MATHEMATICS 152 98-2 Solutions for Assignment 9

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2. If we rotate $y^2 = 4x + 4,\ 0 \le x \le 8$ about the x-axis we can ignore the lower half $y = -2\sqrt{x+1}$, paying attention only to the upper half $y = 2\sqrt{x+1}$. (The two halves of the parabola produce the same surface, one-half turn out of phase with each other.)

If $y = 2\sqrt{x+1}$, $\frac{dy}{dx} = \frac{1}{\sqrt{x+1}}$, and $S = \int_0^8 2\pi \cdot 2\sqrt{x+1}\sqrt{1+\left((x+1)^{-1/2}\right)^2}\,dx =$

$= \int_0^8 4\pi\sqrt{x+2}\,dx = \frac{8\pi}{3}(x+2)^{3/2}\Big]_0^8 = \frac{16\sqrt{2}\,\pi}{3}\left(5\sqrt{5}-1\right).$

Alternatively $x = \frac{y^2-4}{4}$ and $\frac{dx}{dy} = \frac{y}{2}$ so $S = \int_2^6 2\pi y\sqrt{1+\left(\frac{y}{2}\right)^2}\,dy =$

$= \frac{8\pi}{3}\left(1+\left(\frac{y}{2}\right)^2\right)^{3/2}\Bigg]_2^6 = \frac{8\pi}{3}\left(10\sqrt{10}-2\sqrt{2}\right) = \frac{16\sqrt{2}\,\pi}{3}\left(5\sqrt{5}-1\right).$

6. If we rotate $y = \cos x,\ 0 \le x \le \pi/3$ about the x-axis, $\frac{dy}{dx} = -\sin x$.

Thus the surface area is $S = \int_0^{\pi/3} 2\pi\cos x\sqrt{1+(-\sin x)^2}\,dx$.

If we let $u = \sin x$, $du = \cos x\,dx$; $u = 0$ when $x = 0$ and $u = \sqrt{3}/2$ when $x = \pi/3$.

So $S = \int_0^{\sqrt{3}/2} 2\pi\sqrt{u^2+1}\,du = \pi\left(u\sqrt{u^2+1}+\ln\left(u+\sqrt{u^2+1}\right)\right)\Big]_0^{\sqrt{3}/2} = \pi\left(\frac{\sqrt{21}}{4}+\ln\frac{\sqrt{3}+\sqrt{7}}{2}\right).$

Alternatively if we solve for $x = \cos^{-1} y$, then $\frac{dx}{dy} = -\frac{1}{\sqrt{1-y^2}}$ and the surface area is

$S = \int_{1/2}^1 2\pi y\sqrt{\frac{2-y^2}{1-y^2}}\,dy$. Put $u = \sqrt{1-y^2}$, so $du = -\frac{y}{\sqrt{1-y^2}}\,dy$.

Then $u = \sqrt{3}/2$ when $y = 1/2$ and $u = 0$ when $y = 1$. Once again

$S = \int_0^{\sqrt{3}/2} 2\pi\sqrt{u^2+1}\,du = \pi\left(u\sqrt{u^2+1}+\ln\left(u+\sqrt{u^2+1}\right)\right)\Big]_0^{\sqrt{3}/2} = \pi\left(\frac{\sqrt{21}}{4}+\ln\frac{\sqrt{3}+\sqrt{7}}{2}\right).$

14. If we rotate $y = 1 - x^2,\ 0 \le x \le 1$ about the y-axis, $\frac{dy}{dx} = -2x$.

$S = \int_0^1 2\pi x\sqrt{1+(-2x)^2}\,dx = \int_0^1 2\pi x(1+4x^2)^{1/2}\,dx = \frac{\pi}{6}(1+4x^2)^{3/2}\Big]_0^1 = \frac{\pi}{6}\left(5\sqrt{5}-1\right).$

Alternatively $x = \sqrt{1-y}$ and $\frac{dx}{dy} = -\frac{1}{2\sqrt{1-y}}$.

$S = \int_0^1 2\pi\sqrt{1-y}\left(1+\left(-\frac{1}{2\sqrt{1-y}}\right)^2\right)^{1/2} dy = \int_0^1 \pi\sqrt{5-4y}\,dy = -\frac{\pi}{6}(5-4y)^{3/2}\Big]_0^1 = \frac{\pi}{6}\left(5\sqrt{5}-1\right).$

24. Suppose the sphere is formed by rotating $x^2 + y^2 = \left(\frac{D}{2}\right)^2$ about the x-axis in 3-space and the two planes are determined by $y = c$ and $y = c + h$. We can pay attention just to the right half of the circle, $x = \sqrt{\left(\frac{D}{2}\right)^2 - y^2}$, since the two halves of the circle generate the same surface, one-half turn out of phase with each other.

Notice that $2x\frac{dx}{dy} + 2y = 0$ so $\frac{dx}{dy} = -\frac{y}{x}$, and $\frac{ds}{dy} = \sqrt{1+\left(-\frac{y}{x}\right)^2} = \frac{D}{2x}$.

$S = \int_c^{c+h} 2\pi x \cdot \frac{D}{2x}\,dy = \int_c^{c+h} \pi D\,dy = \pi D y\Big]_c^{c+h} = \pi D h.$

28. Let $f(x)$ be positive, with a continuous derivative for $a \le x \le b$.
Let $g(x) = f(x) + c$ for some positive constant c.
Notice that $g'(x) = f'(x)$ so that $\sqrt{1+(g'(x))^2} = \sqrt{1+(f'(x))^2}$.
If S_f is the surface area obtained by rotating $y = f(x)$ around the x-axis and S_g is the surface area obtained by rotating $y = g(x)$ around the x-axis then

$S_g = \int_a^b 2\pi g(x)\sqrt{1+(g'(x))^2}\,dx = \int_a^b 2\pi(f(x)+c)\sqrt{1+(f'(x))^2}\,dx =$

$= \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx + 2\pi c\int_a^b \sqrt{1+(f'(x))^2}\,dx = S_f + 2\pi c L,$

where L is the arc length either along $y = f(x)$ or along $y = g(x)$ between a and b.

The second term can be interpreted as the surface area of a right circular cylinder whose radius is c and whose height is L. You could build such a cylinder, if either of our curves were made out of non-stretching string, by taking one of the curves, straightening it out, placing it along the line $y = c$, and then rotating about the x-axis.

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2. If mass $m_1 = 2$ is located at $P_1 = (5, 1)$, mass $m_2 = 3$ is located at $P_2 = (3, -2)$, and mass $m_3 = 1$ is located at $P_3 = (-2, 4)$, then the total mass is $m = \sum_{i=1}^{3} m_i = 6$, the moment about the x-axis is $M_x = \sum_{i=1}^{3} m_i y_i = 0$, the moment about the y-axis is $M_y = \sum_{i=1}^{3} m_i x_i = 17$, $\bar{x} = \frac{M_y}{m} = \frac{17}{6}$, $\bar{y} = \frac{M_x}{m} = 0$, and the centroid is at $(17/6, 0)$.

8. The moment about the x-axis is

$M_x = \int_0^4 \frac{1}{2}\left(\sqrt{x}\right)^2 dx = \frac{1}{4}x^2\Big]_0^4 = 4,$

the moment about the y-axis is

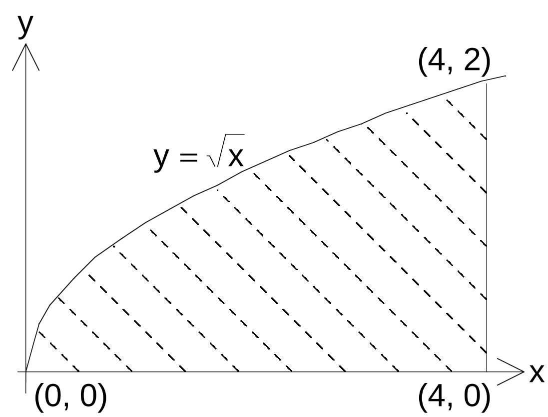
$M_y = \int_0^4 x\sqrt{x}\, dx = \frac{2}{5}x^{5/2}\Big]_0^4 = \frac{64}{5},$ and

the area is $A = \int_0^4 \sqrt{x}\, dx = \frac{2}{3}x^{3/2}\Big]_0^4 = \frac{16}{3}.$

So $\bar{x} = \frac{M_y}{A} = \frac{\frac{64}{5}}{\frac{16}{3}} = 2.4,$

$\bar{y} = \frac{M_x}{A} = \frac{4}{\frac{16}{3}} = 0.75,$ and

the centroid is at $(2.4, 0.75)$.



For Exercise 8

10. The moment about the x-axis is

$M_x = \int_0^{\pi/2} \frac{1}{2}(\sin x)^2\, dx = \int_0^{\pi/2} \frac{1-\cos(2x)}{4}\, dx =$

$= \left(\frac{1}{4}x - \frac{1}{8}\sin(2x)\right)\Big]_0^{\pi/2} = \frac{\pi}{8},$

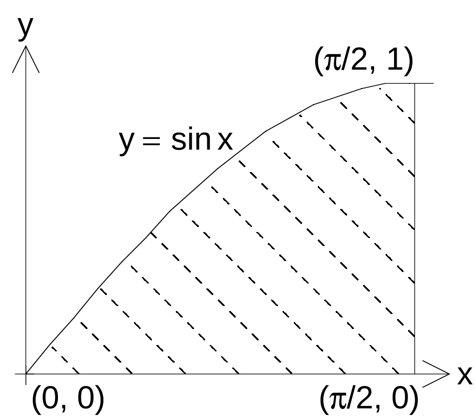
the moment about the y-axis is

$M_y = \int_0^{\pi/2} x\sin x\, dx = [-x\cos x + \sin x]\Big]_0^{\pi/2} = 1,$

and the area is $A = \int_0^{\pi/2} \sin x\, dx = -\cos x\Big]_0^{\pi/2} = 1.$

So $\bar{x} = \frac{M_y}{A} = \frac{1}{1} = 1,$ $\bar{y} = \frac{M_x}{A} = \frac{\frac{\pi}{8}}{1} = \frac{\pi}{8},$

and the centroid is at $(1, \pi/8)$.



For Exercise 10

26. From Example 3, the centroid of a semicircular region of radius r is on the axis of symmetry of the region, at a distance $\frac{4r}{3\pi}$ from the straight base of the region. If we rotate such a region about its base to form a solid sphere, the centroid travels through a distance of $2\pi\frac{4r}{3\pi} = \frac{8}{3}r$. The region's area is $\frac{1}{2}\pi r^2$ so by the Theorem of Pappus the volume of the solid sphere is $V = \frac{1}{2}\pi r^2 \cdot \frac{8}{3}r = \frac{4}{3}\pi r^3$.

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In these exercises g is the acceleration due to gravity (approximately $9.8\ \mathrm{m/s^2}$ at the surface of the earth) and ρ is the mass density of water $(1000\ \mathrm{kg/m^3})$.

4. The width y m below the top of the tank is $2\sqrt{100-y^2}$ m, so the force is

$F = \int_5^{10} \rho g\cdot(y-5)\cdot 2\sqrt{100-y^2}\, dy = \rho g\int_5^{10}\left(2y\sqrt{100-y^2} - 10\sqrt{100-y^2}\right)dy =$

$= \rho g\left(-\frac{2}{3}(100-y^2)^{3/2} - 5y\sqrt{100-y^2} - 500\sin^{-1}\frac{y}{10}\right)\Big]_5^{10} =$

$$= \rho g\left(375\sqrt{3} - \frac{500\pi}{3}\right) \approx 1.23\times 10^6 \text{ N.}$$

8. The width y m below the top of the tank is $\frac{b}{h}y$ m, so the force is

$$F = \int_0^h \rho g y \cdot \frac{b}{h} y\,dy = \frac{\rho g b}{h}\frac{y^3}{3}\Big]_0^h = \frac{\rho g b h^2}{3} \text{ N.}$$

16. The width of the gate y m above its bottom is $2\sqrt{4-y^2}$ m and the corresponding depth below the water surface is $(10 - y)$ m, so the force is

$$F = \int_0^2 \rho g \cdot (10-y)\cdot 2\sqrt{4-y^2}\,dy = \rho g\int_0^2 \left(20\sqrt{4-y^2} - 2y\sqrt{4-y^2}\right)dy =$$

$$= \rho g\left(10y\sqrt{4-y^2} + 40\sin^{-1}\frac{y}{2} + \frac{2}{3}(4-y^2)^{3/2}\right)\Big]_0^2 = \rho g\left(20\pi - \frac{16}{3}\right) \approx 5.63\times 10^5 \text{ N.}$$

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30. $x = t - \frac{1}{t}$, $y = t + \frac{1}{t}$ describes the rectangular hyperbola $y^2 - x^2 = 4$. The hyperbola meets the line $y = 2.5$ where $t + \frac{1}{t} = 2.5$, or $t^2 - 2.5t + 1 = 0$, when $t = 0.5$ or $t = 2$, at the points $(-1.5, 2.5)$ and $(1.5, 2.5)$. The region's area is

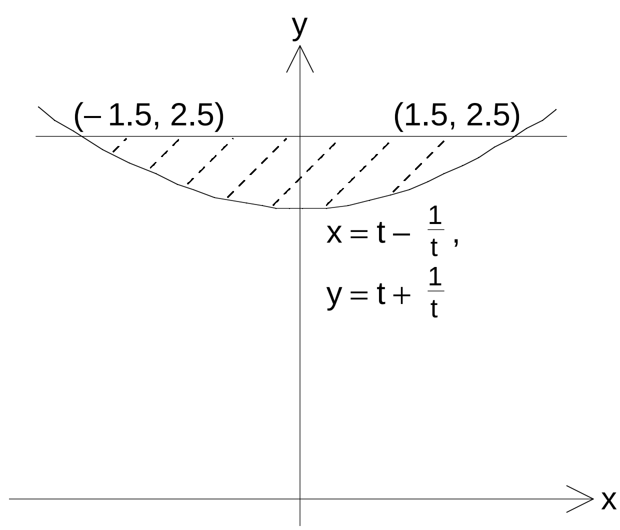
$$A = \int_{0.5}^{2.0}\left[2.5 - \left(t + \frac{1}{t}\right)\right]\cdot\frac{d}{dt}\left(t - \frac{1}{t}\right)dt =$$

$$= \int_{0.5}^{2.0}\left(2.5 - t - \frac{1}{t}\right)\left(1 + \frac{1}{t^2}\right)dt =$$

$$= \int_{0.5}^{2.0}\left[2.5 - t - \frac{2}{t} + \frac{2.5}{t^2} - \frac{1}{t^3}\right]dt =$$

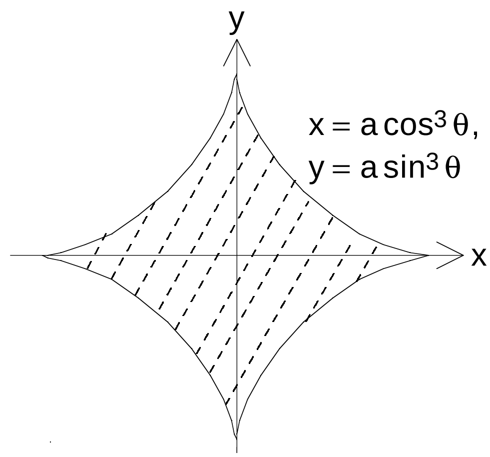
$$= \left[2.5t - \frac{1}{2}t^2 - 2\ln t - \frac{2.5}{t} + \frac{1}{2t^2}\right]\Big]_{0.5}^{2.0} =$$

$$= 3.75 - 2\ln 4.$$



For Exercise 30

32. The astroid $x = a\cos^3\theta,\ y = a\sin^3\theta$ is traversed completely for $0 \le \theta \le 2\pi$. Notice however that the upper half of it is traversed from right to left while the lower half is traversed from left to right, since the motion along the curve is counterclockwise (assuming $a > 0$).



For Exercise 32

So we must calculate $-\int_0^{2\pi} y\,\frac{dx}{dt}\,dt$ to find the area A.

$$A = -\int_0^{2\pi}(a\sin^3\theta)(-3a\cos^2\theta\sin\theta)\,d\theta =$$
$$= \int_0^{2\pi} 3a^2\sin^4\theta\cos^2\theta\,d\theta =$$
$$= \int_0^{2\pi} 3a^2\left[\frac{1-\cos(2\theta)}{2}\right]^2\left[\frac{1+\cos(2\theta)}{2}\right]d\theta =$$
$$= \int_0^{2\pi}\frac{3}{8}a^2\left[1-\cos(2\theta)-\cos^2(2\theta)+\cos^3(2\theta)\right]d\theta =$$
$$= \int_0^{2\pi}\frac{3}{8}a^2\left[1-\cos(2\theta)-\frac{1+\cos(4\theta)}{2}+[1-\sin^2(2\theta)]\cos(2\theta)\right]d\theta =$$
$$= \int_0^{2\pi}\frac{3}{8}a^2\left[\frac{1}{2}-\frac{1}{2}\cos(4\theta)-\sin^2(2\theta)\cos(2\theta)\right]d\theta =$$
$$= \frac{3}{8}a^2\left[\frac{1}{2}\theta-\frac{1}{8}\sin(4\theta)-\frac{1}{6}\sin^3(2\theta)\right]\Big]_0^{2\pi} = \frac{3}{8}\pi a^2.$$

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2. $x = t^2,\ y = 1+4t,\ 0 \le t \le 2$. The integral is easy to evaluate, so I'll do it.

$$L = \int_0^2\sqrt{(2t)^2+4^2}\,dt = \int_0^2 2\sqrt{t^2+4}\,dt = \left(t\sqrt{t^2+4}+4\ln\left|t+\sqrt{t^2+4}\right|\right)\Big]_0^2 = 4\sqrt{2}+4\ln(1+\sqrt{2}).$$

6. $x = a(\cos\theta+\theta\sin\theta),\ y = a(\sin\theta-\theta\cos\theta),\ 0 \le \theta \le \pi$.

$$L = \int_0^\pi\sqrt{(a\theta\cos\theta)^2+(a\theta\sin\theta)^2}\,d\theta = |a|\int_0^\pi\theta\,d\theta = \frac{1}{2}|a|\theta^2\Big]_0^\pi = \frac{1}{2}|a|\pi^2.$$

14. $x = \cos^2 t,\ y = \cos t,\ 0 \le t \le 4\pi$ is part of the parabola $x = y^2$ traced out from $(1, 1)$ to $(1, -1)$ to $(1, 1)$ to $(1, -1)$ to $(1, 1)$. The distance traveled is

$$D = \int_0^{4\pi}\sqrt{(-2\cos t\sin t)^2+(-\sin t)^2}\,dt = \int_0^{4\pi}\sqrt{4\cos^2 t\sin^2 t+\sin^2 t}\,dt = \int_0^{4\pi}|\sin t|\sqrt{4\cos^2 t+1}\,dt.$$

The easiest way to handle this is to integrate from 0 to π (thereby obtaining the arc length) and then quadruple the answer. Let $u = \cos t$ so that $du = -\sin t\,dt$.

$$D = 4\int_0^\pi\sin t\sqrt{4\cos^2 t+1}\,dt = -4\int_1^{-1}\sqrt{4u^2+1}\,du = 8\int_{-1}^{1}\sqrt{u^2+\frac{1}{4}}\,du =$$
$$= 4u\sqrt{u^2+\frac{1}{4}}+\ln\left|u+\sqrt{u^2+\frac{1}{4}}\right|\Big]_{-1}^{1} = 4\sqrt{5}+\ln(9+4\sqrt{5}).$$

16. $x = a\cos^3\theta,\ \ y = a\sin^3\theta,\ \ 0 \le \theta \le 2\pi.$

$$L = \int_0^{2\pi} \sqrt{(-3a\cos^2\theta\sin\theta)^2 + (3a\sin^2\theta\cos\theta)^2}\, d\theta = \int_0^{2\pi} 3|a|\sqrt{\cos^4\theta\sin^2\theta + \cos^2\theta\sin^4\theta}\, d\theta =$$
$$= \int_0^{2\pi} 3|a|\sqrt{\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)}\, d\theta = \int_0^{2\pi} 3|a|\sqrt{\cos^2\theta\sin^2\theta}\, d\theta =$$
$$= \int_0^{2\pi} 3|a||\cos\theta\sin\theta|\, d\theta = \int_0^{2\pi} 1.5|a||\sin(2\theta)|\, d\theta = \int_0^{\pi/2} 6|a||\sin(2\theta)|\, d\theta = -3|a|\cos(2\theta)\Big]_0^{\pi/2} = 6|a|.$$

22. $x = 3t - t^3,\ \ y = 3t^2,\ \ 0 \le t \le 1$ rotated about the x-axis has surface area

$$S = \int_0^1 2\pi\cdot 3t^2\sqrt{(3-3t^2)^2 + (6t)^2}\, dt = \int_0^1 6\pi t^2\sqrt{9 - 18t^2 + 9t^4 + 36t^2}\, dt =$$
$$= \int_0^1 6\pi t^2\sqrt{9 + 18t^2 + 9t^4}\, dt = \int_0^1 6\pi t^2(3 + 3t^2)\, dt = \int_0^1 18\pi(t^2 + t^4)\, dt =$$
$$= 18\pi\left[\frac{1}{3}t^3 + \frac{1}{5}t^5\right]\Big]_0^1 = \frac{48}{5}\pi.$$

26. $x = e^t - t,\ \ y = 4e^{t/2},\ \ 0 \le t \le 1$ rotated about the y-axis has surface area

$$S = \int_0^1 2\pi\cdot(e^t - t)\sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\, dt = \int_0^1 2\pi\cdot(e^t - t)\sqrt{e^{2t} - 2e^t + 1 + 4e^t}\, dt =$$
$$= \int_0^1 2\pi\cdot(e^t - t)\sqrt{e^{2t} + 2e^t + 1}\, dt = \int_0^1 2\pi(e^t - t)(e^t + 1)\, dt = \int_0^1 2\pi[e^{2t} + e^t - te^t - t]\, dt =$$
$$= 2\pi\left(\frac{1}{2}e^{2t} + 2e^t - te^t - \frac{1}{2}t^2\right)\Big]_0^1 = 2\pi\left(\frac{1}{2}e^2 + e - 3\right) = \pi(e^2 + 2e - 6).$$