MATHEMATICS 152 98-2 Solutions for Assignment 1

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2. $\sum_{i=1}^{6} \frac{1}{i+1} = \frac{1}{1+1} + \frac{1}{2+1} + \frac{1}{3+1} + \frac{1}{4+1} + \frac{1}{5+1} + \frac{1}{6+1} =$

$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \frac{669}{420} = \frac{223}{140}$.

8. $\sum_{j=n}^{n+3} j^2 = n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 = 4n^2 + 12n + 14.$

14. $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \cdots + \frac{23}{27} = \sum_{i=3}^{23} \frac{i}{i+4}$.

There are other correct answers, for example $\sum_{i=7}^{27} \frac{i-4}{i}$, or $21 - 4\sum_{i=7}^{27} \frac{1}{i}$.

20. $1 - x + x^2 - x^3 + \cdots + (-1)^n x^n = \sum_{i=0}^{n} (-x)^i = \sum_{i=0}^{n} (-1)^i x^i.$

Again there are other correct answers, for example $\sum_{i=1}^{n+1} (-x)^{i-1}$.

24. $\sum_{k=0}^{8} \cos(k\pi) = \cos 0 + \cos \pi + \cdots + \cos(8\pi) = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 = 1.$

Or notice that $\cos(k\pi) = (-1)^k$. Then $\sum_{k=0}^{8} \cos(k\pi) = \sum_{k=0}^{8} (-1)^k = \frac{1-(-1)^9}{1-(-1)} = \frac{2}{2} = 1.$

(This uses the finite geometric series formula $\sum_{i=0}^{n} a^i = \frac{1-a^{n+1}}{1-a}$, valid if $a \neq 1$.

To prove it, multiply $\sum_{i=0}^{n} a^i$ by $(1-a)$ and notice how all the terms cancel except the 1 at the beginning and the $-a^{n+1}$ at the end.)

28. $\sum_{i=-2}^{4} 2^{3-i} = 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 + 2^{-1} = 63.5.$

Alternatively, putting $j = 4 - i$, $\sum_{i=-2}^{4} 2^{3-i} = \sum_{i=0}^{6} 2^{j-1} = \frac{1}{2}\sum_{i=0}^{6} 2^j = \frac{1}{2} \cdot \frac{1-2^7}{1-2} = \frac{127}{2} = 63.5.$

34. $\sum_{i=1}^{n} i(i+1)(i+2) = \sum_{i=1}^{n} (i^3+3i^2+2i) = \sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + 2\sum_{i=1}^{n} i =$

$= \left(\frac{n(n+1)}{2}\right)^2 + 3\,\frac{n(n+1)(2n+1)}{6} + 2\,\frac{n(n+1)}{2} = \frac{n(n+1)}{4}\left[n(n+1)+2(2n+1)+4\right] =$

$= \frac{n(n+1)}{4}(n^2+5n+6) = \frac{n(n+1)(n+2)(n+3)}{4}$.

44. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^3 + 1\right] = \lim_{n\to\infty}\left[\frac{1}{n^4}\sum_{i=1}^{n} i^3 + \frac{1}{n}\sum_{i=1}^{n} 1\right] = \lim_{n\to\infty}\left[\frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2 + \frac{1}{n}\cdot n\right] =$

$= \lim_{n\to\infty}\left[\frac{1}{4}\left(1+\frac{1}{n}\right)^2 + 1\right] = \frac{5}{4}$.

52. (a) $2\sin u\cos v = \sin(u+v) + \sin(u-v)$, as can be seen by expanding $\sin(u+v) = \sin u\cos v + \cos u\sin v$, $\sin(u-v) = \sin u\cos v - \cos u\sin v in u$, adding, and noticing the cancellation. So putting $u = \frac{1}{2}x$ and $v = ix$,

$2\sin\left(\frac{1}{2}x\right)\cos(ix) = \sin\left(\frac{1}{2}x + ix\right) + \sin\left(\frac{1}{2}x - ix\right) = \sin\left(i+\frac{1}{2}\right)x - \sin\left(i-\frac{1}{2}\right)x$.

(b) Summing the result of part (a) from $i = 1$ to $i = n$,

$\sum_{i=1}^{n} 2\sin\left(\frac{1}{2}x\right)\cos(ix) = \sum_{i=1}^{n}\left[\sin\left(i+\frac{1}{2}\right)x - \sin\left(i-\frac{1}{2}\right)x\right] = \sin\left(n+\frac{1}{2}\right)x - \sin\left(\frac{1}{2}x\right)$,

because all the terms cancel except the second part of the first term, $-\sin\left(\frac{1}{2}x\right)$, and the first part of the last term, $\sin\left(n+\frac{1}{2}\right)x$.

If x is not an integer multiple of 2π, so that $\frac{1}{2}x$ is not an integer multiple of π, then we can divide by $2\sin\left(\frac{1}{2}x\right) \neq 0$, and obtain $\sum_{i=1}^{n}\cos(ix) = \frac{\sin\left(n+\frac{1}{2}\right)x - \sin\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)}$.

Putting $u = \frac{1}{2}nx$ and $v = \frac{1}{2}(n+1)x$ in the formula $2\sin u\cos v = \sin(u+v) + \sin(u-v)$,

$2\sin\left(\frac{1}{2}nx\right)\cos\left(\frac{1}{2}(n+1)x\right) = \sin\left(n+\frac{1}{2}\right)x + \sin\left(-\frac{1}{2}x\right) = \sin\left(n+\frac{1}{2}\right)x - \sin\left(\frac{1}{2}x\right)$.

So $\sum_{i=1}^{n}\cos(ix) = \frac{2\sin\left(\frac{1}{2}nx\right)\cos\left(\frac{1}{2}(n+1)x\right)}{2\sin\left(\frac{1}{2}x\right)} = \frac{\sin\left(\frac{1}{2}nx\right)\cos\left(\frac{1}{2}(n+1)x\right)}{\sin\left(\frac{1}{2}x\right)}$.

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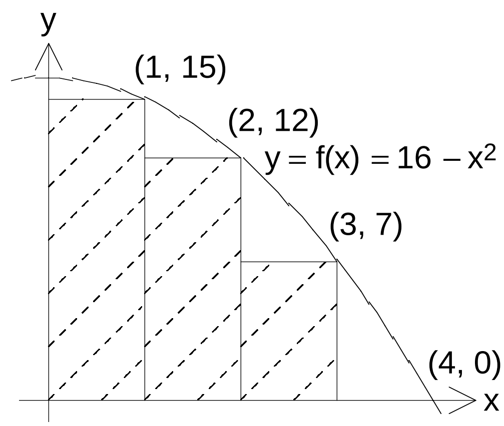
2. $f(x) = 16 - x^2$.
$[a, b] = [0, 4]$.
P is defined by $\{0, 1, 2, 3, 4\}$.
x_i^* is the right-hand endpoint x_i of the i^{th} subinterval $[x_{i-1}, x_i]$.

(a) $\|P\| = Max(1-0, 2-1, 3-2, 4-3) =$
$= Max(1, 1, 1, 1) = 1$.

(b) $\sum_{i=1}^{4} A_i = f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 =$
$= 15\cdot 1 + 12\cdot 1 + 7\cdot 1 + 0\cdot 1 = 34$.

(c) See graph to the right.



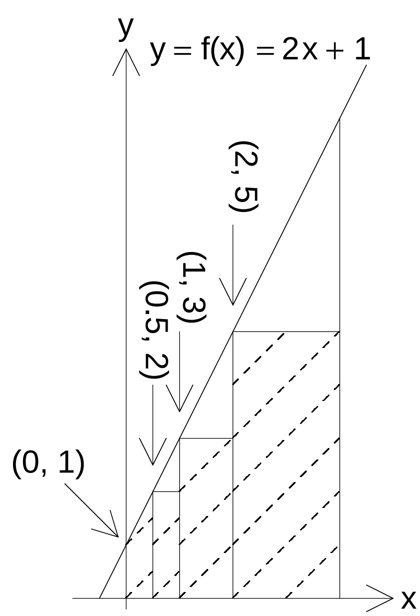
For Exercise 2

4. $f(x) = 2x + 1$.
$[a, b] = [0, 4]$.
P is defined by $\{0, 0.5, 1, 2, 4\}$.
x_i^* is the left-hand endpoint x_{i-1} of the i^{th} subinterval $[x_{i-1}, x_i]$.

(a) $\|P\| = Max(0.5-0, 1-0.5, 2-1, 4-2) =$
$= Max(0.5, 0.5, 1, 2) = 2$.

(b) $\sum_{i=1}^{4} A_i = f(0)\cdot 0.5 + f(0.5)\cdot 0.5 + f(1)\cdot 1 + f(2)\cdot 2 =$
$= 1\cdot 0.5 + 2\cdot 0.5 + 3\cdot 1 + 5\cdot 2 = 14.5$.

(c) See graph below and to the right.



For Exercise 4

6. $f(x) = \frac{1}{x+1}$.

$[a, b] = [0, 2]$.

P is defined by $\{0, 0.5, 1.0, 1.5, 2.0\}$.

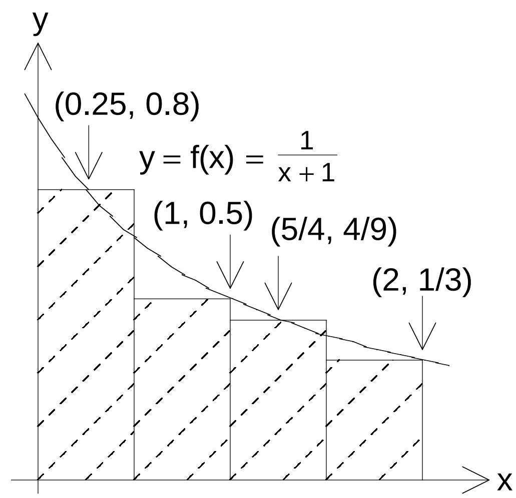
$x_1^* = 0.25$, $x_2^* = 1$, $x_3^* = 1.25$, $x_4^* = 2$.

(a) $\|P\| = \text{Max}\,(0.5, 0.5, 0.5, 0.5) = 0.5$.

(b) $\sum_{i=1}^{4} A_i = [f(0.25) + f(1) + f(1.25) + f(2)]\cdot 0.5 =$

$= \left(0.8 + 0.5 + \frac{4}{9} + \frac{1}{3}\right)\cdot 0.5 = \frac{187}{180} = 1.03\overline{8}$.

(c) See graph on the right.



For Exercise 6

12. $f(x) = x^3$.

$[a, b] = [0, 1]$.

The i^{th} subinterval is $[(i-1)/n, i/n]$.

Its left-hand endpoint is $(i-1)/n$; its right-hand endpoint is i/n; its midpoint is $(2i-1)/2n$.

See graphs below and on the next page.

(a) Using the left-hand endpoints $(i-1)/n$, with $f((i-1)/n) = \left(\frac{i-1}{n}\right)^3$,

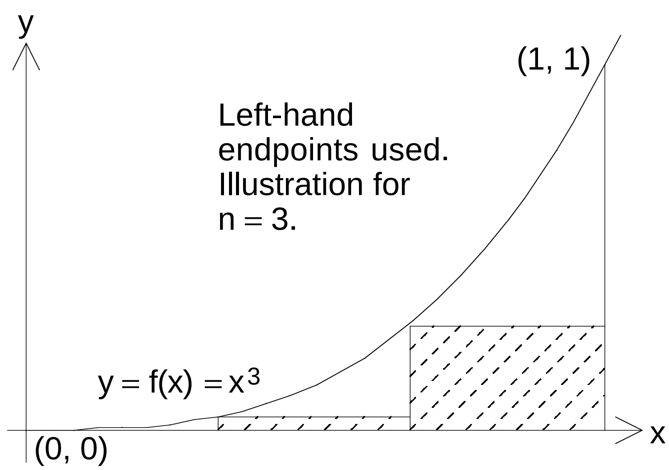
we have $A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^3 \cdot \frac{1}{n} =$

$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{j=1}^{n-1} j^3 = \lim_{n\to\infty} \frac{1}{n^4} \left(\frac{(n-1)n}{2}\right)^2 =$

$= \lim_{n\to\infty} \frac{1}{4} \left(1 - \frac{1}{n}\right)^2 = \frac{1}{4}$.

Here $(i-1)$ has been replaced by j, which goes from 0 to $(n-1)$ when i goes from 1 to n. The $j = 0$ term (which is 0) has been dropped.

See graph to the right.



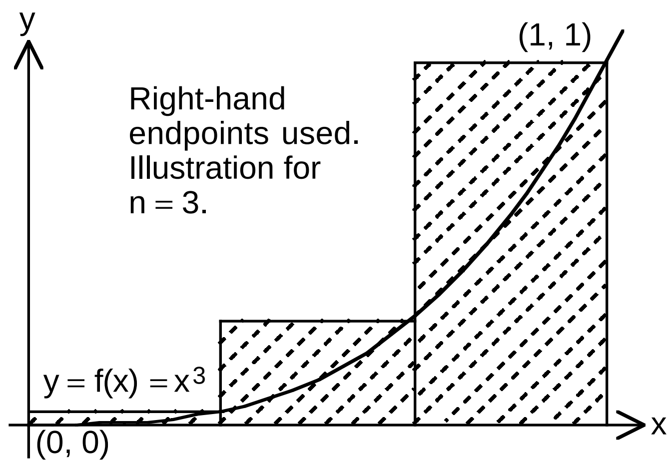
(b) Using the right-hand endpoints i/n, with $f(i/n) = \left(\frac{i}{n}\right)^3$ this time we do not need to make the change of index $j = i - 1$ and the calculation is more straight-forward.

$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \cdot \frac{1}{n} =$

$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3 = \lim_{n\to\infty} \frac{1}{n^4} \left(\frac{n(n+1)}{2}\right)^2 =$

$= \lim_{n\to\infty} \frac{1}{4} \left(1 + \frac{1}{n}\right)^2 = \frac{1}{4}$.

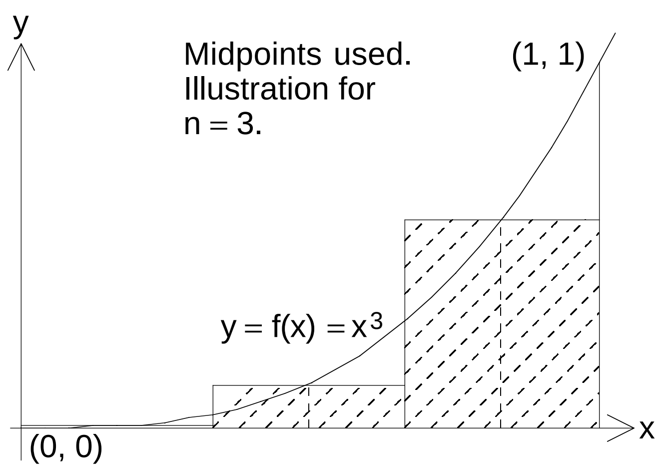
See graph to the right.



(c) Using the midpoints $(2i - 1)/2n$, with $f((2i - 1)/2n) = \left(\frac{2i-1}{2n}\right)^3$, we have $A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i-1}{2n}\right)^3 \cdot \frac{1}{n} =$

$= \lim_{n\to\infty} \frac{1}{8n^4} \sum_{i=1}^{n} (2i - 1)^3$.

Now the problem is to evaluate this sum of cubes of all these **odd** numbers. One way to do that is to sum the cubes of **all** the integers from 1 to $2n$, and then subtract the sum of the cubes of all the **even** integers between 1 and $2n$. Thus $\sum_{i=1}^{n} (2i - 1)^3$



$= \left[1^3 + 2^3 + \cdots + (2n)^3\right] - \left[2^3 + 4^3 + \cdots + (2n)^3\right] =$

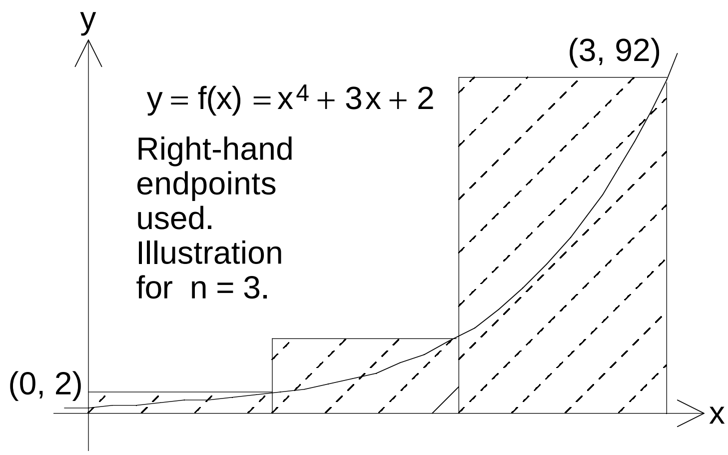
$= \left[1^3 + 2^3 + \cdots + (2n)^3\right] - 2^3\left[1^3 + 2^3 + \cdots + n^3\right] = \left(\frac{(2n)(2n+1)}{2}\right)^2 - 2^3\left(\frac{n(n+1)}{2}\right)^2 =$

$= n^2\left[(2n + 1)^2 - 2(n + 1)^2\right] = n^2(2n^2 - 1)$.

So $A = \lim_{n\to\infty} \frac{1}{8n^4} n^2\left[2n^2 - 1\right] = \frac{1}{4}$. See graph above and to the right.

18. $f(x) = x^4 + 3x + 2$.

Our interval is $[a, b] = [0, 3]$.
The $i^{\underline{th}}$ subinterval is $[3(i-1)/n, 3i/n]$ and has a width of $\frac{3}{n}$.



Since $f(3i/n) =$

$= \left(\frac{3i}{n}\right)^4 + 3\left(\frac{3i}{n}\right) + 2 =$

$= 2 + \frac{9i}{n} + \frac{81i^4}{n^4}$, the area is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[2 + \frac{9i}{n} + \frac{81i^4}{n^4}\right] \cdot \frac{3}{n} =$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{6}{n} + \frac{27i}{n^2} + \frac{243i^4}{n^5}\right] = \lim_{n\to\infty} \left[\frac{6}{n} \cdot n + \frac{27}{n^2} \cdot \frac{n(n+1)}{2} + \frac{243}{n^5} \cdot \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\right] =$$

$$= \lim_{n\to\infty} \left[6 + \frac{27}{2}\left(1 + \frac{1}{n}\right) + \frac{81}{10}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\left(3 + \frac{3}{n} - \frac{1}{n^2}\right)\right] = 68.1.$$

24. One way to interpret the expression $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1+\frac{3i}{n}}$ is as the area of the region between the x-axis and the curve $y = \sqrt{1+x}$ and between the lines $x = 0$ and $x = 3$. Here $x_i = \frac{3i}{n}$ and x_i^* is the right-hand endpoint x_i of the $i^{\underline{th}}$ subinterval $[x_{i-1}, x_i]$. There are other correct interpretations.