MATHEMATICS 152 98-2 Solutions for Assignment 11

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2. $\sum_{n=1}^{\infty}\left[\frac{2}{n\sqrt{n}}+\frac{3}{n^3}\right]$ is the sum of constant multiples of two p-series, with $p = 1.5$ and $p = 3$, and therefore converges.

4. $\sum_{n=1}^{\infty} n^{-0.99}$ is a p-series with $p = 0.99$ and therefore diverges.

8. $\int_2^{\infty}\frac{dx}{x^2-1} = \lim_{t\to\infty}\int_2^t \frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx = \lim_{t\to\infty}\frac{1}{2}\ln\frac{x-1}{x+1}\Big]_2^t =$

$= \lim_{t\to\infty}\frac{1}{2}\left(\ln\frac{t-1}{t+1}-\ln\frac{1}{3}\right) = \frac{1}{2}\ln 3.$

By the integral test, the series $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$ converges also.

Alternatively, $\frac{1}{n^2-1} = \frac{1}{2}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$ using partial fractions.

So the $(n-1)^{\underline{st}}$ partial sum is $s_n = \sum_{i=2}^{n}\frac{1}{2}\left(\frac{1}{i-1}-\frac{1}{i+1}\right) = \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right).$

Since $\lim_{n\to\infty} s_n = \frac{3}{4}$, the series converges to $\frac{3}{4}$.

14. $\int_1^{\infty}\frac{dx}{4x^2+1} = \lim_{t\to\infty}\frac{1}{2}\tan^{-1}(2x)\Big]_1^t = \lim_{t\to\infty}\frac{1}{2}\left[\tan^{-1}(2t)-\tan^{-1}2\right] = \frac{1}{2}\left(\frac{\pi}{2}-\tan^{-1}2\right).$

By the integral test, the series $\sum_{n=1}^{\infty}\frac{1}{4n^2+1}$ converges also.

18. For $x \ge 3$, $\ln x > 1$ and $\ln(\ln x) > 0$. Applying the integral test,

$\int_3^{\infty}\frac{dx}{x\ln x\ln(\ln x)} = \lim_{t\to\infty}\ln(\ln(\ln x))\Big]_3^t = \lim_{t\to\infty}\left[\ln(\ln(\ln t))-\ln(\ln(\ln 3))\right] = +\infty,$

and therefore the series $\sum_{n=3}^{\infty}\frac{1}{n\ln n\ln(\ln n)}$ diverges to $+\infty$ also.

20. We saw in Exercise 18 that when $p = 1$, $\sum_{n=3}^{\infty} \frac{1}{n \ln n\, [\ln(\ln n)]^p}$ diverges to $+\infty$.

If $p > 0$ and $p \neq 1$ then $\frac{1}{x \ln x\, [\ln(\ln x)]^p}$ is positive, continuous, and decreasing for $x \geq 3$, so we may apply the integral test.

$$\int_3^{\infty} \frac{dx}{x \ln x\, (\ln(\ln x))^p} = \lim_{t\to\infty} \frac{1}{1-p} (\ln(\ln x))^{1-p}\Big]_3^t = \frac{1}{1-p} \lim_{t\to\infty} \left[(\ln(\ln t))^{1-p} - (\ln(\ln 3))^{1-p}\right].$$

This converges to $\frac{[\ln(\ln 3)]^{1-p}}{p-1}$ if $p > 1$ but diverges to $+\infty$ if $0 < p < 1$, so the series $\sum_{n=3}^{\infty} \frac{1}{n \ln n\, [\ln(\ln n)]^p}$ converges if $p > 1$ and diverges to $+\infty$ if $0 < p < 1$.

If $p = 0$ the series becomes $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$ and if we apply the integral test we see that

$\int_3^{\infty} \frac{dx}{x \ln x} = \lim_{t\to\infty} \ln(\ln x)\Big]_3^t = \lim_{t\to\infty} \left[\ln(\ln t) - \ln(\ln 3)\right] = +\infty$. Therefore the series diverges to $+\infty$ in this case also.

If $p < 0$, the integrand $\frac{1}{x \ln x\, [\ln(\ln x)]^p}$ is positive, continuous, and **eventually** decreasing (as soon as $\ln(\ln x)\cdot[\ln x + 1] + p > 0$). Applying the integral test (but not starting at 3; starting wherever we must so that the integrand will be decreasing) we obtain the same indefinite integral $\frac{1}{1-p}[\ln(\ln x)]^{1-p}$ as before and since this diverges to $+\infty$ as $x \to +\infty$, the series diverges to $+\infty$ in this case too.

24. (a) For the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$, $s_{10} = \frac{43635917056897}{40327580160000}$ exactly (I used *Maple*).

To 10 significant figures, $s_{10} \approx 1.082036583$ according to *Maple*; by comparison, the entry in the ninth decimal place is 4 according to my TI–36 calculator, with agreement earlier. Asking Maple for 20 significant figures yields 1.0820365834937565468.

If s is the sum of the infinite series then $s_{10} < s < s_{10} + \int_{10}^{\infty} \frac{dx}{x^4} = s_{10} + \frac{1}{3(10)^3}$ and thus $0 < s - s_{10} < \frac{1}{3000}$.

(b) $s_{10} + \frac{1}{3(11)^3} = s_{10} + \int_{11}^{\infty} \frac{dx}{x^4} < s < s_{10} + \int_{10}^{\infty} \frac{dx}{x^4} = s_{10} + \frac{1}{3(10)^3}$.

To nine decimal places this says $1.082287022 < s < 1.082369917$.

The midpoint of this interval is 1.082328470, so $|s - 1.082328470| \leq .000041448$.

(c) If we want a value of n such that $0 < s - s_n < 0.00001$ then we select n so that $\int_n^{\infty} \frac{dx}{x^4} \leq 0.00001$. This means that we want $\frac{1}{3\cdot n^3} \leq 0.00001$, or $3\cdot n^3 \geq 10^5$.

Since $3\cdot 32^3 = 98304 < 10^5$ and $3\cdot 33^3 = 107811 > 10^5$, we need $n = 33$ terms to be sure that $0 < s - s_n < 0.00001$.

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2. Since $0 \le \frac{3}{4^n+5} \le \frac{3}{4^n}$ and $\sum_{n=1}^{\infty} \frac{3}{4^n}$ is a convergent geometric series, $\sum_{n=1}^{\infty} \frac{3}{4^n+5}$ converges by the comparison test.

6. Since $0 \le \frac{\sin^2 n}{n\sqrt{n}} \le \frac{1}{n^{1.5}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ is a convergent p-series, $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$ converges by the comparison test.

8. Since $0 \le \frac{1}{\sqrt{n(n+1)(n+2)}} \le \frac{1}{n^{1.5}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ is a convergent p-series, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$ converges by the comparison test.

12. Since $0 \le \frac{n}{(n+1)\,2^n} \le \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series, $\sum_{n=1}^{\infty} \frac{n}{(n+1)\,2^n}$ converges by the comparison test.

16. Since $0 \le \frac{\arctan n}{n^4} \le \frac{\pi/2}{n^4}$ and $\sum_{n=1}^{\infty} \frac{\pi/2}{n^4}$ is $(\pi/2)$ times a convergent p-series, $\sum_{n=1}^{\infty} \frac{\arctan n}{n^4}$ converges by the comparison test.

22. Notice that $0 \le \frac{(3n-2)n^2}{n^4+n^2+1} = \frac{3n^3-2n^2}{n^4+n^2+1}$ for all positive integers n.

$$\lim_{n\to\infty} \frac{\frac{3n^3-2n^2}{n^4+n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n^4-2n^3}{n^4+n^2+1} = \lim_{n\to\infty} \frac{3-\frac{2}{n}}{1+\frac{1}{n^2}+\frac{1}{n^4}} = 3.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series, $\sum_{n=1}^{\infty} \frac{3n^3-2n^2}{n^4+n^2+1}$ diverges by the limit form of the comparison test.

26. Notice that $0 \le \dfrac{2n^2+7n}{3^n(n^2+5n-1)}$ for all positive integers n.

$$\lim_{n\to\infty} \frac{\frac{2n^2+7n}{3^n(n^2+5n-1)}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{2n^2+7n}{n^2+5n-1} = \lim_{n\to\infty} \frac{2+\frac{7}{n}}{1+\frac{5}{n}-\frac{1}{n^2}} = 2.$$

Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{3^n}$ is a convergent geometric series, $\sum\limits_{n=1}^{\infty} \dfrac{2n^2+7n}{3^n(n^2+5n-1)}$ converges by the limit form of the comparison test.

30. There are many ways to show that $\sum\limits_{n=1}^{\infty} \dfrac{n!}{n^n}$ converges. Here is one.

Notice that $0 \le \dfrac{n!}{n^n} = \dfrac{1\cdot 2\cdots n}{n\cdot n\cdots n} \le \dfrac{2}{n^2}$ for $n \ge 2$ (and even for $n = 1$).

Since $\sum\limits_{n=1}^{\infty} \dfrac{2}{n^2}$ is a constant multiple of a convergent p-series, $\sum\limits_{n=1}^{\infty} \dfrac{n!}{n^n}$ converges by the comparison test.

34. Since $-1 \le \cos n \le 1$, $\sum\limits_{n=1}^{\infty} \dfrac{1+\cos n}{n^5}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{2}{n^5}$.

My TI–36 calculator gives $s_{10} \approx 1.559723537$, and *Maple* agrees. To 20 significant figures, *Maple* gives 1.5597235374638962825 for s_{10}. To estimate the error,

$$0 < \sum_{n=11}^{\infty} \frac{1+\cos n}{n^5} < \sum_{n=11}^{\infty} \frac{2}{n^5} < \int_{10}^{\infty} \frac{2}{x^5}\,dx = \frac{2}{4\cdot 10^4} = 0.00005.$$