and $\theta = \sin^{-1} \frac{x+2}{3}$.

Note $\sin \theta = \frac{x+2}{3}$ and $\cos \theta = \frac{\sqrt{9 - (x+2)^2}}{3} = \frac{\sqrt{5 - 4x - x^2}}{3}$.

$$\begin{aligned} \int \frac{dx}{(5 - 4x - x^2)^{5/2}} &= \int \frac{3 \cos \theta d\theta}{(3 \cos \theta)^5} = \int \frac{1}{81} \sec^4 \theta d\theta = \int \frac{1}{81} (1 + \tan^2 \theta) \sec^2 \theta d\theta = \\ &= \frac{1}{81} \left(\tan \theta + \frac{1}{3} \tan^3 \theta \right) + C = \frac{1}{81} \left(\frac{x+2}{\sqrt{5 - 4x - x^2}} + \frac{1}{3} \frac{(x+2)^3}{(5 - 4x - x^2)^{3/2}} \right) + C. \end{aligned}$$

Alternatively if $x + 2 = 3 \tanh u$, then $u = \tanh^{-1} \frac{x+2}{3} = \frac{1}{2} \ln \frac{1 + \frac{x+2}{3}}{1 - \frac{x+2}{3}} = \frac{1}{2} \ln \frac{5+x}{1-x}$ and

$dx = 3 \operatorname{sech}^2 u du$. Note $\tanh u = \frac{x+2}{3}$, $\operatorname{sech} u = \sqrt{1 - \left(\frac{x+2}{3}\right)^2} = \frac{\sqrt{5 - 4x - x^2}}{3}$,

$\cosh u = \frac{3}{\sqrt{5 - 4x - x^2}}$, and $\sinh u = \frac{x+2}{\sqrt{5 - 4x - x^2}}$.

$$\begin{aligned} \int \frac{dx}{(5 - 4x - x^2)^{5/2}} &= \int \frac{3 \operatorname{sech}^2 u du}{(3 \operatorname{sech} u)^5} = \int \frac{1}{81} \cosh^3 u du = \int \frac{1}{81} (1 + \sinh^2 u) \cosh u du = \\ &= \frac{1}{81} \left(\sinh u + \frac{1}{3} \sinh^3 u \right) + C = \frac{1}{81} \left(\frac{x+2}{\sqrt{5 - 4x - x^2}} + \frac{1}{3} \frac{(x+2)^3}{(5 - 4x - x^2)^{3/2}} \right) + C. \end{aligned}$$

30. (a) If $x = a \tan \theta$, $dx = a \sec^2 \theta d\theta$ and $\theta = \tan^{-1} \frac{x}{a}$.

Note $\tan \theta = \frac{x}{a}$, $\sin \theta = \frac{x}{\sqrt{x^2 + a^2}}$, and $\cos \theta = \frac{a}{\sqrt{x^2 + a^2}}$.

$$\begin{aligned} \int \frac{x^2}{(x^2 + a^2)^{3/2}} dx &= \int \frac{(a \tan \theta)^2 (a \sec^2 \theta d\theta)}{(a \sec \theta)^3} = \int \frac{\sin^2 \theta}{\cos \theta} d\theta = \int \frac{1 - \cos^2 \theta}{\cos \theta} d\theta = \\ &= \int (\sec \theta - \cos \theta) d\theta = \ln |\sec \theta + \tan \theta| - \sin \theta + C = \ln \frac{\sqrt{x^2 + a^2} + x}{a} - \frac{x}{\sqrt{x^2 + a^2}} + C. \end{aligned}$$

(b) If $x = a \sinh t$, $t = \sinh^{-1} \frac{x}{a} = \ln \left(\frac{x}{a} + \sqrt{1 + \left(\frac{x}{a}\right)^2} \right) = \ln \frac{x + \sqrt{x^2 + a^2}}{a}$, and

$dx = a \cosh t dt$. Note $\sinh t = \frac{x}{a}$ and $\cosh t = \frac{\sqrt{x^2 + a^2}}{a}$ so $\tanh t = \frac{x}{\sqrt{x^2 + a^2}}$.

$$\begin{aligned} \int \frac{x^2}{(x^2 + a^2)^{3/2}} dx &= \int \frac{a^2 \sinh^2 t}{a^3 \cosh^3 t} a \cosh t dt = \int \tanh^2 t dt = \int (1 - \operatorname{sech}^2 t) dt = \\ &= t - \tanh t + C = \ln \frac{x + \sqrt{x^2 + a^2}}{a} - \frac{x}{\sqrt{x^2 + a^2}} + C. \end{aligned}$$

32. The upper half of the hyperbola $9x^2 - 4y^2 = 36$ is $y = \frac{3}{2}\sqrt{x^2 - 4}$. We can take the area of the region between the upper half of the hyperbola and the x-axis, between $x = 2$ and $x = 3$, and double it.

$$\text{Area} = 2 \int_2^3 \frac{3}{2} \sqrt{x^2 - 4} \, dx = 3 \int_2^3 \sqrt{x^2 - 4} \, dx.$$

If $x = 2 \sec \theta$, $dx = 2 \sec \theta \tan \theta \, d\theta$ and $\theta = \sec^{-1} \frac{x}{2}$. Note $\sec \theta = \frac{x}{2}$, $\cos \theta = \frac{2}{x}$, $\sin \theta = \frac{\sqrt{x^2 - 4}}{x}$, and $\tan \theta = \frac{\sqrt{x^2 - 4}}{2}$.

$$\text{Let } \alpha = \sec^{-1} \frac{3}{2} = \cos^{-1} \frac{2}{3}.$$

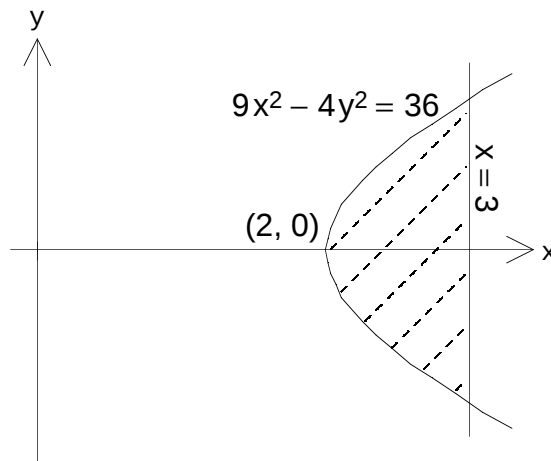
$$\begin{aligned} \text{Area} &= 3 \int_2^3 \sqrt{x^2 - 4} \, dx = 3 \int_0^\alpha (2 \tan \theta) (2 \sec \theta \tan \theta \, d\theta) = 12 \int_0^\alpha \tan^2 \theta \sec \theta \, d\theta = \\ &= 12 \int_0^\alpha (\sec^2 \theta - 1) \sec \theta \, d\theta = 12 \int_0^\alpha (\sec^3 \theta - \sec \theta) \, d\theta = \\ &= 12 \left(\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| - \ln |\sec \theta + \tan \theta| \right) \Big|_0^\alpha = \\ &= 6 \left(\sec \theta \tan \theta - \ln |\sec \theta + \tan \theta| \right) \Big|_0^\alpha = 6 \left(\frac{3\sqrt{5}}{4} - \ln \frac{3+\sqrt{5}}{2} \right) = \frac{9}{2} \sqrt{5} - 6 \ln \frac{3+\sqrt{5}}{2}. \end{aligned}$$

Alternatively if $x = 2 \cosh u$, $dx = 2 \sinh u \, du$ and $u = \cosh^{-1} \frac{x}{2} =$

$$= \ln \left(\frac{x}{2} + \sqrt{\left(\frac{x}{2}\right)^2 - 1} \right) = \ln \frac{x + \sqrt{x^2 - 4}}{2}. \text{ Note } \cosh u = \frac{x}{2} \text{ and } \sinh u = \frac{\sqrt{x^2 - 4}}{2}.$$

$$\text{Let } \beta = \ln \frac{3+\sqrt{5}}{2} = \cosh^{-1} \frac{3}{2}.$$

$$\begin{aligned} \text{Area} &= 3 \int_2^3 \sqrt{x^2 - 4} \, dx = 3 \int_0^\beta (2 \sinh u) (2 \sinh u \, du) = 12 \int_0^\beta \sinh^2 u \, du = \\ &= 12 \left(\frac{1}{2} \sinh u \cosh u - \frac{1}{2} u \right) \Big|_0^\beta = 6 \left(\frac{\sqrt{5}}{2} \frac{3}{2} - \ln \frac{3+\sqrt{5}}{2} \right) = \frac{9}{2} \sqrt{5} - 6 \ln \frac{3+\sqrt{5}}{2}. \end{aligned}$$



For Exercise 32

36. We need to look at the circular disk forming a vertical cross-section of the cylindrical tank. If the disk is described by $x^2 + y^2 \leq 25$ then we must find the ratio of the shaded area to that of the whole disk, 25π . It is easiest to integrate with respect to y and determine $A = 2 \int_{-5}^2 \sqrt{25 - y^2} dy$.

If $y = 5 \sin \theta$, $dy = 5 \cos \theta d\theta$, and $\theta = \sin^{-1} \frac{y}{5}$.

When $y = -5$, $\theta = -\frac{\pi}{2}$.

When $y = 2$, $\theta = \sin^{-1} \frac{2}{5} = \alpha$.

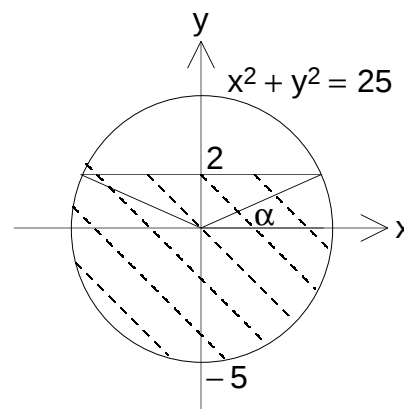
Notice that $\sin \alpha = 0.4$ and $\cos \alpha = \sqrt{0.84}$,

while $\sin(2\alpha) = 2 \sin \alpha \cos \alpha = 0.8\sqrt{0.84}$.

$$\begin{aligned} A &= 2 \int_{-\pi/2}^{\alpha} (5 \cos \theta)(5 \cos \theta d\theta) = 25 \int_{-\pi/2}^{\alpha} (1 + \cos(2\theta)) d\theta = 25 \left(\theta + \frac{1}{2} \sin(2\theta) \right) \Big|_{-\pi/2}^{\alpha} = \\ &= 25 \left(\alpha + \frac{\pi}{2} + \frac{0.8\sqrt{0.84}}{2} \right) = 25 \left(\sin^{-1} 0.4 + \frac{\pi}{2} + \frac{0.8\sqrt{0.84}}{2} \right). \end{aligned}$$

The ratio of this area to the total disk area 25π is $\frac{\sin^{-1} 0.4 + 0.4\sqrt{0.84}}{\pi} + \frac{1}{2} \approx 0.74768412$, or about 75%.

One can find the area A using geometry. It is easy to find both the area of the circular sector with central angle $2\alpha + \pi$ below the isosceles triangle and the area of the triangle itself. If you do this, compare the various terms involved to those in the answer obtained above.



For Exercise 36

$$4. \quad \frac{x^3 - x^2}{(x-6)(5x+2)^3} = \frac{A}{x-6} + \frac{B}{5x+3} + \frac{C}{(5x+3)^2} + \frac{D}{(5x+3)^3}.$$

$$\begin{aligned} 8. \quad \frac{x^4 + x^3 - x^2 - x + 1}{x^3 - x} &= x + 1 + \frac{1}{x^3 - x} = x + 1 + \frac{1}{x(x+1)(x-1)} = \\ &= x + 1 + \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}. \end{aligned}$$

$$12. \quad \frac{1 + 16x}{(2x+3)(x+5)^2(x^2+x+1)} = \frac{A}{2x-3} + \frac{B}{x+5} + \frac{C}{(x+5)^2} + \frac{Dx+E}{x^2+x+1}.$$

$$16. \quad \frac{1}{x^6 - x^3} = \frac{1}{x^3(x^3 - 1)} = \frac{1}{x^3(x-1)(x^2+x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{Ex+F}{x^2+x+1}.$$

$$32. \frac{18-2x-4x^2}{x^3+4x^2+x-6} = \frac{18-2x-4x^2}{(x+3)(x+2)(x-1)} = \frac{A}{x+3} + \frac{B}{x+2} + \frac{C}{x-1}.$$

$$\text{So } 18-2x-4x^2 = A(x+2)(x-1) + B(x+3)(x-1) + C(x+3)(x+2).$$

Put $x = -3$ to see that $A = -3$.

Put $x = -2$ to see that $B = -2$.

Put $x = 1$ to see that $C = 1$.

$$\begin{aligned} \int \frac{18-2x-4x^2}{x^3+4x^2+x-6} dx &= \int \left(\frac{-3}{x+3} + \frac{-2}{x+2} + \frac{1}{x-1} \right) dx = \\ &= -3\ln|x+3| - 2\ln|x+2| + \ln|x-1| + K = \ln \frac{|x-1|}{|x+3|^3|x+2|^2} + K. \end{aligned}$$

$$36. \frac{x^3}{(x+1)^3} = 1 + \frac{-3x^2-3x-1}{(x+1)^3} = 1 + \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

$$\text{So } -3x^2-3x-1 = A(x+1)^2 + B(x+1) + C.$$

Put $x = -1$ to see that $C = -1$.

Differentiate to obtain $-6x-3 = 2A(x+1) + B$, and then put $x = -1$ to see that $B = 3$.

Differentiate again to obtain $-6 = 2A$; thus $A = -3$.

$$\int \frac{x^3}{(x+1)^3} dx = \int \left(1 - \frac{3}{x+1} + \frac{3}{(x+1)^2} - \frac{1}{(x+1)^3} \right) dx = x - 3\ln|x+1| - \frac{3}{x+1} + \frac{1}{2(x+1)^2} + K.$$

For an easier way to get the partial fraction expansion write

$$\frac{x^3}{(x+1)^3} = \frac{[(x+1)-1]^3}{(x+1)^3} = \frac{(x+1)^3 - 3(x+1)^2 + 3(x+1) - 1}{(x+1)^3} = 1 - \frac{3}{x+1} + \frac{3}{(x+1)^2} - \frac{1}{(x+1)^3}.$$

$$46. \frac{x^3}{x^3+1} = 1 - \frac{1}{x^3+1} = 1 - \frac{1}{(x+1)(x^2-x+1)} = 1 + \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}.$$

$$\text{So } -1 = A(x^2-x+1) + (Bx+C)(x+1).$$

Put $x = -1$ to see that $A = -\frac{1}{3}$.

$$\text{Replace } A \text{ by } -\frac{1}{3}, \text{ obtaining } \frac{1}{3}x^2 - \frac{1}{3}x - \frac{2}{3} = (Bx+C)(x+1) = Bx^2 + (B+C)x + C.$$

$$\text{Then it is clear that } B = \frac{1}{3} \text{ and } C = -\frac{2}{3}.$$

But $\frac{\frac{1}{3}x - \frac{2}{3}}{x^2 - x + 1}$ is unpleasant to integrate. Replace it by

$$\frac{\frac{1}{6}(2x-1) - \frac{1}{2}}{x^2 - x + 1} = \frac{1}{6} \cdot \frac{2x-1}{x^2-x+1} - \frac{1}{2} \cdot \frac{1}{(x-0.5)^2 + 0.75} = \frac{1}{6} \cdot \frac{2x-1}{x^2-x+1} - \frac{2}{3} \cdot \frac{1}{((2x-1)/\sqrt{3})^2 + 1}.$$

$$\int \frac{x^3}{x^3+1} dx = \int \left(1 - \frac{1}{3} \cdot \frac{1}{x+1} + \frac{1}{6} \cdot \frac{2x-1}{x^2-x+1} - \frac{2}{3} \cdot \frac{1}{((2x-1)/\sqrt{3})^2 + 1} \right) dx =$$

$$= x - \frac{1}{3}\ln|x+1| + \frac{1}{6}\ln(x^2-x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + K.$$

$$60. \frac{2x+1}{4x^2+12x-7} = \frac{\frac{1}{4}(8x+12)-2}{4x^2+12x-7} = \frac{1}{4} \cdot \frac{8x+12}{4x^2+12x-7} - \frac{1}{2} \cdot \frac{1}{x^2+3x-\frac{7}{4}} =$$

$$= \frac{1}{4} \cdot \frac{8x+12}{4x^2+12x-7} - \frac{1}{2} \cdot \frac{1}{\left(x+\frac{3}{2}\right)^2-2^2}.$$

$$\int \frac{2x+1}{4x^2+12x-7} dx = \frac{1}{4} \ln|4x^2+12x-7| - \frac{1}{8} \ln \left| \frac{x-\frac{1}{2}}{x+\frac{7}{2}} \right| + K = \frac{1}{4} \ln|(2x+3)^2-4^2| - \frac{1}{8} \ln \left| \frac{2x-1}{2x+7} \right| + K.$$

If $|2x+3| < 4$, this can be rewritten as $\frac{1}{4} \ln(4^2-(2x+3)^2) + \frac{1}{4} \tanh^{-1} \frac{x+\frac{3}{2}}{2} + K =$

$$= \frac{1}{4} \ln(4^2-(2x+3)^2) + \frac{1}{4} \tanh^{-1} \frac{2x+3}{4} + K.$$

If $|2x+3| > 4$, this can be rewritten as $\frac{1}{4} \ln((2x+3)^2-4^2) + \frac{1}{4} \coth^{-1} \frac{x+\frac{3}{2}}{2} + K =$

$$= \frac{1}{4} \ln((2x+3)^2-4^2) + \frac{1}{4} \coth^{-1} \frac{2x+3}{4} + K.$$

64. Using the method of disks, if the region under the curve $y = \frac{1}{x^2+3x+2}$ from $x = 0$ to $x = 1$ is rotated about the x-axis the resulting volume is

$$V = \pi \int_0^1 \left(\frac{1}{x^2+3x+2} \right)^2 dx = \pi \int_0^1 \frac{1}{(x+1)^2(x+2)^2} dx.$$

$$\frac{1}{(x+1)^2(x+2)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2}.$$

$$\text{So } A(x+1)(x+2)^2 + B(x+2)^2 + C(x+1)^2(x+2) + D(x+1)^2 = 1.$$

Putting $x = -1$, $B = 1$; putting $x = -2$, $D = 1$.

$$\text{Differentiating, } A(x+2)(3x+4) + 2B(x+2) + C(x+1)(3x+5) + 2D(x+1) = 0.$$

$$\text{Putting } x = -1, A + 2B = 0 \text{ so } A = -2B = -2; \text{ putting } x = -2, C - 2D = 0 \text{ so } C = 2D = 2.$$

$$\text{Thus } \frac{1}{(x+1)^2(x+2)^2} = \frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{2}{x+2} + \frac{1}{(x+2)^2}.$$

Integrating, the volume is

$$V = \pi \left(-2 \ln(x+1) - \frac{1}{x+1} + 2 \ln(x+2) - \frac{1}{x+2} \right) \Big|_0^1 = \pi \left(\ln \frac{9}{16} + \frac{2}{3} \right).$$

The method of cylindrical shells could also be used, but one would have to treat the subintervals $0 \leq y \leq \frac{1}{6}$ and $\frac{1}{6} \leq y \leq \frac{1}{2}$ separately, and for the latter region expressing x as a function of y would involve a square root with radicand containing $\frac{1}{y}$, an unpleasant prospect.

2. If $u = \sqrt[3]{x}$, $x = u^3$ and $dx = 3u^2 du$.

$$\int_0^1 \frac{1}{1+\sqrt[3]{x}} dx = \int_0^1 \frac{1}{1+u} \cdot 3u^2 du = \int_0^1 \left(3u - 3 + \frac{3}{1+u} \right) du = \left(\frac{3}{2} u^2 - 3u + 3 \ln(1+u) \right) \Big|_0^1 = -\frac{3}{2} + 3 \ln 2.$$

6. If $u = \sqrt{x+2}$, $x = u^2 - 2$ and $dx = 2u du$.

$$\int \frac{1}{x - \sqrt{x+2}} dx = \int \frac{2u du}{u^2 - u - 2} = \int \left(\frac{4/3}{u-2} + \frac{2/3}{u+1} \right) du = \frac{4}{3} \ln|u-2| + \frac{2}{3} \ln|u+1| + C = \frac{4}{3} \ln|\sqrt{x+2}-2| + \frac{2}{3} \ln|\sqrt{x+2}+1| + C.$$

10. If $u = \sqrt{x}$, $x = u^2$ and $dx = 2u du$.

$$\int_{1/3}^3 \frac{\sqrt{x}}{x^2 + x} dx = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{u}{u^4 + u^2} 2u du = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{2 du}{u^2 + 1} = 2 \tan^{-1} u \Big|_{1/\sqrt{3}}^{\sqrt{3}} = \frac{\pi}{3}.$$

14. If $u = \sqrt[6]{x}$, $x = u^6$ and $dx = 6u^5 du$.

$$\begin{aligned} \int \frac{\sqrt{x}}{\sqrt{x} - \sqrt[3]{x}} dx &= \int \frac{u^3}{u^3 - u^2} \cdot 6u^5 du = \int \frac{6u^6}{u-1} du = \int 6(u^5 + u^4 + u^3 + u^2 + u + 1 + \frac{1}{u-1}) du = \\ &= u^6 + \frac{6}{5} u^5 + \frac{3}{2} u^4 + 2u^3 + 3u^2 + 6u + 6 \ln|u-1| + C = \\ &= x + \frac{6}{5} x^{5/6} + \frac{3}{2} x^{2/3} + 2x^{1/2} + 3x^{1/3} + 6x^{1/6} + 6 \ln|\sqrt[6]{x}-1| + C. \end{aligned}$$

18. If $u = \sin x$, $x = \sin^{-1} u$ and $du = \cos x dx$.

$$\int \frac{\cos x}{\sin^2 x + \sin x} dx = \int \frac{du}{u^2 + u} = \int \left(\frac{1}{u} - \frac{1}{u+1} \right) du = \ln|u| - \ln|u+1| + C = \ln \left| \frac{\sin x}{\sin x + 1} \right| + C.$$

20. If $u = \sqrt{1+e^x}$, $x = \ln(u^2 - 1)$ and $dx = \frac{2u du}{u^2 - 1}$.

$$\int \frac{dx}{\sqrt{1+e^x}} = \int \frac{2u du}{(u^2 - 1)u} = \int \frac{2 du}{u^2 - 1} = \int \left(\frac{1}{u-1} - \frac{1}{u+1} \right) du = \ln|u-1| - \ln|u+1| + C = \ln \frac{\sqrt{1+e^x} - 1}{\sqrt{1+e^x} + 1} + C.$$

26. If $t = \tan \frac{x}{2}$, $x = 2 \tan^{-1} t$ and $dx = \frac{2 dt}{1+t^2}$.

Then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, and $\tan x = \frac{2t}{1-t^2}$.

$$\begin{aligned} \int \frac{dx}{\sin x + \tan x} &= \int \left(\frac{2t}{1+t^2} + \frac{2t}{1-t^2} \right)^{-1} \frac{2 dt}{1+t^2} = \int \frac{(1-t^2) dt}{2t} = \frac{1}{2} \int \left(\frac{1}{t} - t \right) dt = \\ &= \frac{1}{2} \left(\ln|t| - \frac{1}{2} t^2 \right) + C = \frac{1}{2} \ln \left| \tan \frac{x}{2} \right| - \frac{1}{4} \tan^2 \frac{x}{2} + C. \end{aligned}$$