MATHEMATICS 152 98-2 Solutions for Assignment 8

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4. $\int_{2}^{+\infty} \frac{1}{(x+3)^{3/2}}\,dx = \lim_{t\to+\infty}\int_{2}^{t} \frac{1}{(x+3)^{3/2}}\,dx = \lim_{t\to+\infty}\left(-\frac{2}{(x+3)^{1/2}}\right)\Bigg]_{2}^{t} =$

$= \lim_{t\to+\infty}\left(-\frac{2}{(t+3)^{1/2}}+\frac{2}{5^{1/2}}\right)=\frac{2}{\sqrt{5}}.$

6. $\int_{-\infty}^{-1} \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{s\to-\infty}\int_{s}^{-1} \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{s\to-\infty} \frac{3}{2}(x-1)^{2/3}\Big]_{s}^{-1} =$

$= \lim_{s\to-\infty}\left(\frac{3}{2}\left((-2)^{2/3}-(s-1)^{2/3}\right)\right)=-\infty.$

12. $\int_{-\infty}^{+\infty} x^2e^{-x^3}\,dx = \int_{-\infty}^{0} x^2e^{-x^3}\,dx + \int_{0}^{+\infty} x^2e^{-x^3}\,dx =$

$= \lim_{s\to-\infty}\int_{s}^{0} x^2e^{-x^3}\,dx + \lim_{t\to+\infty}\int_{0}^{t} x^2e^{-x^3}\,dx = \lim_{s\to-\infty}\left[-\frac{1}{3}e^{-x^3}\right]\Bigg]_{s}^{0} + \lim_{t\to+\infty}\left[-\frac{1}{3}e^{-x^3}\right]\Bigg]_{0}^{t} =$

$= \lim_{s\to-\infty}\left[-\frac{1}{3}\left[1-e^{-s^3}\right]\right] + \lim_{t\to+\infty}\left[-\frac{1}{3}\left[e^{-t^3}-1\right]\right] = +\infty$

because of the first term; the second term is just $\frac{1}{3}$.

20. $\int_{0}^{+\infty} xe^{-x}\,dx = \lim_{t\to+\infty}\int_{0}^{t} xe^{-x}\,dx = \lim_{t\to+\infty}\left[-xe^{-x}-e^{-x}\right]\Big]_{0}^{t} = \lim_{t\to+\infty}\left[-\frac{t+1}{e^t}+1\right]=$

$= \lim_{t\to+\infty}\left[-\frac{1}{e^t}+1\right]=1$, by L'Hospital's Rule.

26. $\int_{1}^{+\infty} \frac{\ln x}{x^3}\,dx = \lim_{t\to+\infty}\int_{1}^{t} \frac{\ln x}{x^3}\,dx = \lim_{t\to+\infty}\left[-\frac{1}{4}\,\frac{2\ln x+1}{x^2}\right]\Bigg]_{1}^{t} = \lim_{t\to+\infty}\left[-\frac{1}{4}\left\{\frac{2\ln t+1}{t^2}-1\right\}\right]=$

$= \lim_{t\to+\infty}\left[-\frac{1}{4}\left\{\frac{2t^{-1}}{2t}-1\right\}\right]=\frac{1}{4}$, by L'Hospital's Rule.

32. $\int_{0}^{2} \frac{1}{4x-5}\,dx = \int_{0}^{5/4} \frac{1}{4x-5}\,dx + \int_{5/4}^{2} \frac{1}{4x-5}\,dx.$

$\int_{0}^{5/4} \frac{1}{4x-5}\,dx = \lim_{s\to(5/4)^-}\int_{0}^{s} \frac{1}{4x-5}\,dx = \lim_{s\to(5/4)^-} \frac{1}{4}\ln|4x-5|\Big]_{0}^{s} = -\infty.$

Likewise $\int_{5/4}^{2} \frac{1}{4x-5}\,dx = \lim_{t\to(5/4)^+}\int_{t}^{2} \frac{1}{4x-5}\,dx = \lim_{t\to(5/4)^+} \frac{1}{4}\ln|4x-5|\Big]_{t}^{2} = +\infty.$

But that doesn't mean that $\int_{0}^{2} \frac{1}{4x-5}\,dx = 0$; you can't do arithmetic with ∞ that way.

38. $\int_0^4 \frac{dx}{x^2+x-6} = \int_0^4 \left(\frac{0.2}{x-2} - \frac{0.2}{x+3}\right) dx = \int_0^2 \frac{0.2}{x-2}\,dx + \int_2^4 \frac{0.2}{x-2}\,dx - \int_0^4 \frac{0.2}{x+3}\,dx.$

$\int_0^2 \frac{0.2}{x-2}\,dx = \lim_{s\to 2^-} \int_0^s \frac{0.2}{x-2}\,dx = \lim_{s\to 2^-} 0.2\ln|x-2|\,\Big]_0^s = -\infty.$

Likewise $\int_2^4 \frac{0.2}{x-2}\,dx = \lim_{t\to 2^+} \int_t^4 \frac{0.2}{x-2}\,dx = \lim_{t\to 2^+} 0.2\ln|x-2|\,\Big]_t^4 = +\infty.$

Although $\int_0^4 \frac{0.2}{x+3}dx = 0.2\ln|x+3|\,\Big]_0^4 = 0.2\ln\frac{7}{3}$ is tame that doesn't help; all three integrals would have to converge to make our original improper integral $\int_0^4 \frac{dx}{x^2+x-6}$ exist.

42. $\int_0^1 \frac{\ln x}{\sqrt{x}}dx = \lim_{s\to 0^+} \int_s^1 \frac{\ln x}{\sqrt{x}}dx = \lim_{s\to 0^+} 2\sqrt{x}\,(\ln x - 2)\,\Big]_s^1 = \lim_{s\to 0^+} \left(-4 - 2\sqrt{s}\,(\ln s - 2)\right) =$

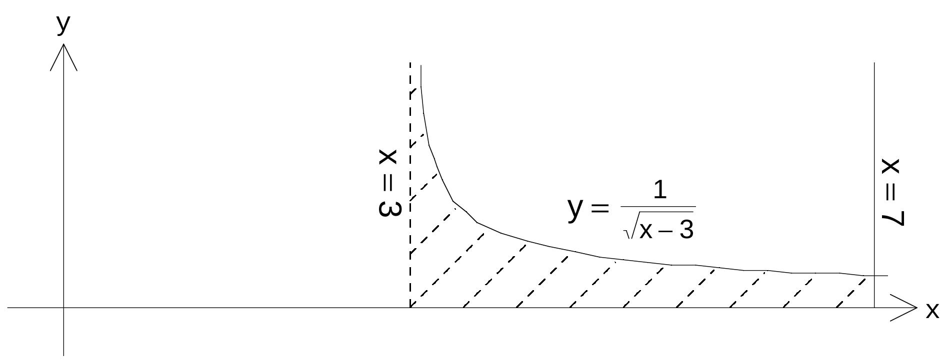
$= \lim_{s\to 0^+} \left[-4 - \frac{2\,(\ln s - 2)}{s^{-1/2}}\right] = \lim_{s\to 0^+} \left[-4 - \frac{2\,s^{-1}}{-0.5\,s^{-3/2}}\right] = \lim_{s\to 0^+} (-4 + 4s^{1/2}) = -4,$

by L'Hospital's Rule.

48. $S = \{(x, y) : 3 < x \le 7,\ 0 \le y \le (x-3)^{-1/2}\}.$

Area $= \int_3^7 \frac{1}{\sqrt{x-3}}dx = \lim_{s\to 3^+} \int_s^7 \frac{1}{\sqrt{x-3}}dx = \lim_{s\to 3^+} 2\sqrt{x-3}\,\Big]_s^7 = \lim_{s\to 3^+} \left(4 - 2\sqrt{s-3}\right) = 4.$

See graph below.



50. Since $\frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} \ge \frac{1}{\sqrt{x}} \ge 0$ and $\int_1^{+\infty} \frac{1}{\sqrt{x}}dx = \int_1^{+\infty} \frac{1}{x^{1/2}}dx$ is known to diverge to $+\infty$ because $\frac{1}{2} \le 1$, $\int_1^{+\infty} \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}dx$ also diverges to $+\infty$.

52. Since $0 \le \frac{1}{\sqrt{x^3+1}} \le \frac{1}{\sqrt{x^3}} = \frac{1}{x^{1.5}}$ and $\int_1^{+\infty} \frac{1}{x^{1.5}}dx$ is known to converge because $1.5 > 1$, $\int_1^{+\infty} \frac{1}{\sqrt{x^3+1}}dx$ also converges.

58. If $p \neq 1$, $\int_{e}^{+\infty} \frac{1}{x(\ln x)^{p}}\,dx = \lim_{t\to+\infty}\int_{e}^{t} \frac{1}{x(\ln x)^{p}}\,dx = \lim_{t\to+\infty}\left(\frac{1}{1-p}\frac{1}{(\ln x)^{p-1}}\right)\Big]_{e}^{t} =$

$= \lim_{t\to+\infty} \frac{1}{1-p}\left[\frac{1}{(\ln t)^{p-1}} - 1\right] = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ +\infty & \text{if } p < 1 \end{cases}.$

If $p = 1$, $\int_{e}^{+\infty} \frac{1}{x(\ln x)^{p}}\,dx = \int_{e}^{+\infty} \frac{1}{x\ln x}\,dx = \lim_{t\to+\infty}\int_{e}^{t} \frac{1}{x\ln x}\,dx = \lim_{t\to+\infty} \ln(\ln x)\Big]_{e}^{t} =$

$= \lim_{t\to+\infty} \ln(\ln t) = +\infty.$

So the integral converges to $\frac{1}{p-1}$ just when $p > 1$, and diverges to $+\infty$ otherwise.

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2. If $\frac{dy}{dx} = \frac{x+\sin x}{3y^2}$ then $\int 3y^2\,dy = \int (x+\sin x)\,dx.$

Integrating, $y^3 = \frac{1}{2}x^2 - \cos x + C$, and thus $y = \left(\frac{1}{2}x^2 - \cos x + C\right)^{1/3}$.

Conversely, substitution shows that $y = \left(\frac{1}{2}x^2 - \cos x + C\right)^{1/3}$ will solve $\frac{dy}{dx} = \frac{x+\sin x}{3y^2}$.

4. If $y' = xy$ then $\int y^{-1}\,dy = \int x\,dx$ if $y \neq 0$. ($y = 0$ is a solution too.)

Integrating, $\ln|y| = \frac{1}{2}x^2 + C_1$, so $|y| = C_2 e^{x^2/2}$, where $0 < C_2 = e^{C_1}$.

This is equivalent to $y = C_3 e^{x^2/2}$, where $|C_3| = C_2$, so that $C_3 \neq 0$.

If we allow $C_3 = 0$, we capture the singular solution $y = 0$ as well.

Conversely, substitution shows that $y = C_3 e^{x^2/2}$ will solve $y' = xy$.

8. If $\frac{dx}{dt} = 1 + t - x - tx$ then $\int \frac{dx}{1-x} = \int (1+t)\,dt$ if $x \neq 1$. ($x = 1$ is a solution too.)

Integrating, $-\ln|x-1| = t + \frac{1}{2}t^2 + C_1$, so $|x-1| = C_2 e^{-(t+t^2/2)}$, where $0 < C_2 = e^{-C_1}$.

So $x = 1 + C_3 e^{-(t+t^2/2)}$, where $|C_3| = C_2$, so that $C_3 \neq 0$.

If we allow $C_3 = 0$, we capture the singular solution $x = 1$ too.

Conversely, substitution shows that $x = 1 + C_3 e^{-(t+t^2/2)}$ will solve $\frac{dx}{dt} = 1 + t - x - tx$.

14. If $\frac{dy}{dt} = \frac{ty+3t}{t^2+1}$ and $y(2) = 2$ then $\int \frac{dy}{y+3} = \int \frac{t\,dt}{t^2+1}$ if $y \neq -3$.

Note $y = -3$ is a solution of the differential equation $\frac{dy}{dt} = \frac{ty+3t}{t^2+1}$ too, but it does not pass through $(2, 2)$.

Integrating, $\ln|y+3| = \frac{1}{2}\ln(t^2+1) + C.$

Putting $t = 2$ and $y = 2$, $\ln 5 = \frac{1}{2}\ln 5 + C$ and thus $C = \frac{1}{2}\ln 5.$

Thus $\ln(y+3) = \frac{1}{2}\ln(5(t^2+1))$, so $y + 3 = \sqrt{5(t^2+1)}$, and $y = -3 + \sqrt{5(t^2+1)}$.

Conversely, substitution shows that $y = -3 + \sqrt{5(t^2+1)}$ will solve $\frac{dy}{dt} = \frac{ty+3t}{t^2+1}$ and satisfies the condition $y(2) = 2$.

30. (a) Let y be the mass (measured in kg) of the salt in the tank t minutes after the two brine sources begin to fill the tank. Then $y = 0$ when $t = 0$. The first source adds salt at a rate of 0.05 kg/l $\times$ 5 l/min $= 0.25$ kg/min and the second source adds salt at a rate of 0.04 kg/l $\times$ 10 l/min $= 0.4$ kg/min, so the rate at which new salt is added is 0.65 kg/min. The amount of solution in the tank is always 1000 l since the outflow rate, 15 l/min, equals the sum of the two inflow rates, 5 l/min + 10 l/min. The rate at which salt is lost through the outflow is $\frac{y}{1000}$ kg/l $\times$ 15 l/min $= \frac{3y}{200}$ kg/min and the net rate at which salt accumulates in the tank is $\left(0.65 - \frac{3y}{200}\right)$ kg/min, or $\frac{130-3y}{200}$ kg/min.

This gives us the differential equation $\frac{dy}{dt} = \frac{130-3y}{200}$ with initial condition $y(0) = 0$.

Thus $\int \frac{dy}{130-3y} = \int \frac{dt}{200}$, if $y \neq \frac{130}{3}$. Note $y = \frac{130}{3}$ is a solution of the differential equation $\frac{dy}{dt} = \frac{130-3y}{200}$ too, but it does not satisfy the initial condition $y(0) = 0$.

Integrating, $-\frac{1}{3}\ln|130-3y| = \frac{t}{200} + C.$

Putting $t = 0$ and $y = 0$, $-\frac{1}{3}\ln 130 = 0 + C$. Consequently $C = -\frac{1}{3}\ln 130.$

Thus $\ln\frac{130-3y}{130} = -\frac{3t}{200}$, so $130 - 3y = 130e^{-3t/200}$, and $y = \frac{130}{3}[1 - e^{-3t/200}]$ kg.

(b) When $t = 60$ min, $y = \frac{130}{3}(1 - e^{-0.9}) \approx 25.715$ kg.

Notice that $\lim_{t\to+\infty} y = \frac{130}{3}$ kg and the concentration after a long time will be nearly equal to the weighted average $\left(\frac{5}{15}\times 0.05 + \frac{10}{15}\times 0.04\right)$ kg/l of the concentrations of salt in the two intakes. Put another way, if you wait long enough nearly all the original pure water will be gone.

42. If $m(t)$ is the mass of the raindrop at time t, we are told that $\frac{dm}{dt} = km$ for some positive constant k. Newton's second law then requires that the force on the raindrop due to gravity, gm, and the rate of change of momentum, $\frac{d}{dt}(mv)$, must be equal, so $gm = \frac{d}{dt}(mv) = v\frac{dm}{dt} + m\frac{dv}{dt} = vkm + m\frac{dv}{dt} = m\left(vk + \frac{dv}{dt}\right)$. We assume $m(t) \neq 0$. So $g = vk + \frac{dv}{dt}$ and thus $\int \frac{dv}{g-kv} = \int dt$, if $v \neq \frac{g}{k}$. Note $v(t) = \frac{g}{k}$ is a solution too.

Integrating, $-\frac{1}{k}\ln|g-kv| = t + C_1$, so $|g-kv| = C_2 e^{-kt}$ where $C_2 = e^{-kC_1}$ is positive.

Thus $v(t) = \frac{1}{k}[g + C_3 e^{-kt}]$ where $|C_3| = C_2 \neq 0$.

If we allow $C_3 = 0$, we capture the singular solution $v(t) = \frac{g}{k}$ too.

Conversely, substitution shows that $v(t) = \frac{1}{k}[g + C_3 e^{-kt}]$ will solve $g = vk + \frac{dv}{dt}$.

Since $k > 0$, $\lim_{t\to+\infty} v(t) = \frac{g}{k}$. The terminal velocity is $\frac{g}{k}$.

Now what keeps the raindrop mass from approaching $+\infty$?

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4. We can solve $12xy = 4y^4 + 3$ for $x = \frac{1}{3}y^3 + \frac{1}{4}y^{-1}$ and observe that this curve does pass through A(7/12, 1) and B(67/24, 2). Then $\frac{dx}{dy} = y^2 - \frac{1}{4}y^{-2}$ so

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\left(y^2 - \frac{1}{4}y^{-2}\right)^2} = \sqrt{1+\left(y^4 - \frac{1}{2} + \frac{1}{16}y^{-4}\right)} = \sqrt{y^4 + \frac{1}{2} + \frac{1}{16}y^{-4}} = y^2 + \frac{1}{4}y^{-2}.$$

$$L = \int_1^2 \left(y^2 + \frac{1}{4}y^{-2}\right) dy = \left(\frac{1}{3}y^3 - \frac{1}{4}y^{-1}\right)\Big]_1^2 = \frac{59}{24}.$$

6. If $y = \frac{x^3}{6} + \frac{1}{2x}$, $1 \le x \le 2$, then $\frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2}$ and $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{x^2}{2} + \frac{1}{2x^2}$.

$$L = \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = \left(\frac{x^3}{6} - \frac{1}{2x}\right)\Big]_1^2 = \frac{17}{12}.$$

10. If $y = \ln(\sin x)$, $\frac{\pi}{6} \le x \le \frac{\pi}{3}$ then $\frac{dy}{dx} = \cot x$.

$$L = \int_{\pi/6}^{\pi/3} \sqrt{1+\cot^2 x}\, dx = \int_{\pi/6}^{\pi/3} \csc x\, dx = -\ln|\csc x + \cot x|\Big]_{\pi/6}^{\pi/3} = \ln\left(\frac{2}{\sqrt{3}} + 1\right).$$

Alternatively $x = \sin^{-1}(e^y)$ and $\frac{dx}{dy} = \frac{e^y}{\sqrt{1-e^{2y}}}$.

$$L = \int_{\ln 1/2}^{\ln\sqrt{3}/2} \left[1 + \frac{e^{2y}}{1-e^{2y}}\right]^{1/2} dy = \int_{\ln 1/2}^{\ln\sqrt{3}/2} \frac{e^{-y}}{\sqrt{(e^{-y})^2 - 1}}\, dy = -\ln\left(e^{-y} + \sqrt{(e^{-y})^2 - 1}\right)\Big]_{\ln 1/2}^{\ln\sqrt{3}/2} =$$

$$= \ln\left(\frac{2}{\sqrt{3}} + 1\right).$$

16. The curve $y^2 = 4x$, $0 \le y \le 2$ runs between (0, 0) and (1, 2).

Writing $y = 2\sqrt{x}$, $\frac{dy}{dx} = x^{-1/2}$ and $L = \int_0^1 \sqrt{1 + (x^{-1/2})^2}\, dx$.

Letting $u = \sqrt{x}$, $x = u^2$ and $dx = 2u\, du$; $u = 0$ when $x = 0$ and $u = 1$ when $x = 1$.

So $L=\int_0^1 2\sqrt{u^2+1}\,du=\left(u\sqrt{u^2+1}+\ln\left(u+\sqrt{u^2+1}\right)\right)\Big]_0^1=\sqrt{2}+\ln\left(1+\sqrt{2}\right)$.

Alternatively $x=\frac{1}{4}y^2$, $\frac{dx}{dy}=\frac{y}{2}$, and $L=\int_0^2\sqrt{1+\left(\frac{y}{2}\right)^2}\,dy=\frac{1}{2}\int_0^2\sqrt{4+y^2}\,dy=$

$=\left(\frac{y}{4}\sqrt{4+y^2}+\ln\left(y+\sqrt{4+y^2}\right)\right)\Big]_0^2=\sqrt{2}+\ln\left(1+\sqrt{2}\right)$.

20. $y=\frac{b}{a}\sqrt{a^2-x^2}$ and $\frac{dy}{dx}=-\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}}$ on the upper right quarter of the ellipse.
Here $0\le x\le a$. Multiplying the length of this quarter of the ellipse by 4,

$$L=4\int_0^a\left(1+\left(-\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}}\right)^2\right)^{1/2}dx=\frac{4}{a}\int_0^a\left[\frac{(b^2-a^2)x^2+a^4}{a^2-x^2}\right]^{1/2}dx.$$

(In general, such integrals cannot be evaluated in closed form in terms of elementary functions of a and b, unless $a=b$ and the ellipse is really a circle.)

It is just as bad if one solves for x in terms of y; the letters a and b trade places.

32. (a) If $y=a\cosh\frac{x}{a}$, $-b\le x\le b$, then $\frac{dy}{dx}=\sinh\frac{x}{a}$ and $\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\cosh\frac{x}{a}$.

$L=\int_{-b}^{b}\cosh\frac{x}{a}\,dx=a\sinh\frac{x}{a}\Big]_{-b}^{b}=2a\sinh\frac{b}{a}$.

(b) If the poles are 50 ft apart and the wire length is 56 ft then $b=25$ and $56=2a\sinh\frac{25}{a}$, so we need to solve $\sinh\frac{25}{a}=\frac{28}{a}$. This requires a numerical approximation technique such as Newton's method. To simplify the setup, let $t=\frac{25}{a}$ so that the equation to be solved becomes $\sinh t=1.12t$.

Since $\cosh t$, the derivative of $\sinh t$, has smallest value, 1, at $t=0$ and since $\cosh t$ becomes steadily larger than 1 as we retreat from $t=0$ in either direction, it is clear that the graphs of $y=\sinh t$ and $y=1.12t$ will cross at three places in the ty-plane: the origin, and two more points symmetrical with respect to the origin. Probing with my TI–36 calculator, $\sinh 1\approx 1.175201194>1.12$; the positive root is less than 1.
$\sinh 0.5\approx 0.521095305<0.56$; the positive root is more than 0.5.

Let $F(t)=\sinh t-1.12t$. Then $F'(t)=\cosh t-1.12$. The recursion relationship for Newton's method is $t_{n+1}=t_n-\frac{F(t_n)}{F'(t_n)}=t_n-\frac{\sinh t_n-1.12t_n}{\cosh t_n-1.12}=\frac{t_n\cosh t_n-\sinh t_n}{\cosh t_n-1.12}$.

I chose as first guess $t_1=0.8$, and using the TI–36 found t_4 and t_5 agreeing at 0.833915825. Checking, $\sinh 0.833915825-1.12\cdot 0.833915825\approx 1.09\cdot 10^{-10}$.

The corresponding value for $a=\frac{25}{t}$ is 29.97904495.

The lowest point on the wire is about 30 ft above the ground.