

$$2. \sum_{i=1}^6 \frac{1}{i+1} = \frac{1}{1+1} + \frac{1}{2+1} + \frac{1}{3+1} + \frac{1}{4+1} + \frac{1}{5+1} + \frac{1}{6+1} = \\ = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \frac{669}{420} = \frac{223}{140}.$$

$$8. \sum_{j=n}^{n+3} j^2 = n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 = 4n^2 + 12n + 14.$$

$$14. \frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \dots + \frac{23}{27} = \sum_{i=3}^{23} \frac{i}{i+4}.$$

There are other correct answers, for example $\sum_{i=7}^{27} \frac{i-4}{i}$, or $21 - 4 \sum_{i=7}^{27} \frac{1}{i}$.

$$20. 1 - x + x^2 - x^3 + \dots + (-1)^n x^n = \sum_{i=0}^n (-x)^i = \sum_{i=0}^n (-1)^i x^i.$$

Again there are other correct answers, for example $\sum_{i=1}^{n+1} (-x)^{i-1}$.

$$24. \sum_{k=0}^8 \cos(k\pi) = \cos 0 + \cos \pi + \dots + \cos(8\pi) = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 = 1.$$

Or notice that $\cos(k\pi) = (-1)^k$. Then $\sum_{k=0}^8 \cos(k\pi) = \sum_{k=0}^8 (-1)^k = \frac{1 - (-1)^9}{1 - (-1)} = \frac{2}{2} = 1.$

(This uses the finite geometric series formula $\sum_{i=0}^n a^i = \frac{1 - a^{n+1}}{1 - a}$, valid if $a \neq 1$.)

To prove it, multiply $\sum_{i=0}^n a^i$ by $(1 - a)$ and notice how all the terms cancel except the 1 at the beginning and the $-a^{n+1}$ at the end.)

$$28. \sum_{i=-2}^4 2^{3-i} = 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 + 2^{-1} = 63.5.$$

Alternatively, putting $j = 4 - i$, $\sum_{i=-2}^4 2^{3-i} = \sum_{i=0}^6 2^{j-1} = \frac{1}{2} \sum_{i=0}^6 2^j = \frac{1}{2} \cdot \frac{1 - 2^7}{1 - 2} = \frac{127}{2} = 63.5.$

$$\begin{aligned}
34. \quad \sum_{i=1}^n i(i+1)(i+2) &= \sum_{i=1}^n (i^3 + 3i^2 + 2i) = \sum_{i=1}^n i^3 + 3 \sum_{i=1}^n i^2 + 2 \sum_{i=1}^n i = \\
&= \left(\frac{n(n+1)}{2} \right)^2 + 3 \frac{n(n+1)(2n+1)}{6} + 2 \frac{n(n+1)}{2} = \frac{n(n+1)}{4} [n(n+1) + 2(2n+1) + 4] = \\
&= \frac{n(n+1)}{4} (n^2 + 5n + 6) = \frac{n(n+1)(n+2)(n+3)}{4}.
\end{aligned}$$

$$\begin{aligned}
44. \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \left[\left(\frac{i}{n} \right)^3 + 1 \right] &= \lim_{n \rightarrow \infty} \left[\frac{1}{n^4} \sum_{i=1}^n i^3 + \frac{1}{n} \sum_{i=1}^n 1 \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{n^4} \left(\frac{n(n+1)}{2} \right)^2 + \frac{1}{n} \cdot n \right] = \\
&= \lim_{n \rightarrow \infty} \left[\frac{1}{4} \left(1 + \frac{1}{n} \right)^2 + 1 \right] = \frac{5}{4}.
\end{aligned}$$

52. (a) $2 \sin u \cos v = \sin(u+v) + \sin(u-v)$, as can be seen by expanding $\sin(u+v) = \sin u \cos v + \cos u \sin v$, $\sin(u-v) = \sin u \cos v - \cos u \sin v$, adding, and noticing the cancellation. So putting $u = \frac{1}{2}x$ and $v = ix$,

$$2 \sin\left(\frac{1}{2}x\right) \cos(ix) = \sin\left(\frac{1}{2}x + ix\right) + \sin\left(\frac{1}{2}x - ix\right) = \sin\left(i + \frac{1}{2}\right)x - \sin\left(i - \frac{1}{2}\right)x.$$

(b) Summing the result of part (a) from $i = 1$ to $i = n$,

$$\sum_{i=1}^n 2 \sin\left(\frac{1}{2}x\right) \cos(ix) = \sum_{i=1}^n \left[\sin\left(i + \frac{1}{2}\right)x - \sin\left(i - \frac{1}{2}\right)x \right] = \sin\left(n + \frac{1}{2}\right)x - \sin\left(\frac{1}{2}x\right),$$

because all the terms cancel except the second part of the first term, $-\sin\left(\frac{1}{2}x\right)$,

and the first part of the last term, $\sin\left(n + \frac{1}{2}\right)x$.

If x is not an integer multiple of 2π , so that $\frac{1}{2}x$ is not an integer multiple of π , then

$$\text{we can divide by } 2 \sin\left(\frac{1}{2}x\right) \neq 0, \text{ and obtain } \sum_{i=1}^n \cos(ix) = \frac{\sin\left(n + \frac{1}{2}\right)x - \sin\left(\frac{1}{2}x\right)}{2 \sin\left(\frac{1}{2}x\right)}.$$

Putting $u = \frac{1}{2}nx$ and $v = \frac{1}{2}(n+1)x$ in the formula $2 \sin u \cos v = \sin(u+v) + \sin(u-v)$,

$$2 \sin\left(\frac{1}{2}nx\right) \cos\left(\frac{1}{2}(n+1)x\right) = \sin\left(n + \frac{1}{2}\right)x + \sin\left(-\frac{1}{2}x\right) = \sin\left(n + \frac{1}{2}\right)x - \sin\left(\frac{1}{2}x\right).$$

$$\text{So } \sum_{i=1}^n \cos(ix) = \frac{2 \sin\left(\frac{1}{2}nx\right) \cos\left(\frac{1}{2}(n+1)x\right)}{2 \sin\left(\frac{1}{2}x\right)} = \frac{\sin\left(\frac{1}{2}nx\right) \cos\left(\frac{1}{2}(n+1)x\right)}{\sin\left(\frac{1}{2}x\right)}.$$

2. $f(x) = 16 - x^2$.

$[a, b] = [0, 4]$.

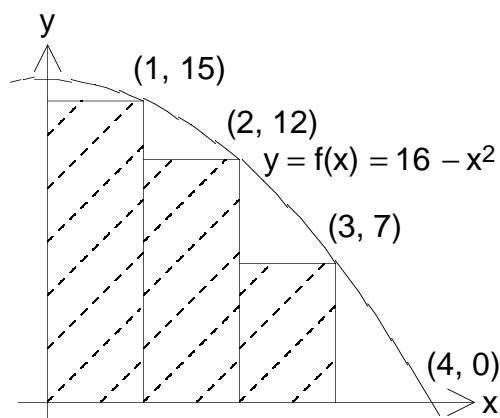
P is defined by $\{0, 1, 2, 3, 4\}$.

x_i^* is the right-hand endpoint x_i of the i^{th} subinterval $[x_{i-1}, x_i]$.

(a) $\|P\| = \text{Max}(1-0, 2-1, 3-2, 4-3) = \text{Max}(1, 1, 1, 1) = 1$.

(b) $\sum_{i=1}^4 A_i = f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 = 15 \cdot 1 + 12 \cdot 1 + 7 \cdot 1 + 0 \cdot 1 = 34$.

(c) See graph to the right.



For Exercise 2

4. $f(x) = 2x + 1$.

$[a, b] = [0, 4]$.

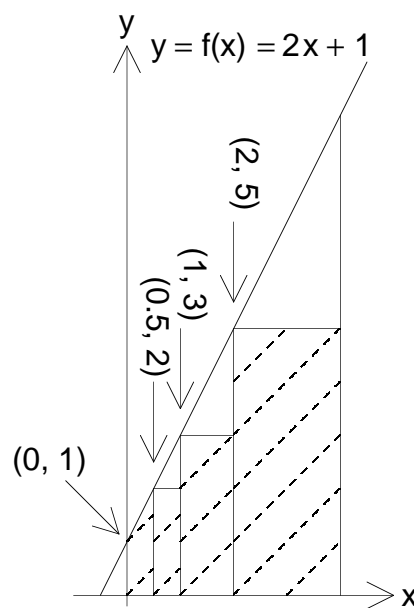
P is defined by $\{0, 0.5, 1, 2, 4\}$.

x_i^* is the left-hand endpoint x_{i-1} of the i^{th} subinterval $[x_{i-1}, x_i]$.

(a) $\|P\| = \text{Max}(0.5-0, 1-0.5, 2-1, 4-2) = \text{Max}(0.5, 0.5, 1, 2) = 2$.

(b) $\sum_{i=1}^4 A_i = f(0) \cdot 0.5 + f(0.5) \cdot 0.5 + f(1) \cdot 1 + f(2) \cdot 2 = 1 \cdot 0.5 + 2 \cdot 0.5 + 3 \cdot 1 + 5 \cdot 2 = 14.5$.

(c) See graph below and to the right.



For Exercise 4

$$6. f(x) = \frac{1}{x+1}.$$

$$[a, b] = [0, 2].$$

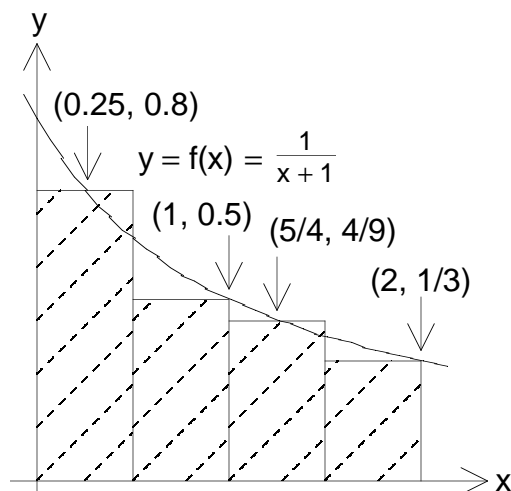
P is defined by $\{0, 0.5, 1.0, 1.5, 2.0\}$.

$$x_1^* = 0.25, \quad x_2^* = 1, \quad x_3^* = 1.25, \quad x_4^* = 2.$$

$$(a) \quad \|P\| = \text{Max}(0.5, 0.5, 0.5, 0.5) = 0.5.$$

$$(b) \quad \sum_{i=1}^4 A_i = [f(0.25) + f(1) + f(1.25) + f(2)] \cdot 0.5 = \\ = \left(0.8 + 0.5 + \frac{4}{9} + \frac{1}{3}\right) \cdot 0.5 = \frac{187}{180} = 1.03\bar{8}.$$

(c) See graph on the right.



For Exercise 6

$$12. f(x) = x^3.$$

$$[a, b] = [0, 1].$$

The i^{th} subinterval is $[(i-1)/n, i/n]$.

Its left-hand endpoint is $(i-1)/n$; its right-hand

endpoint is i/n ; its midpoint is $(2i-1)/2n$.

See graphs below and on the next page.

(a) Using the left-hand endpoints

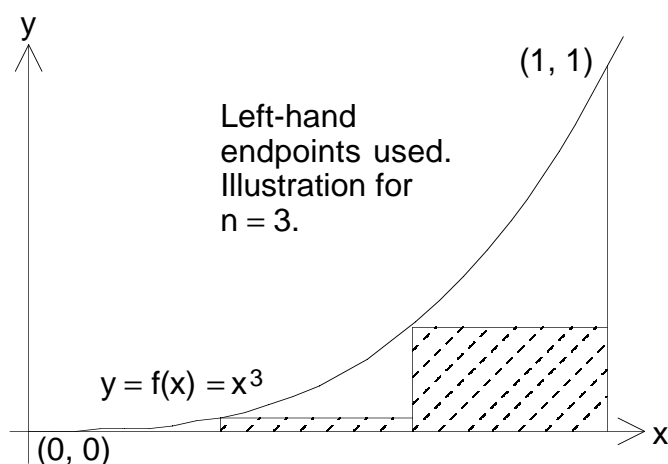
$$(i-1)/n, \text{ with } f((i-1)/n) = \left(\frac{i-1}{n}\right)^3,$$

$$\text{we have } A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{i-1}{n}\right)^3 \cdot \frac{1}{n} =$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{j=1}^{n-1} j^3 = \lim_{n \rightarrow \infty} \frac{1}{n^4} \left(\frac{(n-1)n}{2}\right)^2 =$$

$$= \lim_{n \rightarrow \infty} \frac{1}{4} \left(1 - \frac{1}{n}\right)^2 = \frac{1}{4}.$$

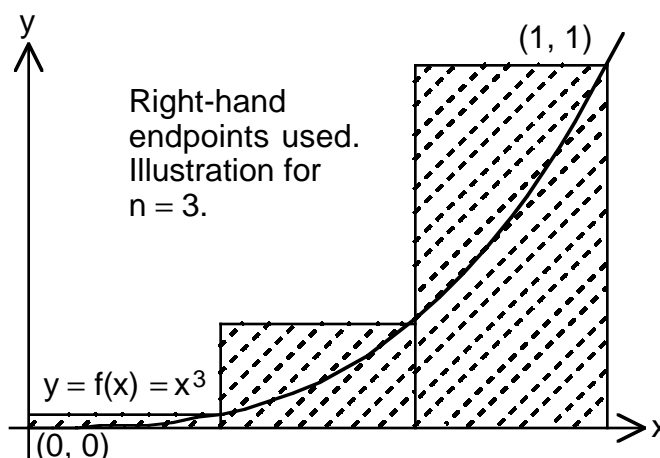
Here $(i-1)$ has been replaced by j , which goes from 0 to $(n-1)$ when i goes from 1 to n . The $j=0$ term (which is 0) has been dropped. See graph to the right.



(b) Using the right-hand endpoints i/n , with $f(i/n) = \left(\frac{i}{n}\right)^3$ this time we do not need to make the change of index $j = i - 1$ and the calculation is more straight-forward.

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{i}{n}\right)^3 \cdot \frac{1}{n} = \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n i^3 = \lim_{n \rightarrow \infty} \frac{1}{n^4} \left(\frac{n(n+1)}{2}\right)^2 = \\ &= \lim_{n \rightarrow \infty} \frac{1}{4} \left(1 + \frac{1}{n}\right)^2 = \frac{1}{4}. \end{aligned}$$

See graph to the right.



(c) Using the midpoints $(2i - 1)/2n$, y with $f((2i - 1)/2n) = \left(\frac{2i - 1}{2n}\right)^3$, we

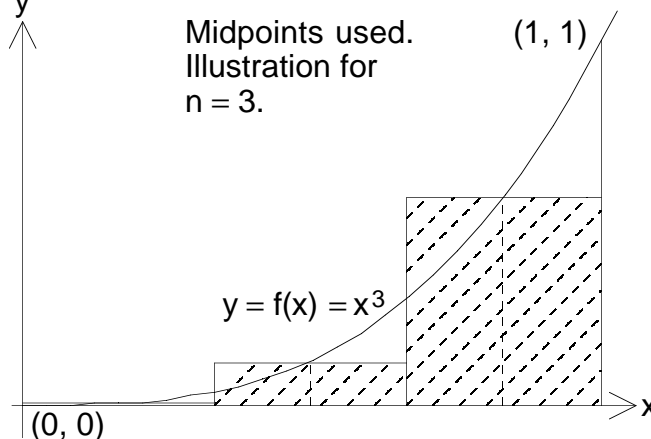
$$\begin{aligned} \text{have } A &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2i - 1}{2n}\right)^3 \cdot \frac{1}{n} = \\ &= \lim_{n \rightarrow \infty} \frac{1}{8n^4} \sum_{i=1}^n (2i - 1)^3. \end{aligned}$$

Now the problem is to evaluate this sum of cubes of all these **odd** numbers. One way to do that is to sum the cubes of **all** the integers from 1 to $2n$, and then subtract the sum of the cubes of all the **even** integers

between 1 and $2n$. Thus $\sum_{i=1}^n (2i - 1)^3$

$$\begin{aligned} &= [1^3 + 2^3 + \dots + (2n)^3] - [2^3 + 4^3 + \dots + (2n)^3] = \\ &= [1^3 + 2^3 + \dots + (2n)^3] - 2^3[1^3 + 2^3 + \dots + n^3] = \left(\frac{(2n)(2n+1)}{2}\right)^2 - 2^3 \left(\frac{n(n+1)}{2}\right)^2 = \\ &= n^2[(2n+1)^2 - 2(n+1)^2] = n^2(2n^2 - 1). \end{aligned}$$

So $A = \lim_{n \rightarrow \infty} \frac{1}{8n^4} n^2 [2n^2 - 1] = \frac{1}{4}$. See graph above and to the right.



18. $f(x) = x^4 + 3x + 2$.

Our interval is $[a, b] = [0, 3]$.

The i^{th} subinterval is $[3(i-1)/n, 3i/n]$ and has a width of $\frac{3}{n}$.

Since $f(3i/n) =$

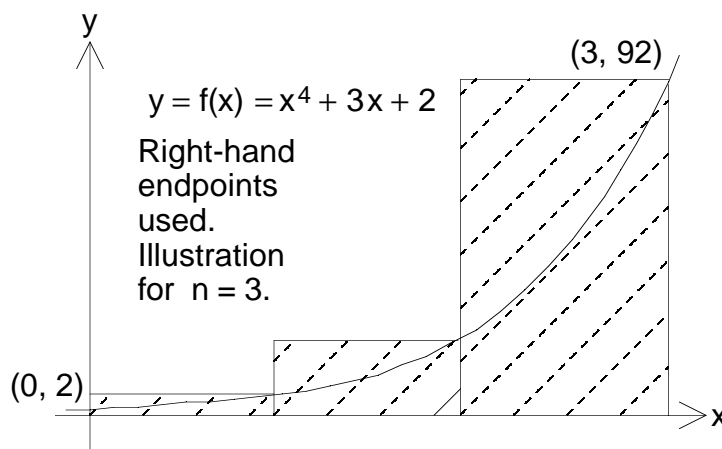
$$= \left(\frac{3i}{n}\right)^4 + 3\left(\frac{3i}{n}\right) + 2 =$$

$$= 2 + \frac{9i}{n} + \frac{81i^4}{n^4}, \text{ the area is}$$

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[2 + \frac{9i}{n} + \frac{81i^4}{n^4} \right] \cdot \frac{3}{n} =$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\frac{6}{n} + \frac{27i}{n^2} + \frac{243i^4}{n^5} \right] = \lim_{n \rightarrow \infty} \left[\frac{6}{n} \cdot n + \frac{27}{n^2} \cdot \frac{n(n+1)}{2} + \frac{243}{n^5} \cdot \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \right] =$$

$$= \lim_{n \rightarrow \infty} \left[6 + \frac{27}{2} \left(1 + \frac{1}{n} \right) + \frac{81}{10} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \left(3 + \frac{3}{n} - \frac{1}{n^2} \right) \right] = 68.1.$$



24. One way to interpret the expression $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \sqrt{1 + \frac{3i}{n}}$ is as the area of the region between the x -axis and the curve $y = \sqrt{1+x}$ and between the lines $x = 0$ and $x = 3$. Here $x_i = \frac{3i}{n}$ and x_i^* is the right-hand endpoint x_i of the i^{th} subinterval $[x_{i-1}, x_i]$. There are other correct interpretations.