MATHEMATICS 152 98-2 Solutions for Assignment 5

Page 421

2. $\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$

4. $\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C.$

10. $\int \theta\sec^2\theta\,d\theta = \theta\tan\theta - \int \tan\theta\,d\theta = \theta\tan\theta - \int \frac{\sin\theta}{\cos\theta}\,d\theta = \theta\tan\theta + \ln|\cos\theta| + C.$

12. $\int t^3e^t\,dt = t^3e^t - \int 3t^2e^t\,dt = t^3e^t - 3t^2e^t + \int 6te^t\,dt = t^3e^t - 3t^2e^t + 6te^t - \int 6e^t\,dt =$
$= t^3e^t - 3t^2e^t + 6te^t - 6e^t + C = (t^3 - 3t^2 + 6t - 6)e^t + C.$
(You could do something similar for $\int P(t)e^t\,dt$, where $P(t)$ is any polynomial; eventually you would differentiate the polynomial to death.)

18. $\int_1^4 \sqrt{t}\ln t\,dt = \frac{2}{3}t^{3/2}\ln t\Big]_1^4 - \int_1^4 \frac{2}{3}t^{1/2}\,dt = \frac{16}{3}\ln 4 - \frac{4}{9}t^{3/2}\Big]_1^4 = \frac{16}{3}\ln 4 - \frac{28}{9}.$

24. $\int x^3e^{x^2}\,dx = \int \left(\frac{1}{2}x^2\right)\cdot(2xe^{x^2})\,dx = \frac{1}{2}x^2e^{x^2} - \int xe^{x^2}\,dx = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C =$
$= \frac{1}{2}(x^2-1)e^{x^2} + C.$

30. $\int \sin(\ln x)\,dx = \int x\sin(\ln x)\cdot\frac{1}{x}\,dx = -x\cos(\ln x) + \int \cos(\ln x)\,dx =$
$= -x\cos(\ln x) + \int x\cos(\ln x)\cdot\frac{1}{x}\,dx = -x\cos(\ln x) + x\sin(\ln x) - \int \sin(\ln x)\,dx.$
So $2\int \sin(\ln x)\,dx = -x\cos(\ln x) + x\sin(\ln x) + K$, and
$\int \sin(\ln x)\,dx = -\frac{1}{2}x\cos(\ln x) + \frac{1}{2}x\sin(\ln x) + C = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C.$

38. (a) $\int \cos^n x\,dx = \int \cos^{n-1}x\cos x\,dx = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\sin^2 x\,dx =$
$= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,(1-\cos^2 x)\,dx =$
$= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx.$
So $n\int \cos^n x\,dx = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx$, and
$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx.$

(b) $\int \cos^2 x\,dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}\int 1\,dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C =$

$= \frac{1}{4}\sin(2x) + \frac{1}{2}x + C.$

(c) $\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x\,dx =$

$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left(\frac{1}{2}\cos x \sin x + \frac{1}{2}x\right) + C =$

$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C = \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + \frac{3}{8}x + C.$

42. $\int x^n e^x\,dx = x^n e^x - \int n x^{n-1} e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx.$

46. $\int x^4 e^x\,dx = x^4 e^x - 4\int x^3 e^x\,dx = x^4 e^x - 4x^3 e^x + 12\int x^2 e^x\,dx =$

$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24\int x e^x\,dx = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24\int e^x\,dx =$

$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24e^x + C = (x^4 - 4x^3 + 12x^2 - 24x + 24)\,e^x + C.$

48. $y = 5\ln x$ meets $y = x\ln x$ at the point $(1, 0)$, where $\ln x = 0$, and also at the point $(5, 5\ln 5)$, where $x = 5$. Between these points $\ln x > 0$ and $x < 5$, so $y = x\ln x$ is the lower curve and $y = 5\ln x$ is the upper curve. The enclosed area is given by

$\int_1^5 (5\ln x - x\ln x)\,dx = \int_1^5 (5 - x)\ln x\,dx =$

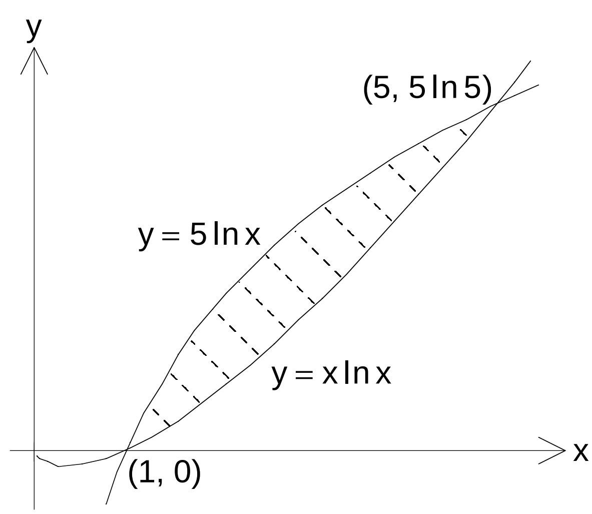
$= -\frac{1}{2}(5 - x)^2 \ln x\Big]_1^5 + \frac{1}{2}\int_1^5 \frac{(5 - x)^2}{x}\,dx =$

$= 0 + \frac{1}{2}\int_1^5 \left(\frac{25}{x} - 10 + x\right) dx =$

$= \frac{1}{2}\left(25\ln x - 10x + \frac{1}{2}x^2\right)\Big]_1^5 =$

$= \frac{1}{2}(25\ln 5 - 28) = 12.5\ln 5 - 14.$

See graph above and to the right.



For Exercise 48

Page 428

2. $\int_0^{\pi/2} \cos^2 x\,dx = \frac{1}{2}\cos x \sin x\Big]_0^{\pi/2} + \int_0^{\pi/2}\frac{1}{2}\,dx = 0 + \frac{1}{2}x\Big]_0^{\pi/2} = \frac{\pi}{4}.$

Alternatively $\int_0^{\pi/2} \cos^2 x\,dx = \int_0^{\pi/2}\frac{1+\cos(2x)}{2}\,dx = \left(\frac{1}{2}x + \frac{1}{4}\sin(2x)\right)\Big]_0^{\pi/2} = \frac{\pi}{4}.$

Alternatively (and the laziest way of all) if $u = \frac{\pi}{2} - x$ so that $du = -dx$, then

$\int_0^{\pi/2} \cos^2 x\,dx = -\int_{\pi/2}^{0}\sin^2 u\,du = \int_0^{\pi/2}\sin^2 u\,du.$

The name of the variable of integration is irrelevant, so $\int_0^{\pi/2}\sin^2 u\,du = \int_0^{\pi/2}\sin^2 x\,dx,$

and thus $\int_0^{\pi/2} \cos^2 x\,dx = \int_0^{\pi/2}\sin^2 u\,du = \int_0^{\pi/2}\sin^2 x\,dx = \int_0^{\pi/2}[1 - \cos^2 x]\,dx.$

Consequently $2\int_0^{\pi/2} \cos^2 x\,dx = \int_0^{\pi/2} 1\,dx = x\Big]_0^{\pi/2} = \frac{\pi}{2}$, and $\int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4}.$

6. $\int \sin^4 x \cos^3 x\,dx = \int \sin^4 x\,(1 - \sin^2 x)\cos x\,dx = \int (\sin^4 x - \sin^6 x)\cos x\,dx =$

$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C.$

10. $\int \sin\left(x + \frac{\pi}{6}\right)\cos x\,dx = \int \frac{1}{2}\left[\sin\frac{\pi}{6} + \sin\left(2x + \frac{\pi}{6}\right)\right]dx =$

$= \left(\frac{1}{2}\sin\frac{\pi}{6}\right)\cdot x - \frac{1}{4}\cos\left(2x + \frac{\pi}{6}\right) + C = \frac{1}{4}x - \frac{1}{4}\cos\left(2x + \frac{\pi}{6}\right) + C.$

Alternatively $\int \sin\left(x + \frac{\pi}{6}\right)\cos x\,dx = \int\left[\sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6}\right]\cos x\,dx =$

$= \int\left(\frac{\sqrt{3}}{2}\sin x \cos x + \frac{1}{2}\cos^2 x\right)dx = \int\left(\frac{\sqrt{3}}{4}\sin(2x) + \frac{1+\cos(2x)}{4}\right)dx =$

$= -\frac{\sqrt{3}}{8}\cos(2x) + \frac{1}{4}x + \frac{1}{8}\sin(2x) + C.$ These answers really are the same.

14. $\int x \sin^3(x^2)\,dx = \int x \sin^2(x^2)\sin(x^2)\,dx = \int x\,[1 - \cos^2(x^2)]\sin(x^2)\,dx =$

$= \int [(\sin(x^2)\cdot x - \cos^2(x^2)\sin(x^2)\cdot x]\,dx = -\frac{1}{2}\cos(x^2) + \frac{1}{6}\cos^3(x^2) + C.$

26. $\int \tan^3 x \sec^3 x\,dx = \int \tan^2 x \sec^2 x\,(\sec x \tan x)\,dx =$

$= \int [\sec^2 x - 1]\sec^2 x\,(\sec x \tan x)\,dx = \int [\sec^4 x - \sec^2 x](\sec x \tan x)\,dx =$

$= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.$

36. $\int \cot^3 x \csc^4 x\,dx = \int \cot^3 x\,[\cot^2 x + 1]\csc^2 x\,dx = \int [\cot^5 x + \cot^3 x]\csc^2 x\,dx =$
$= -\frac{1}{6}\cot^6 x - \frac{1}{4}\cot^4 x + C.$

42. $\int \sin(3x)\sin(6x)\,dx = \int \sin(3x)\,[2\sin(3x)\cos(3x)]\,dx = \int 2\sin^2(3x)\cos(3x)\,dx =$
$= \frac{2}{9}\sin^3(3x) + C.$
Alternatively $\int \sin(3x)\sin(6x)\,dx = \int \frac{1}{2}[\cos(3x) - \cos(9x)]\,dx =$
$= \frac{1}{6}\sin(3x) - \frac{1}{18}\sin(9x) + C.$ These answers really are the same.

48. (a) $\int \sin x\cos x\,dx = \int (-\cos x)\left(\frac{d}{dx}\cos x\right)dx = -\frac{1}{2}\cos^2 x + C_1.$

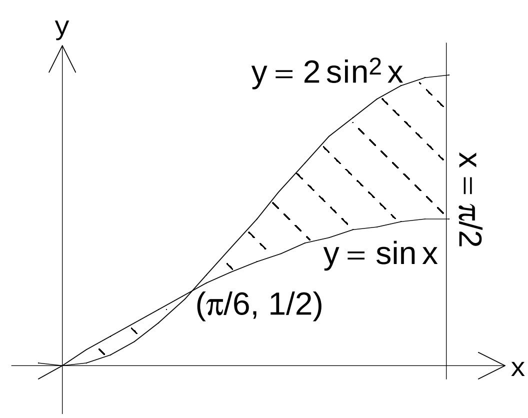
(b) $\int \sin x\cos x\,dx = \int \sin x\left(\frac{d}{dx}\sin x\right)dx = \frac{1}{2}\sin^2 x + C_2.$

(c) $\int \sin x\cos x\,dx = \int \frac{1}{2}\sin(2x)\,dx = -\frac{1}{4}\cos(2x) + C_3.$

(d) $\int \sin x\cos x\,dx = -\cos x\cdot\cos x - \int \cos x\cdot\sin x\,dx$, hence

$2\int \sin x\cos x\,dx = -\cos^2 x + K_1$ and $\int \sin x\cos x\,dx = -\frac{1}{2}\cos^2 x + C_1.$
Alternatively $\int \sin x\cos x\,dx = \sin x\cdot\sin x - \int \cos x\cdot\sin x\,dx$, hence
$2\int \sin x\cos x\,dx = \sin^2 x + K_2$ and $\int \sin x\cos x\,dx = \frac{1}{2}\sin^2 x + C_2.$
Here $C_2 = C_1 - \frac{1}{2}$ and $C_3 = C_1 - \frac{1}{4}$.

50. In the region $0 \le x \le \frac{\pi}{2}$, the curves $y = \sin x$ and $y = 2\sin^2 x$ meet at $(0, 0)$ and at $(\pi/6, 1/2)$, since $\sin 0 = 0$ and $2\sin(\pi/6) = 1$.
For $0 < x < \frac{\pi}{6}$, $y = 2\sin^2 x$ is the lower curve; for $\frac{\pi}{6} < x \le \frac{\pi}{2}$, $y = \sin x$ is the lower curve. Thus the height of the region is $\sin x - 2\sin^2 x$ for $0 \le x \le \frac{\pi}{6}$.



The height is $2\sin^2 x - \sin x$ for $\frac{\pi}{6} \le x \le \frac{\pi}{2}$.
The area is $\int_0^{\pi/6}[\sin x - 2\sin^2 x]\,dx + \int_{\pi/6}^{\pi/2}[2\sin^2 x - \sin x]\,dx =$
$= \int_0^{\pi/6}[\sin x + \cos(2x) - 1]\,dx + \int_{\pi/6}^{\pi/2}[-\sin x - \cos(2x) + 1]\,dx =$
$= \left(-\cos x + \frac{1}{2}\sin(2x) - x\right)\Big]_0^{\pi/6} + \left(\cos x - \frac{1}{2}\sin(2x) + x\right)\Big]_{\pi/6}^{\pi/2} = 1 - \frac{\sqrt{3}}{2} + \frac{\pi}{6}.$
See graph above and to the right.

56. Using the method of washers, the volume is

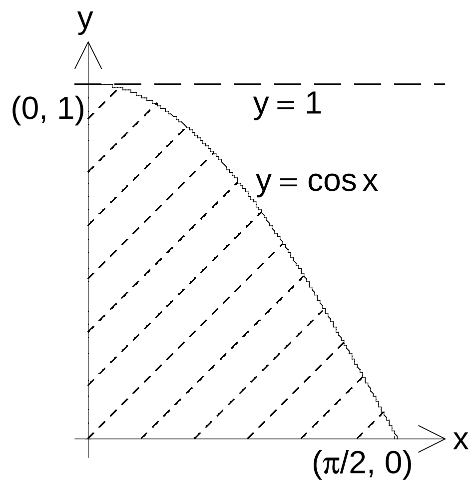
$V = \pi\int_0^{\pi/2}[(1-0)^2 - (1-\cos x)^2]\,dx =$

$= \pi\int_0^{\pi/2}[2\cos x - \cos^2 x]\,dx =$

$= \pi\int_0^{\pi/2}\left(2\cos x - \frac{1+\cos(2x)}{2}\right)dx =$

$= \pi\left(2\sin x - \frac{1}{2}x - \frac{1}{4}\sin(2x)\right)\Big]_0^{\pi/2} = 2\pi - \frac{\pi^2}{4}.$

The method of cylindrical shells can also be used, but it involves integrating $(1-y)\cos^{-1} y$. That integration (by parts) is somewhat tricky.



For Exercise 56