16. If $u_n = \frac{(-1)^n x^{2n-1}}{(2n-1)!}$ then $\left|\frac{u_{n+1}}{u_n}\right| = \frac{x^2}{2n(2n+1)} \to 0$ as $n \to \infty$ for all real x.
Thus $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}$ converges absolutely for all real x, and $R = \infty$.
The convergence interval is $(-\infty, +\infty)$.

20. If $u_n = \frac{(-3)^n (x-1)^n}{\sqrt{n+1}}$ then $\left|\frac{u_{n+1}}{u_n}\right| = 3\sqrt{\frac{n+1}{n+2}}\,|x-1| \to 3|x-1|$ as $n \to \infty$.
Thus $\sum_{n=0}^{\infty} \frac{(-3)^n (x-1)^n}{\sqrt{n+1}}$ converges absolutely for $|x-1| < \frac{1}{3}$, diverges for $|x-1| > \frac{1}{3}$,
and $R = \frac{1}{3}$.
When $x = \frac{2}{3}$ the series becomes $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{0.5}}$, a divergent p-series.
When $x = \frac{4}{3}$ the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$, which is conditionally convergent by the alternating series test. The convergence interval is (2/3, 4/3].

28. If $u_n = \frac{2\cdot 4\cdot 6\cdot \cdots \cdot(2n)}{1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1)} x^n$ then $\left|\frac{u_{n+1}}{u_n}\right| = \frac{2n+2}{2n+1}|x| \to |x|$ as $n \to \infty$. Thus
$\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdot \cdots \cdot(2n)}{1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1)} x^n$ converges absolutely for $|x| < 1$, diverges for $|x| > 1$, and
$R = 1$.
When $x = 1$ the series becomes $\sum_{n=0}^{\infty} \frac{2\cdot 4\cdot 6\cdot \cdots \cdot(2n)}{1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1)}$. But $\frac{2\cdot 4\cdot 6\cdot \cdots \cdot(2n)}{1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1)} \geq \frac{2}{1} = 2$,
the terms $\frac{2\cdot 4\cdot 6\cdot \cdots \cdot(2n)}{1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1)}$ of the series do not approach 0, and the series diverges.
When $x = -1$ the terms $(-1)^n \frac{2\cdot 4\cdot 6\cdot \cdots \cdot(2n)}{1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1)}$ of the series are at least 2
in absolute value and thus do not approach 0, so again the series diverges.
The convergence interval is $(-1, 1)$.

32. $A(x) = 1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \cdots$.
If $u_n = \frac{x^{3n}}{2\cdot 3\cdot 5\cdot 6\cdot \cdots \cdot(3n-1)\cdot(3n)}$ then $\left|\frac{u_{n+1}}{u_n}\right| = \frac{x^3}{(3n+2)(3n+3)} \to 0$ for all real x so the domain of $A(x)$ is $(-\infty, +\infty)$.
Plots are omitted; the Airy functions available in *Maple* are defined in a slightly different way. You might plot a few partial sums over the intervals $[-1, 1]$, $[-2, 2]$, and $[-3, 3]$.

36. If $\sum c_n x^n$ has radius of convergence R then $\sum c_n x^{2n} = \sum c_n (x^2)^n$ converges absolutely if $|x^2| < R$, i.e. if $|x| < \sqrt{R}$, and diverges if $|x^2| > R$, i.e. if $|x| > \sqrt{R}$.
So $\sum c_n x^{2n}$ has radius of convergence $\sqrt{R}$.

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2. $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$, so $4 \le R \le 6$.

(a) $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} c_n 1^n$ converges absolutely, since $|1| < 4 \le R$.

(b) $\sum_{n=0}^{\infty} c_n 8^n$ diverges, since $|8| > 6 \ge R$.

(c) $\sum_{n=0}^{\infty} c_n (-3)^n$ converges absolutely, since $|-3| < 4 \le R$.

(d) $\sum_{n=0}^{\infty} (-1)^n c_n 9^n = \sum_{n=0}^{\infty} c_n (-9)^n$ diverges, since $|-9| > 6 \ge R$.

4. If $u_n = \frac{(-1)^n x^n}{\sqrt[3]{n}}$ then $\left|\frac{u_{n+1}}{u_n}\right| = \sqrt[3]{\frac{n}{n+1}}|x| \to |x|$ as $n \to \infty$.

Thus $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$ converges absolutely for $|x| < 1$, diverges for $|x| > 1$, and $R = 1$.

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ is a divergent p-series while $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ converges by the alternating series test, so $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$ diverges at $x = -1$ and converges conditionally at $x = 1$.

The convergence interval is $(-1, 1]$.

6. If $u_n = \frac{x^n}{n^2}$ then $\left|\frac{u_{n+1}}{u_n}\right| = \frac{n^2}{(n+1)^2}|x| \to |x|$ as $n \to \infty$.

Thus $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges absolutely for $|x| < 1$, diverges for $|x| > 1$, and $R = 1$.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series so $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is absolutely convergent at both endpoints.

The convergence interval is $[-1, 1]$.

12. If $u_n = \frac{n^2 x^n}{10^n}$ then $\left|\frac{u_{n+1}}{u_n}\right| = \frac{(n+1)^2}{n^2}\frac{|x|}{10} \to \frac{|x|}{10}$ as $n \to \infty$.

Thus $\sum_{n=0}^{\infty} \frac{n^2 x^n}{10^n}$ converges absolutely for $|x| < 10$, diverges for $|x| > 10$, and $R = 10$.

For $x = \pm 10$, $|u_n| = n^2 \to \infty$ as $n \to \infty$ and since the terms of the series do not approach 0, the series diverges.

The convergence interval is $(-10, 10)$.

$\int_1^{\infty} \frac{\sqrt{x}}{e^{\sqrt{x}}}dx = \lim_{t\to\infty} \frac{-2x-4\sqrt{x}-4}{e^{\sqrt{x}}}\Big]_1^t = \frac{10}{e}$, so the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{e^{\sqrt{n}}}$ converges.

Note: One might be tempted to try the ratio test or the root test. But in this case both yield the useless limit value of 1.

20. The divergent p-series $\sum_{n=1}^{\infty} \frac{1}{n}$ may be used in the limit form of the comparison test to show that $\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n+1)(n+2)}$ is not absolutely convergent.

Since $n(n+3) \ge (n+1)^2$, $\frac{n}{(n+1)(n+2)} \ge \frac{n+1}{(n+2)(n+3)}$ so the alternating series test shows that $\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n+1)(n+2)}$ does converge. Thus the convergence is conditional.

22. The divergent p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ may be used in the limit form of the comparison test to show that $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^5+n^2+2}}$ diverges.

30. $\lim_{x\to\infty} \frac{2^{3x-1}}{x^2+1} = \lim_{x\to\infty} \frac{3\cdot\ln 2\cdot 2^{3x-1}}{2x} = \lim_{x\to\infty} \frac{(3\cdot\ln 2)^2\cdot 2^{3x-1}}{2} = +\infty$ (L'Hospital's Rule twice).
So $\sum_{n=1}^{\infty} \frac{2^{3n-1}}{n^2+1}$ diverges to $+\infty$ with a vengeance!

Alternatively, $\lim_{n\to\infty} \frac{\frac{2^{3(n+1)-1}}{(n+1)^2+1}}{\frac{2^{3n-1}}{n^2+1}} = \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1}\cdot 8 = 8 > 1$, so $\sum_{n=1}^{\infty} \frac{2^{3n-1}}{n^2+1}$ diverges by the ratio test.

Alternatively, $\lim_{n\to\infty} \left(\frac{2^{3n-1}}{n^2+1}\right)^{1/n} = 8 > 1$ since $\lim_{n\to\infty} 2^{3-1/n} = 8$ and $\lim_{n\to\infty} (n^2+1)^{1/n} = 1$.
(The latter can be seen by taking the limit of the logarithm, using L'Hospital's Rule when necessary.) So $\sum_{n=1}^{\infty} \frac{2^{3n-1}}{n^2+1}$ diverges by the root test.

34. The divergent p-series $\sum_{j=1}^{\infty} \frac{1}{\sqrt{j}}$ may be used in the limit form of the comparison test to show that $\sum_{j=1}^{\infty} \frac{(-1)^j\sqrt{j}}{j+5}$ is not absolutely convergent.

But $\frac{\sqrt{j}}{j+5} \ge \frac{\sqrt{j+1}}{j+6}$ for $j \ge 5$ because $j(j+6)^2 \ge (j+1)(j+5)^2$ for $j \ge 5$, so the alternating series test shows that $\sum_{j=1}^{\infty} \frac{(-1)^j\sqrt{j}}{j+5}$ converges conditionally.

34. Since $-1 \le \cos n \le 1$, $0 < \frac{1}{\sqrt{n}} \le \frac{2+\cos n}{\sqrt{n}} \le \frac{3}{\sqrt{n}}$. Consequently $a_n > 0$ for all n.

If we define a series $\sum b_n$ by $b_1 = 1$, $b_{n+1} = \frac{3}{\sqrt{n}} b_n$, then $0 < a_n \le b_n$ for all n.

$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{3}{\sqrt{n}} = 0 < 1$, so $\sum b_n$ is absolutely convergent by the ratio test, and thus $\sum a_n$ is absolutely convergent by the comparison test.

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4. $\lim_{i\to\infty} \frac{\frac{(i+1)^4}{4^{i+1}}}{\frac{i^4}{4^i}} = \lim_{i\to\infty} \frac{1}{4}\left(1 + \frac{1}{i}\right)^4 = \frac{1}{4} < 1$, so $\sum_{i=1}^{\infty} \frac{i^4}{4^i}$ is convergent by the ratio test.

Alternatively $\lim_{i\to\infty} \left(\frac{i^4}{4^i}\right)^{1/i} = \frac{1}{4}\left(\lim_{i\to\infty} i^{1/i}\right)^4 = \frac{1}{4}\cdot 1^4 < 1$; $\sum_{i=1}^{\infty} \frac{i^4}{4^i}$ is convergent by the root test.

(To see that $\lim_{i\to\infty} i^{1/i} = 1$, take the logarithm and observe that $\lim_{x\to\infty} \frac{\ln x}{x} = 0$ by L'Hospital's Rule.)

6. $\lim_{n\to\infty} \frac{(n+1)^2 e^{-(n+1)^3}}{n^2 e^{-n^3}} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} e^{-3n^2-3n-1} = 0 < 1$, so $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges by the ratio test.

Alternatively, $\lim_{n\to\infty} (n^2 e^{-n^3})^{1/n} = \lim_{n\to\infty} (n^{1/n})^2 e^{-n^2} = 1\cdot 0 = 0$, so $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges by the root test.

Alternatively, $\frac{d}{dx} x^2 e^{-x^3} = (2x - 3x^4)e^{-x^3} = x(2 - 3x^3)e^{-x^3} < 0$ for $x \ge 1$, so $x^2 e^{-x^3}$ is decreasing on $[1, +\infty)$. Since $\lim_{x\to\infty} x^2 e^{-x^3} = 0$ by L'Hospital's Rule, we may apply the integral test. Since $\int_1^{\infty} x^2 e^{-x^3}\,dx = \lim_{t\to\infty} \left(-\frac{1}{3} e^{-x^3}\right)\Big]_1^t = \frac{1}{3e} < +\infty$, $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges.

12. $\lim_{n\to\infty} \left(\left(\frac{n^2+1}{2n^2+1}\right)^n\right)^{1/n} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} = \frac{1}{2} < 1$, so $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$ converges by the root test.

14. $\frac{d}{dx} \frac{\sqrt{x}}{e^{\sqrt{x}}} = \frac{1}{e^{\sqrt{x}}}\left(-\frac{1}{2} + \frac{1}{2\sqrt{x}}\right) < 0$ for $x > 1$, so $\frac{\sqrt{x}}{e^{\sqrt{x}}}$ is decreasing for $x \ge 1$.

Also $\lim_{x\to\infty} \frac{\sqrt{x}}{e^{\sqrt{x}}} = 0$ by L'Hospital's Rule, so we may use the integral test.
Integrating by parts and using L'Hospital's Rule once more we see that

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+1}$ is absolutely convergent since we can compare $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ to the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

10. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ diverges, using the limit form of the comparison test to compare it with the divergent p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Thus $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{\sqrt{n}}{n+1}$ is not absolutely convergent.
But $\frac{\sqrt{n}}{n+1} \geq \frac{\sqrt{n+1}}{n+2}$ since $n(n+2)^2 \geq (n+1)^3$, so the conditions of the alternating series test are met, and the series is conditionally convergent.

12. Since $\lim_{n\to\infty} \frac{2^n}{n^2+1} = +\infty$ and $\lim_{n\to\infty} (-1)^n \frac{2^n}{n^2+1}$ does not exist at all,
$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2+1}$ diverges.
Alternatively $\lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)^2+1}}{\frac{2^n}{n^2+1}} = \lim_{n\to\infty} \frac{2(n^2+1)}{(n+1)^2+1} = 2$. By the ratio test the series diverges.
Alternatively $\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2+1}} = \lim_{n\to\infty} \frac{2}{\sqrt[n]{n^2+1}} = 2$ since $\lim_{n\to\infty} \sqrt[n]{n^2+1} = 1$.
One way to see that $\lim_{n\to\infty} \sqrt[n]{n^2+1} = 1$ is to change to a real variable x. Observe that

$$\lim_{x\to\infty} \ln((x^2+1)^{1/x}) = \lim_{x\to\infty} \frac{\ln(x^2+1)}{x} = \lim_{x\to\infty} \frac{\frac{2x}{x^2+1}}{1} = 0, \text{ so that } \lim_{x\to\infty} (x^2+1)^{1/x} = e^0 = 1.$$

20. Since $\lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)\,n^n}{(n+1)^{(n+1)}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1$, $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is absolutely convergent.
To see that $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$, notice that $\lim_{x\to\infty} \left[\ln\left(\frac{x}{x+1}\right)^x\right] = \lim_{x\to\infty} x \ln \frac{x}{x+1} =$

$$= \lim_{x\to\infty} \frac{\ln x - \ln(x+1)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x+1}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \left[-\frac{x}{x+1}\right] = -1, \text{ so } \lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x = e^{-1} = \frac{1}{e}.$$

24. $\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{t\to\infty} \ln(\ln x)\Big]_2^t = +\infty$, hence $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the integral test, and $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ does not converge absolutely.
The conditions of the alternating series test are met, so the series is conditionally convergent.

24. For $p \le 0$, $\frac{(\ln n)^p}{n} \ge \frac{(\ln(n+1))^p}{n+1}$ for all $n \ge 1$, and $\lim_{n \to \infty} \frac{(\ln n)^p}{n} = 0$.

So $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n}$ converges by the alternating series test when $p \le 0$.

In fact the series also converges for all values of $p > 0$.

By L'Hospital's Rule, $\lim_{x \to \infty} \frac{(\ln x)^p}{x} = \lim_{x \to \infty} \frac{p\,(\ln x)^{p-1} \cdot x^{-1}}{1} = p \lim_{x \to \infty} \frac{(\ln x)^{p-1}}{x}$.

The exponent on $(\ln x)$ decreases by 1, and a constant coefficient p appears.

Now if $(p-1) \le 0$, we observed that $\lim_{x \to \infty} \frac{(\ln x)^{p-1}}{x} = 0$ in the first part of the exercise.

If $(p-1) > 0$, repeat the argument to see that $\lim_{x \to \infty} \frac{(\ln x)^p}{x} = p(p-1) \lim_{x \to \infty} \frac{(\ln x)^{p-2}}{x}$.

Continuing until the exponent has finally been reduced to 0 or less, we see that $\lim_{x \to \infty} \frac{(\ln x)^p}{x} = 0$ regardless of how large p may be! Changing from a real variable x to an integer variable n, $\lim_{n \to \infty} \frac{(\ln n)^p}{n} = 0$ for all values of p. Although for $p > 0$, the function $\frac{(\ln x)^p}{x}$ is **not** decreasing for **all** $x \ge 1$, it **is** decreasing for $x \ge e^p$ (take the first derivative to see why). So by chopping off the first part of the series (where the terms may be temporarily increasing in size) we can use the alternating series test on the rest of the series to see that it will converge regardless of the size of p. Once we know that, we can put the early terms back, changing the final sum but not the convergence status of the series.

26. To approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ with error less than 0.001, we use the partial sum $s_n = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i^4}$ with $\frac{1}{(n+1)^4} < 0.001$. This first occurs with $n = 5$.

Since $s_5 = \frac{12280111}{12960000} \approx 0.947539429$, the desired approximation is 0.948.

30. To approximate $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$ to four decimal places the error must be less than 0.00005. We use $s_n = \sum_{i=0}^{n} \frac{(-1)^i}{(2i)!}$ with $\frac{1}{[2(n+1)]!} < 0.00005$. This first occurs with $n = 3$. Since $s_3 = \frac{389}{720} \approx 0.540277777$, the desired approximation is 0.5403.

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2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series so $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is not absolutely convergent.
The conditions of the alternating series test are met, so the series is conditionally convergent.

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2. $-5 - \frac{5}{2} + \frac{5}{5} - \frac{5}{8} + \frac{5}{11} - \frac{5}{14} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 5}{3n-4}$ converges.

Except for the first term, the signs of the terms alternate; $\frac{5}{3n-4} \geq \frac{5}{3(n+1)-4}$ for $n \geq 1$; and $\lim_{n\to\infty} \frac{5}{3n-4} = 0$.

4. $\frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \frac{1}{\ln 5} + \frac{1}{\ln 6} - \cdots = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges.

The signs of the terms alternate, $\frac{1}{\ln n} \geq \frac{1}{\ln(n+1)}$ for $n \geq 2$, and $\lim_{n\to\infty} \frac{1}{\ln n} = 0$.

8. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n \ln n}$ converges by the alternating series test.

The signs of the terms alternate, $\frac{1}{n \ln n} \geq \frac{1}{(n+1)\ln(n+1)}$ for $n \geq 2$, and $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$.

10. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+1}$ diverges.

Although it is an alternating series, the other two conditions are not met, since $\frac{n^2}{n^2+1} < \frac{(n+1)^2}{(n+1)^2+1}$, and $\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0$ so that $\lim_{n\to\infty} (-1)^n \frac{n^2}{n^2+1}$ does not exist.

16. $\sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n!} = \frac{1}{1!} + \frac{0}{2!} - \frac{1}{3!} - \frac{0}{4!} + \frac{1}{5!} + \frac{0}{6!} - \frac{1}{7!} - \frac{1}{8!} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!}$ converges.

It is an alternating series with $\frac{1}{(2k-1)!} \geq \frac{1}{(2k+1)!}$ for all $k \geq 1$, and $\lim_{n\to\infty} \frac{1}{(2k-1)!} = 0$.

22. For $p > 0$, $\frac{1}{n^p} \geq \frac{1}{(n+1)^p}$ and $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

Thus $\sum_{k=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges by the alternating series test.

For $p = 0$, $\lim_{n\to\infty} \frac{1}{n^p} = \lim_{n\to\infty} \frac{1}{n^0} = \lim_{n\to\infty} \frac{1}{1} = 1 \neq 0$, so $\lim_{n\to\infty} \frac{(-1)^{n-1}}{n^p}$ does not exist, and worse yet for $p < 0$, $\lim_{n\to\infty} \frac{1}{n^p} = +\infty$ and $\lim_{n\to\infty} \frac{(-1)^{n-1}}{n^p}$ does not exist.

Thus $\sum_{k=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ diverges when $p \leq 0$.