Appendix B

HINTS FOR SELECTED EXERCISES

1:3.3 Let F be the set of all numbers of the form $x+y\sqrt{2}$ where x, $y \in \mathbb{Q}$. Again to be sure that nine properties of a field hold it is enough to check, here, that $a+b$ and $a \cdot b$ are in F if both a and b are.

1:3.5 As a first step define what x^2 and $2x$ really mean. In fact define 2. Then multiply $(x+1)\cdot(x+1)$ using only the rules given here. Since your proof uses only the field axioms it must be valid in any situation in which these axioms are true, not just for $\mathbb{R}$.

1:4.3 Suppose $a > 0$ and $b > 0$ and $a \neq b$. Establish that $\sqrt{a} \neq \sqrt{b}$. Establish that

$$(\sqrt{a} - \sqrt{b})^2 > 0.$$

Carry on. What have you proved? Now what if $a = b$?

1:6.4 You can use induction on the size of E, i.e., prove for every positive integer n that if E has n elements then $\sup E = \max E$.

1:7.3 Suppose not, then the set $\{1/n : n = 1, 2, 3, \dots\}$ has a positive lower bound, etc.. You will have to use the existence of a greatest lower bound.

1:7.7 Not that easy to show. Rule out the possibilities $\alpha^2 < 2$ and $\alpha^2 > 2$ using the Archimedean property to assist.

1:9.8 To find a number in (x, y) find a rational in $(x//\sqrt{2}, y//\sqrt{2})$. Conclude from this that the set of all (irrational) numbers of the form $\pm m\sqrt{2}/n$ is dense.

2:2.3 Here is a formula that generates the first five terms of the sequence 0, 0, 0, 0, c, ...

$$f(n) = c(n-1)(n-2)(n-3)(n-4)/4!.$$

2:2.10 The formula is

$$f_n = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right\}.$$

It can be verified by induction.

2:3.1 Find a function $f : (a, b) \to (0, 1)$ one-one onto and consider the sequence $f(s_n)$ where $\{s_n\}$ is a sequence that is claimed to have all of (a, b) as its range.

2:3.4 We can consider that the elements of each of the sets S_i can be listed, say,

$$S_1 = \{x_{11}, x_{12}, x_{13}, \dots\}$$

$$S_2 = \{x_{21}, x_{22}, x_{23}, \dots\}$$

and so on. Now try to think of a way of listing all of these items, i.e., making one big list that contains them all.

2:3.6 We need (i) every number has a decimal expansion, (ii) if a and b are numbers such that in the nth decimal place one has a 5 (or a 6) and the other does not then $a \neq b$, and (iii) every string of 5's and 6's defines a real number with that decimal expansion. Do you think that the fact that some numbers have *two* different decimal expansions (e.g., $12/100 = 0.12000 \cdots = 0.19999 \dots$) interferes with this proof?

2:3.10 Try to find a way of ranking the algebraic numbers in the same way that the rational numbers were ranked.

2:5.6 To establish a correct converse reword. If all $x_n > 0$ and $\frac{x_n}{x_n+1} \to 1$ then $x_n \to \infty$. Prove that this is true. The converse of the statement in the exercise is false (e.g., $x_n = 1/n$).

2:6.5 Use the same method as used in the proof of Theorem 2.11.

2:8.8 Take any number r strictly between 1 and that limit. Show that for some N, $s_{n+1} < rs_n$ if $n \geq N$. Deduce that $s_{N+2} < r^2 s_N$ and $s_{N+3} < r^3 s_N$. Carry on.

2:8.9 Take any number r strictly between 1 and that limit. Show that for some N, $s_{n+1} > rs_n$ if $n \geq N$. Deduce that $s_{N+2} > r^2 s_N$ and $s_{N+3} > r^3 s_N$. Carry on.

2:11.12 If a sequence contains subsequences converging to every number in $(0, 1)$ show that it also contains a subsequence converging to 0.

2:12.5 Consider the sequence $s_n = 1 + 1/2 + 1/3 + \dots 1/n$.

2:12.10 Compare to

$$1 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{6} - \dots$$

which is the sum of a geometric progression.

2:13.14 Consider separately the cases where the sequence is bounded or not.

2:14.12 For (a) show that

$$|s_{n+1} - s_n| \leq \frac{1}{17}|s_n - s_{n-1}|$$

for all $n = 2,\ 3,\ 4,\ \ldots$. For (b) you will need to use the fact that the sum of geometric progressions is bounded, in fact that

$$1 + r + r^2 + \ldots r^n < (1-r)^{-1}$$

if $0 < r < 1$. Express for $m > n$,

$$|s_m - s_n| \leq |s_{n+1} - s_n| + |s_{n+2} - s_{n+1}| + \cdots + \ldots |s_m - s_{m-1}|$$

and then use the contractive hypothesis. Note that

$$|s_4 - s_3| \leq r|s_3 - s_2| \leq r^2|s_2 - s_1|.$$

For (d) you might have to wait for the study of series in order to find an appropriate example of a convergent sequence that is not contractive.

2:14.14 This is from the 1947 Putnam Mathematical Competition.

2:14.15 This is from the 1949 Putnam Mathematical Competition.

2:14.16 This is from the 1950 Putnam Mathematical Competition.

2:14.17 This is from the 1953 Putnam Mathematical Competition.

2:14.18 Problem posed by A.. Emerson in the Amer. Math. Monthly 85 (1978), p. 496.

3:2.2 Define $\sum_{i\in I} a_i$ for I with zero or one elements. Suppose it is defined for I with n elements. Define it for I with $n+1$ elements and show well defined.

3:2.4 The answer is yes if I and J are disjoint. Otherwise the correct formula would be

$$\sum_{i\in I\cup J} a_i + \sum_{i\in I\cap J} a_i = \sum_{i\in I} a_i + \sum_{i\in J} a_i.$$

3:2.8 Try to interpret the "difference" $\Delta s_k = s_{k+1} - s_k = a_{k+1}$ as the analog of a derivative.

3:2.11 Use a telescoping sum method. Even if you cannot remember your trigonometric identities you can work backwards to see which one is needed. Check the formula for values of θ with $\sin\theta/2 = 0$ and see that it can be interpreted by taking limits.

3:3.1 This is similar to the statement that convergent sequences have unique limits. Try to imitate that proof.

3:3.2 This is similar to the statement that convergent sequences are bounded. Try to imitate that proof.

3:3.3 This is similar to the statement that monotone, bounded sequences are convergent. Try to imitate that proof.

3:3.9 Compare with the sum $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$ given in the introduction to this chapter.

3:4.10 Handle the case where each $a_k \geq 0$ separately from the general case.

3:4.14 Using properties of the log function you can view this series as a telescoping one.

3:4.15 Consider that

$$\frac{1}{r-1} - \frac{1}{r+1} = \frac{2}{r^2-1}.$$

3:4.25 As a first step show that

$$\int_{2k\pi+\pi/4}^{2k\pi+3\pi/4} \frac{|\sin x|}{x}\,dx \geq \frac{1}{\sqrt{2}} \int_{2k\pi+\pi/4}^{2k\pi+3\pi/4} \frac{1}{x}\,dx.$$

(Remember that in the calculus an integral $\int_0^\infty$ is interpreted as $\lim_{X\to\infty} \int_0^X$.)

3:5.5 Add up the terms containing p digits in the denominator. Note that our deletions leave only $8 \times 9^{p-1}$ of them. The total sum is bounded by $8(1/1 + 9/10 + 9^2/100 + \dots) = 80$.

3:5.14 Use the Cauchy-Schwartz inequality.

3:5.15 Use the Cauchy-Schwartz inequality.

3:6.3 The answer for (d) is $x < 1/e$.

3:6.24 The integral test should occur to you while thinking of this problem.

3:7.12 Imitate the proof of the first part of Theorem 3.49 but arrange for the partial sums to go larger than α before inserting a term q_k. You must take the *first* opportunity to insert q_k when this occurs.

3:12.3 For (h) consider the series $\sum_{k=1}^{\infty}(s_{k+1}-s_k)/s_{k+1}$ where s_k is the sequence of partial sums of the series given.

b Use Abel's method and the computation in Exercise 3:2.11.

3:12.5 This is from the 1948 Putnam Mathematical Competition.

3:12.6 This is from the 1952 Putnam Mathematical Competition.

3:12.7 This is from the 1954 Putnam Mathematical Competition.

3:12.8 This is from the 1955 Putnam Mathematical Competition.

3:12.9 This is from the 1964 Putnam Mathematical Competition.

3:12.10 This is from the 1988 Putnam Mathematical Competition.

3:12.11 This is from the 1994 Putnam Mathematical Competition.

3:12.12 Problem posed by A.. Torchinsky in the Amer. Math. Monthly 82 (1975), p. 936.

3:12.13 Problem posed by Jan Mycielski in the Amer. Math.Monthly, 83 (1976), p. 284.

4:2.25 Let $\{q_n\}$ be an enumeration of the rationals. If x is isolated then there is an open interval I_x containing x and containing no other point of the set. Pick the least integer n so that $q_n \in I_x$. This associates integers with the isolated points in a set.

4:3.1 Consider the set $\{1/n : n \in Nats\}$.

4:4.6 Consider the intersection of the family of *all* closed sets that contain the set E.

4:4.7 Consider the union of the family of *all* open sets that are contained in the set E.

4:5.1 Try this one: define $f(x) = 0$ for x irrational and $f(x) = q$ if $x = p/q$ where p/q is a rational with p, q integers and with no common factors.

4:5.5 Take compact to mean closed and bounded. Show that a finite union or arbitrary intersection of compact sets is again compact. Check that an arbitrary union of compact sets need not be compact. Show that any closed subset of a compact set is compact. Show that any finite set is compact.

4:5.8 For a course in functions of one variable open covers can consist of intervals. In more general settings there may be nothing that corresponds to an "interval", thus the more general covering by open sets is needed. Your task is just to look through the proof and spot where an "open interval" needs to be changed to an "open set".

4:5.9 Cousin's lemma offers the easiest proof, although any other compactness argument would work. Take the family of all intervals $[c, d]$ for which $f(c) < f(d)$ and check that the hypotheses of that lemma hold on any interval $[x, y]$.

4:5.18 Let $\mathcal{C} = \{V_\alpha : \alpha \in A\}$ be the open cover. Let N_1, N_2, ... be a listing of all open intervals with rational endpoints. For each $x \in E$ there is a $x \in V_\alpha$ and a k so that the interval N_k satisfies $x \in N_k \subset V_\alpha$. Call this choice $k(x)$. Thus

$$\mathcal{N} = \{N_{k(x)} : x \in E\}$$

is a countable open cover of E (but not the countable open cover that we want). But corresponding to each member of $\mathcal{N}$ is a member of $\mathcal{C}$ that contains it. Using that correspondence we construct the countable subcollection of $\mathcal{C}$ that forms a cover of E.

4:5.19 Lindelöff's theorem asserts that an open cover of any set of reals can be reduced to a countable subcover. The Heine-Borel theorem asserts that an open cover of any compact set of reals can be reduced to a finite subcover.

4:5.20 For (b)⇒(d) and for (c)⇒(d). Suppose there is an open cover of A but no finite subcover. Step 1: You may assume that the open cover is just a sequence of open sets. (This is because of Exercise 4:5.18.) Step 2: You may assume that the open cover is an increasing sequence of open sets $G_1 \subset G_2 \subset G_3 \subset \dots$ [just take the union of the first terms in the sequence you were given]. Step 3: Now choose points x_i to be in $G_i \cap A$ but not in any previous G_j for $j < i$. Step 4: Now apply (b) (or (c)) to get a point $z \in A$ that is an accumulation point of the points x_i. This would have to be a point in some set G_N (since these cover A) but for $n > N$ none of the points x_n can belong to G_N.

4:6.5 This result may seem surprising at first since the Cantor set, at first sight, seems to contain only the endpoints of the open intervals that are removed at each stage, and that set of endpoints would be countable. (That view is mistaken; there are many more points.) Show that a point x in $[0, 1]$ belongs to the Cantor set if and only if it can be written as a ternary expansion $x = 0.c_1, c_2, c_3 \dots$ (base 3) in such a way that only 0s and 2s occur. This is now a simple characterization of the Cantor set (in terms of string of 0s and 2s) and you should be able to come up with some argument as to why it is now uncountable.

4:6.9 You will need the Bolzano-Weierstrass theorem (Theorem 4.21). But this uncountable set E might be unbounded. How could we prove that an uncountable set would have to contain an infinite bounded subset? Consider $E = \bigcup_{n=1}^{\infty} E \cap [-n, n]$.

4:6.10 Select a rational number from each member of the family and use that to place them in an order.

4:7.5 For part (b) look ahead to part (c): any such example must have A and B unbounded. For part (c) assume $\delta(A, B) = 0$. Then there must be points $x_n \in A$ and $y_n \in B$ with $|x_n - y_n| < 1/n$. As A is compact there is a convergent subsequence x_{n_k} converging to a point z in A. What is happening to y_{n_k}? (Be sure to use here the fact that B is closed.)

5:1.1 Model your answer after Example 5.2.

5:1.2 Consider the cases $a = 0$ and $a \neq 0$ separately. If it is easier for you, break into the three cases $a > 0$, $a < 0$ and $a = 0$.

5:1.3 Model your answer after Example 5.3.

5:1.4 Consider the cases $x_0 = 0$ and $x_0 \neq 0$ separately. Use the factoring trick in Example 5.3 and the device of restricting x to be close to x_0 by assuming that $|x - x_0| < 1$ at least.

5:1.8 Don't forget to exclude $x_0 < 0$ from your answer since it is not a point of accumulation of the domain of this function. Consider the cases $x_0 = 0$ and $x_0 > 0$ separately.

5:1.12 If $B \subset A$ then the existence of $\lim_{x\to x_0} g(x)$ can be deduced from the existence of $\lim_{x\to x_0} f(x)$. Can you find other conditions? If x_0 is a point of accumulation of $A \cap B$ then the equality of the two limits can be deduced, assuming that both exist.

5:1.16 Either find a single sequence $x_n \to 0$ with $x_n \neq 0$ so that the limit $\lim_{n\to\infty} |x_n|/x_n$ does not exist or else find two such sequences with different limits.

5:1.22 You could assume (i) that $L > 0$ or (ii) that $f(x) \geq 0$ for all x in its domain. Then convert to a statement about sequences.

5:1.28 At $x_0 \neq 0$ the two one sided limits are equal. What are they? At $x_0 = 0$ they differ.

5:1.29 On one side the limit is zero and on the other the limit fails to exist. (Look ahead to Exercise 5:1.38 where you are asked to show that the limit is ∞ which means that the limit does not exist.) You may use the elementary inequality $0 < z < e^z$ (which is valid for all $z > 0$) in your argument. Consider the sequences $1/n \to 0$ and $-1/n \to 0$.

5:1.30 Check the definition: there would be no distinction. The limit $\lim_{x\to 0-} \sqrt{x}$, however, would be meaningless since 0 is not a point of accumulation of the domain of the square root function on the left.

5:1.34 Use the definitions in this section as a model. You will need a replacement for the "x_0 is a point of accumulation" of the domain condition. If you cannot think of anything better then simply use the assumption that f is defined in some interval (a, ∞).

5:1.38 On one side at 0 the limit is zero and on the other the limit is ∞. See Exercise 5:1.29.

5:2.1 Model your proof after Theorem 2.8 for sequences.

5:2.3 If the theorem were false then in every interval $(x_0 - 1/n, x_0 + 1/n)$ there would be a point x_n for which $|f(x_n)| > n$.

5:2.9 If x_0 is not a point of accumulation of $dom(f) \cap dom(g)$ then the statement $\lim_{x\to x_0} f(x) + g(x) = L$ does not have any meaning even though the two statements about $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ may have.

5:2.11 What exactly is the domain of the function $f(x)/g(x)$? Show that x_0 would be a point of accumulation of that domain provided $g(x) \to C$ as $x \to x_0$ and $C \neq 0$.

5:2.28 It is enough to assume that $\lim_{x\to x_0} f(x)$ exists and to apply Theorem 5.24 with $F(x) = |x|$. Be sure to explain why this function F has the properties expressed in that theorem.

5:2.29 It is enough to assume that $\lim_{x\to x_0} f(x)$ exists and is positive and then apply Theorem 5.24 with $F(x) = \sqrt{x}$. Alternatively assume that $f(x) \geq$ for all x in a neighborhood of x_0. Again be sure to explain why this function F has the properties expressed in that theorem.

5:2.32 Use the property of exponentials that $e^{a+b} = e^a e^b$ and the product rule for limits.

5:2.33 Use a trigonometric identity for $\sin(x-x_0+x_0)$ and the sum and products rule for limits.

5:2.34 Take the function $H(x)$ of the text and consider instead $H(x) + x$.

5:2.36 This would be trivial if the sets A_i were disjoint. So it is the case where these are not disjoint that you need to address.

5:2.42 If x_0 is not in the Cantor set K then it is in some open interval complementary to that set. Use that to prove the existence of the limit. If x_0 is in the Cantor set then there must be sequences $x_n \to x_0$ and $y_n \to x_0$ with $x_n \in K$ and $y_n \not{i}nK$. Use that to prove the nonexistence of the limit.

5:3.5 Consider separately the cases $x_0 \in E$ and $x_0 \notin E$. Under what circumstances in the latter case would the lim sup be larger according to this revised definition?

5:4.15 One of the definitions treats isolated points in a special way. Note that each point in the domain of f is isolated.

5:4.17 You must arrange for $f(0)$ to be the limit of the sequence of values $f(2^{-n})$. No other condition is necessary.

5:4.19 At an isolated point x_0 of the domain the limit $\lim_{x\to x_0} f(x)$ has no meaning. But if x_0 is not an isolated point in the domain of f it must be a point of accumulation and then $\lim_{x\to x_0} f(x)$ is defined and it must be equal to $f(x_0)$.

5:4.20 For the converse consider the function $f(x) = \sqrt{x}$ on $[0, 1]$.

5:4.33 The equation $f(x+y) = f(x) + f(y)$ is called a functional equation. You are told about this function only that it satisfies such a relationship and has a nice property at one point. Now you must show that this implies more. Show first that $f(0) = 0$ and that $f(x-y) = f(x) - f(y)$.

5:4.34 This continues Exercise 5:4.33. Show first that $f(r) = rf(1)$ for all $r = m/n$ rational. Then make use of the continuity of f that you had already established in the other exercise.

5:4.35 Show that either f is always zero or else $f(0) = 1$. Establish $f(x - y) = f(x)/f(y)$.

5:6.1 Let $a = \inf K$ and $b = \sup K$ and apply Cousin's Lemma to the interval $[a, b]$ by taking the same collection nearly, namely $\mathcal{C}$ consist of all closed subintervals $[t, s]$ such that $|f(t') - f(s')| < \varepsilon/2$ for all t', $s' \in K \cap [t, s]$. You will have to find a different choice of δ to make your argument work.

5:6.3 If the set X has no points of accumulation this is possible. If the set X does have a point of accumulation then it is possible to give an example of a function defined on X that is not uniformly continuous on X.

5:6.5 You need consider only two compact sets X_1, X_2. Since they are compact there is a positive distance between them which you can use to help define your δ. For not closed consider $X_1 = (0, 1)$ and $X_2 = (1, 2)$ and define f appropriately. For not bounded use $X_1 = \{1, 2, 3, \dots\}$ and $X_2 = \{1, 2+1/2, 3+1/3, 4+1/4, \dots$ and define f appropriately.

5:6.6 For the converse consider the function $f(x) = \sqrt{x}$ on $[0, 1]$. By Theorem 5.43 we know that this function is uniformly continuous on $[0, 1]$.

5:6.7 Show that any function defined on a set X containing just one element is uniformly continuous. Then consider the sequence $X_i = \{x_i\}$, $i = 1, 2, \dots, n$.

5:6.8 For the sequence of intervals you might choose [1, 2], [2, 3], [3, 4], ... (Why would you not be able to choose [1/2, 1], [1/4, 1/2], [1/8, 1/4], ... ?)

5:6.9 Use an indirect proof. Show that if f is not uniformly continuous then there are sequences $\{x_n\}$ and $\{y_n\}$ so that $x_n - y_n \to 0$ but $|f(x_n) - f(y_n)| > c$ for some positive c. Now apply the Bolzano-Weierstrass property to obtain subsequences and get a contradiction.

5:6.10 Using the local continuity property claim that there are open intervals I_x containing any point x so that $|f(y) - f(x)| < \varepsilon$ for any $y \in I_x$. Now apply the Heine-Borel property to this open cover. Obtain uniform continuity from the finite subcover.

5:6.12 Let $\mathcal{C}$ be the collection of all closed intervals $I \subset [a, b]$ so that f is bounded on I. Use Cousin's lemma to find a partition of $[a, b]$ using intervals in $\mathcal{C}$.

5:6.13 Use an indirect proof. Show that if f is not bounded then there is a sequence $\{x_n\}$ of points in $[a, b]$ so that $|f(x_n)| > n$ for all n. Now apply the Bolzano-Weierstrass property to obtain subsequences and get a contradiction.

5:6.14 Using the local continuity property claim that there are open intervals I_x containing any point x so that $|f(y) - f(x)| < 1$ for any $y \in I_x$. Now apply the Heine-Borel property to this open cover. Obtain boundedness of f from the finite subcover.

5:7.2 That is, prove that the image set $f(K)$ is compact if K is compact and f is a continuous function defined at every point of K. Apply Theorem 5.47.

5:7.3 Let $M = \sup\{f(x) : a \leq x \leq b\}$. Explain why you can choose a sequence of points $\{x_n\}$ from $[a, b]$ so that $f(x_n) > M - 1/n$. Now apply the Bolzano-Weierstrass theorem and use the continuity of f.

5:7.5 If $f(x_0) = c > 0$ then there in an interval $[-N, N]$ so that $x_0 \in [N, N]$ and $|f(x)| < c/2$ for all $x > N$ and $x < -N$.

5:8.3 Suppose that the theorem is false and explain, then, why there should exist sequences $\{x_n\}$ and $\{y_n\}$ from $[a, b]$ so that $f(x_n) > c$, $f(y_n) < c$ and $|x_n - y_n| < 1/n$.

5:8.4 Suppose that the theorem is false and explain, then, why there should exist at each point $x \in [a, b]$ an open interval I_x centered at x so that either $f(t) > c$ for all $t \in I_x \cap [a, b]$ or else $f(t) < c$ for all $t \in I_x \cap [a, b]$.

5:8.5 The function must be onto. Hence there is a point x_1 with $f(x_1) = a$ and a point x_2 with $f(x_2) = b$. Now convince yourself that there is a point on the graph of the function that is also on the line $y = x$.

5:8.7 Condition (a) is the intermediate value property IVP according to Definition 5.26 while (b) can be interpreted as saying that connectedness is preserved by continuous functions. This latter interpretation requires a careful definition of connectedness in $\mathbb{R}$.

5:8.8 That is, prove that the image set $f([c, d])$ is a compact interval for any interval $[c, d]$ if f is a continuous function defined at every point of $[c, d]$. Apply Theorem 5.47 and Theorem 5.48.

5:9.14 You wish to show that (i) f is discontinuous at every point in C, indeed has a jump discontinuity at each such point, (ii) f is continuous at every point not in C, (iii) f is nondecreasing, (iv) f is increasing on any interval in which C is dense, and (v) f is constant on any interval containing no point of C.

The most direct and easiest proof that f is continuous at every point not in C would be to use "uniform convergence" but that is in a later chapter. Here you will have to use an ε, δ argument.

5:9.16 How large can the set of discontinuity points be?

5:9.17 The function f^{-1} is defined on the interval $J = [f(a), f(b)]$. Explain first why it exists (not all functions must have an inverse). Prove that it is increasing. Prove that it is continuous (using the fact that it is increasing).

5:10.11 You will need to use the fact that

$$\{x : \limsup_{x \to x_0^-} f(x) > \limsup_{x \to x_0^+} f(x)\}$$

is countable. See Exercise 5:10.2.

6:2.10 To make this true assume that f is onto or else show that if E is dense then $f(E)$ is dense in the set (interval) $f(\mathbb{R})$.

6:3.1 If $q_1, q_2, q_3, \ldots$ is an enumeration of the rationals, then each of the sets $\{q_i\}$, $i \in \mathbb{N}$, is nowhere dense, but $\bigcup_{i=1}^{n} \{q_i\} = \mathbb{Q}$ is not nowhere dense. (Indeed it is dense.)

6:3.2 All of (a)–(e) and (h) are true. Find counterexamples for (f) and (g). The proofs that the others are true follow routinely from the definition.

6:4.1 Suppose that $A_n = \bigcup_{k=1}^{\infty} A_{nk}$ with each of the sets A_{nk} nowhere dense. Then

$$\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{nk} = \bigcup_{n,k=1}^{\infty} A_{nk}$$

expresses that union as a first category set.

6:4.2 Let $\{B_n\}$ be a sequence of residual subsets of $\mathbb{R}$. Thus each of the sets B_n is the complement of a first category set A_n. For each n write $A_n = \bigcup_{k=1}^{\infty} A_{nk}$ with each of the sets A_{nk} nowhere dense. Then $B_n = \mathbb{R} \setminus \bigcup_{k=1}^{\infty} A_{nk}$. Now use the de Morgan laws.

6:4.3 Suppose that X is residual, i.e.,

$$X = \mathbb{R} \setminus \bigcup_{n=1}^{\infty} Q_n$$

where each Q_n is nowhere dense. Show that for any interval $[a, b]$ there is a point in $X \cap [a, b]$ by constructing an appropriate descending sequence of closed subintervals of $[a, b]$.

6:4.4 Make sure your sets are dense but not both residual (e.g., $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$).

6:4.5 This follows, with the correct interpretation, directly from the Baire Category theorem.

6:4.7 Consider the sequence

$$A_N = \{x \in [0, 1] : |f_n(x)| \leq 1 \text{ for all } n \geq N\}.$$

Check that $\bigcup_{N=1}^{\infty} A_N = [0, 1]$.

6:5.7 It is clear that there must be many irrational numbers in the Cantor ternary set, since that set is uncountable and the rationals are countable. Your job is to find just one.

6:5.10 Consider $G = (0, 1) \setminus C$ where C is the Cantor ternary set.

6:6.7 Often to prove a set identity such as this the best way is to start with a point x that belongs to the set on the right and then show that point must be in the set on the left. After that is successful start with a point x that belongs to the set on the left. For example here, if $f(x) > \alpha$ then $f(x) \geq \alpha + 1/m$ for some integer m. But $f_n(x) \to f(x)$ and so there must be an integer R so that $f_n(x) > \alpha + 1/m$ for all $n \geq R$. Etc.

This exercise shows how unions and intersections of sequences of open and closed sets might arise in analysis. Note that the sets $\{x : f_n(x) \geq \alpha + 1/m\}$ would be closed if the functions f_n are continuous. Thus it would follow that the set

$$\{x : f(x) > \alpha\}$$

must be of type $\mathcal{F}_\sigma$. This says something very interesting about a function f that is the limit of a sequence of continuous functions $\{f_n\}$.

6:7.3 You need to recall Theorem 5.55 which asserts that monotone functions have left and right hand limits.

6:9.1 This is from the 1964 Putnam Mathematical Competition.

7:2.1 Write $x = x_0 + h$.

7:2.6 Write $f(x+h) - f(x-h) = [f(x+h) - f(x)] + [f(x) - f(x-h)]$.

7:2.7 Use $1 - \cos x = 2\sin^2 x/2$. When you take the square root be sure to use the absolute value.

7:2.12 Just use the definition of the derivative. Give a counterexample with $f(0) = 0$ and $f'(0) > 0$ but so that f is not increasing in any interval containing 0.

7:2.13 Even for polynomials $p(x)$ increasing does not imply that $p'(x) > 0$ for all x. For example take $p(x) = x^3$. That has only one point where the derivative is not positive. Can you do any better?

7:2.14 Actually the assumptions are different. Here we assume $f'(x_0)$ does exist, whereas in the trapping principle we had to assume more inequalities to deduce that it exists.

7:2.15 Review Exercise 5:4.35 first.

7:2.15 Advanced (very advanced) methods would allow you to find a function continuous on $[0, 1]$ that is differentiable at *no* point of that interval. For the purpose of this exercise just try to find one that is not differentiable at $1/2$, $1/3$, $1/4$, $\ldots$. (Novices constructing examples often feel they need to give a simple formula for functions. Here, for example you can define the function on $[1/2, 1]$, then on $[1/4, 1/2]$, then on $[1/8, 1/4]$, and so on $\ldots$ and then finally at 0.)

7:2.17 Find two examples of functions, one continuous and one discontinuous at 0, with an infinite derivative there.

7:2.18 Imitate the proof of Theorem 7.6. Find a counterexample to the question.

7:3.5 Use Theorem 7.7 (the product rule) and for the induction step consider

$$\frac{d}{dx}x^n = \frac{d}{dx}[x][x^{n-1}].$$

7:3.10 This formula is known as Leibnitz's rule (which should indicate its age). It is

$$(fg)^{(n)}(x_0) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^{(k)}(x_0) g^{(n-k)}(x_0).$$

7:3.11 Consider a sequence $x_n \to x_0$ with $x_n \neq x_0$ and $f(x_n) = f(x_0)$.

7:3.12 Let $f(x) = x^2 \sin x^{-1}$ ($f(0) = 0$) and take $x_0 = 0$. Utilize the fact that 0 is a limit point of the set $\{x : f(x) = 0\}$.

7:3.17 If $I(x)$ is the inverse function then $I(\sin x) = x$. The chain rule gives derivative as $I'(\sin x) = 1/\cos x$. This needs some work. Use $\cos x = \sqrt{1 - \sin^2 x}$ and obtain

$$I'(\sin x) = \frac{1}{\sqrt{1 - \sin^2 x}}.$$

Now replace the $\sin x$ by some other variable. Caution: while doing this exercise make very sure that you know how the arcsin function $\sin^{-1} x$ is actually defined. It is not the inverse of the function $\sin x$ since that function has no inverse.

7:3.19 Draw a good picture. The graph of $y = g(x)$ is the reflection in the line $y = x$ of the graph of $y = f(x)$. What is the slope of the reflected tangent line?

7:3.21 Use the idea in the example. If $f(x) = x^{1/m}$ then $[f(x)]^m = x$ and use the chain rule. If $F(x) = x^{n/m}$ then $[F(x)]^m = x^n$ and use the chain rule.

7:3.22 Once you know that $\frac{d}{dx}e^x = e^x$ you can determine that $\frac{d}{dx}\ln x = 1/x$ using inverse functions. Then consider

$$\frac{d}{dx}x^p = e^{(\ln p)x}.$$

7:3.23 The formula you should obtain is

$$a_k = \frac{p^{(k)}(0)}{k!}$$

for $k = 0, 1, 2, \ldots$.

7:3.24 If you succeed then you have proved the binomial theorem using derivatives. Of course you need to compute $p(0)$, $p'(0)$, $p''(0)$, $p'''(0)$, ... to do this.

7:5.6 Consider sets of the form

$$A_n = \left\{x : f(t) < f(x) \text{ for } t \in \left(x - \frac{1}{n}, x\right) \cup \left(x, x + \frac{1}{n}\right)\right\},$$

and observe that $\bigcup_{n=1}^{\infty} A_n$ is the set in question.

7:5.7 Modify the hint in Exercise 7:5.6.

7:6.3 Use Rolle's theorem to show that if x_1 and x_2 are distinct solutions of $p(x) = 0$ then between them is a solution of $p'(x) = 0$.

7:6.4 Use Rolle's Theorem twice. See Exercise 7:6.6 for another variant on the same theme.

7:6.5 Since f is continuous we already know (look it up) that f maps $[a, b]$ to some closed bounded interval $[c, d]$. Use Rolle's Theorem to show that there cannot be two values in $[a, b]$ mapping to the same point.

7:6.6 cf. Exercise 7:6.4.

7:6.8 First show directly from the definition that the Lipshitz condition will imply a bounded derivative. Then use the mean value theorem to get the converse, i.e., apply the mean value theorem to f on the interval $[x, y]$ for any $a \leq x < y \leq b$.

7:6.9 Note that an increasing function f would allow only positive numbers in S.

7:6.12 Apply the mean value theorem to f on the interval $[x, x + a]$ to obtain a point ξ in $[x, x + a]$ with $f(x + a) - f(x) = af'(\xi)$.

7:6.13 Use the mean value theorem to compute

$$\lim_{x \to a+} \frac{f(x) - f(a)}{x - a}.$$

7:6.14 This is just a variant on Exercise 7:6.13. Show that under these assumptions f' is continuous at x_0.

7:6.15 Use the mean value theorem to relate $\sum_{i=1}^{\infty}(f(i+1) - f(i))$ to $\sum_{i=1}^{\infty} f'(i)$. Note that f is increasing and treat the former series as a telescoping series.

7:6.16 The proof of the mean value theorem was obtained by applying Rolle's theorem to the function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

For this mean value theorem apply Rolle's theorem twice to a function of the form

$$h(x) = f(x) - f(a) - f'(a)(x - a) - \alpha(x - a)^2$$

for an appropriate number α.

7:6.18 Write $f(x + h) + f(x - h) - 2f(x) = [f(x + h) - f(x)] + [f(x - h) - f(x)]$ and apply the mean value theorem to each term.

7:6.19 Let

$$\phi(x) = \begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f(x) & g(x) & h(x) \end{vmatrix}$$

and imitate the proof of Theorem 7.20.

7:7.1 Interpret as a monotonicity statement about the function $f(x) = (1-x)e^x$.

7:7.5 Interpret this as a monotonicity property for the function $F(x) = f(x)/x$. We need to show that F' is positive. Show that this is true if $f'(x) > f(x)/x$ for all x. But how can we show this? Apply the mean value theorem to f on the interval $[0, x]$ (and don't forget to use the hypothesis that f' is an increasing function).

7:7.6 If not there is an interval $[a, b]$ with $f(a) = f(b) = 0$ and neither f nor g vanish on (a, b). Show that $f(x)/g(x)$ is monotone (strictly) on $[a, b]$.

7:8.7 Let $\varepsilon > 0$ and consider $f(x) + \varepsilon x$.

7:8.9 For part (a) fingure out a way to express $\mathbb{R}$ as a countable union of disjoint dense sets A_n and then let $f(x) = n$ for all $x \in A_n$. For part (b) subtract an appropriate linear function F from f such that $f - F$ is not an increasing function, and apply Theorem 7.28.

7:8.10 In connection with this exercise we should make this remark. If $A = \{a_k\}$ is any countable set, then the function defined by the series

$$\sum_{k=0}^{\infty} \frac{-|x - a_k|}{2^k}$$

has $D^+f(x) < D_-f(x)$ for all $x \in A$. This can be verified using the results in Chapter9 on uniform convergence.

7:9.1 For the third part use the function $F(x) = x^2 \sin x^{-1}$, $F(0) = 0$ to show that there exists a differentiable function f such that $f'(x) = \cos x^{-1}$, $f(0) = 0$). Consider $g(x) = f(x) - x^3$ on an appropriate interval.

7:9.3 If either FG' or GF' were a derivative, so would the other be since $(FG)' = FG' + GF'$. In that case $FG' - GF'$ is also a derivative. But now show that this is impossible (because of (c)).

7:9.4 Use $fg' = (fg)' - f'g$. You need to know the Fundamental Theorem of Calculus to continue.

7:10.1 Show that at every point of continuity of f'_+ the function is differentiable. How many discontinuities does the (nondecreasing) function f'_+ have?

7:10.7 The methods of Chapter 9 would help here. There we learn in general how to check for the differentiability of functions defined by series. For now just use the definitions and compute carefully.

7:10.8 For (d) let

$$f(x) = \begin{cases} e^{-1/x^2}(\sin 1/x)^2, & \text{for } x > 0 \\ 0, & \text{for } x = 0, \\ -e^{-1/x^2}(\sin 1/x)^2, & \text{for } x < 0 \end{cases}$$

7:11.1 Use L'Hôpital's rule to find that $f(0)$ should be $\ln(3/2)$. Use the definition of the derivative and L'Hôpital's rule twice to compute $f'(0) = [(\ln 3)^2 - (\ln 2)^2]/2$. Exercise 7:6.13 shows that the technique in (c) part does in fact compute the derivative provided only that you can show that this limit exists.

7:11.2 Treat the cases $A > 0$ and $A < 0$ separately.

7:11.10 We must have $\lim_{x\to\infty} f'(x) = 0$ in this case. (Why?)

7:13.3 Consider the function $H(x) = p(x) + p'(x) + p''(x) + \cdots + p^{(n)}(x)$ and note, in particular, the relation between H, H' and p.

7:13.9 If g does not vanish on (x_1, x_2), then Rolle's theorem applied to the quotient f/g provides a contradiction.

7:13.11 This is from the 1939 Putnam Mathematical Competition.

7:13.12 This is from the 1946 Putnam Mathematical Competition.

7:13.13 This is from the 1958 Putnam Mathematical Competition.

7:13.14 This is from the 1962 Putnam Mathematical Competition.

7:13.15 This is from the 1992 Putnam Mathematical Competition.

7:13.16 This is from the 1998 Putnam Mathematical Competition.

8:2.1 You will need to find a formula for $\sum_{k=1}^{n} k^3$.

8:2.9 Be sure, first, to check that these associated points are legitimate. Show that each of these sums has the same value (think of telescoping sums!). What, then would be the limit of the Riemann sums?

8:4.5 It would converge for all continuous functions.

8:5.2 Define $\int_{-\infty}^{\infty} f(x)\,dx$ to be the sum of $\int_{-\infty}^{a} f(x)\,dx$ and $\int_{a}^{\infty} f(x)\,dx$. Be sure to prove that this definition would not depend on the choice of a.

8:5.10 Compare with Exercise 3:4.25.

8:6.1 Note that this function is discontinuous everywhere and that $\omega f([c,d]) = 1$ for every interval $[c,d]$.

8:6.3 The answer is no. It would be true if $|f| > c > 0$ everywhere. Equivalently it is true if $1/f$ is bounded.]

8:6.4 Step functions were defined in Section 5.2.6. If you sketch a picture of what the approximating sums look like the step functions needed should be apparent.

8:6.6 The fact that the oscillation of a function f is smaller than η at each point of an interval $[c, d]$ is a local condition. Express it by using a $\delta(x)$ at each point. Now use a compactness argument (e.g., Heine-Borel) to get a uniform size that works.

8:7.1 Make ϕ' integrable and f continuous at each point $\phi(t)$ for $t \in [a, b]$.

8:7.7 For (a): What if F is discontinuous? For (b): Consider the Cantor function (Section 6.5.3). For (d): This is not easy! We will discuss this in Section 9.7.

8:10.1 This is from the 1947 Putnam Mathematical Competition.

9:3.15 Use the Cauchy criterion for convergence of sequences of real numbers to obtain a candidate for the limit function f. Note that if $\{f_n\}$ is uniformly Cauchy on a set D then for each $x \in D$, the sequence of real numbers $\{f_n(x)\}$ is a Cauchy sequence and hence convergent.

9:3.16 $S_n(x) = \sum_{k=0}^{n} x^k = \frac{1-x^n}{1-x}$.

9:4.10 For part (b) consider $F_n(x) = f_n(x) + Mx$ and apply Exercise 9:4.6.

9:7.15 Suppose h' were integrable. Explain why $h(x) - h(a) = \displaystyle\int_a^x h'(t)\, dt$ for all $x \in [a, b]$. Now by considering an appropriate Riemann sum, since $h' = 0$ on a dense set, we would have $h(x) - h(a) = 0$ for all $x \in [a, b]$. That should be a contradiction.

9:8.1 What properties would F' have to have if the convergence were uniform?

9:9.1 You will need to use the Baire category theorem for the second part of this.

10:2.2 This follows immediately from the inequalities

$$\liminf_k \left|\frac{a_{k+1}}{a_k}\right| \le \liminf \sqrt[k]{|a_k|} \le \limsup_k \sqrt[k]{|a_k|} \le \limsup_k \left|\frac{a_{k+1}}{a_k}\right|$$

that we obtained in Exercise 2:13.15.

10:3.3 Write out the Cauchy criterion for uniform convergence on $(-r, r)$ and deduce that the Cauchy criterion for uniform convergence on $[-r, r]$ must then also hold.

10:4.4 $\int_0^1 \frac{1-e^{-sx}}{s}\, ds = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k(k!)} x^k$.

10:5.4 It is clear that $f^{(k)}$ exists for all $x \neq 0$. For $x = 0$ verify the following assertions:

1. $f^{(k)}(0)$ is of the form $R(x)e^{-1/x^2}$ for $x \neq 0$, where R is a rational function.
2. Show that $\lim_{x\to 0} \frac{1}{x^n}e^{-1/x^2} = 0$ for all $n = 1, 2, \ldots$.
3. Conclude that $\lim_{x\to 0} f^{(k)}(x) = 0$ for all $k = 1, 2, \ldots$.
4. Conclude that $f^{(k)}(0) = 0$ for all k.

10:6.2 Just use Theorem 10.32.

10:7.1 Just use Theorem 10.33.

10:8.1 Quite easy really. Just substitute $u = x + t$ in the integral $\int_0^\pi f(x + t)D_n(t)\,dt$ and expand the terms $\cos(ku - kx)$ using standard trigonometric identities.

10:8.5 First obtain a polynomial q so that $|f(x) - q(x)| < \varepsilon/2$. Then find a polynomial p with rational coefficients so that $|p(x) - q(x)| < \varepsilon/2$.

10:8.6 Try $f(x) = e^x$.

10:8.7 Try $f(x) = 1/x$.

10:8.8 Show that f must be identically equal to zero. Use Theorem 10.37.

10:8.9 Define $G(t) = f(t/\pi)$ for $t \in [0, \pi]$ and extend to $[-\pi, 0]$ by $G(-t) = -G(t)$. Consider the Fourier series of G and show that it contains only sin terms (no cosine terms). Show that f must be identically equal to zero. Use Theorem 10.36.

A:2.4 For (c) and (d): All numbers do not have a unique decimal expansion; for example $1/2$ can be written as $0.5000000\ldots$ or as $0.499999999\ldots$. For (e): take the domain as the set $\mathbb{N}$. Are you troubled (some people might be) by the fact that nobody knows how to determine if x is a prime number when x is very large?

A:2.13 As a project, research the topic of Russell's paradox (named after Bertrand Russell (1872-1969) who discovered this in the early days of set theory and caused a crisis thereby.)

A:5.1 Suppose not. Then $\sqrt{2}$ is rational. This means $\sqrt{2} = m/n$ where m and n are not both even. Square both sides to obtain $2n^2 = m^2$. Continue arguing until you can show that both m and n are even. That is your contradiction and the proof is complete.

A:5.2 Suppose not. Then it is possible to list all the primes

$$2, 3, 5, 7, 11, 13, \ldots P$$

where P is the last of the primes. Consider the number

$$1 + (2 \times 3 \times 5 \times 7 \times 11 \times \ldots P).$$

From this obtain your contradiction and the proof is complete. [To be completely accurate here one needs to know the prime factorization theorem: every number can be written as a product of primes.] This is a famous proof known in ancient Greece.

A:6.1 The contrapositive statement reads "if $x+r$ is not irrational for all rational numbers r then x is not irrational". Translate this to "if $x + r$ is rational for some rational number r then x is rational". Now this statement is easy enough to prove.

A:8.1 Check for $n = 1$. Assume that

$$1^2 + 2^2 + 3^2 + \ldots n^2 = \frac{n(n+1)(2n+1)}{6}$$

is true for some fixed value of n. Using this assumption (called the induction hypothesis in this kind of proof) try to find an expression for

$$1^2 + 2^2 + 3^2 + \ldots n^2 + (n+1)^2.$$

It should turn out to be exactly the correct formula for the sum of the first $n+1$ squares. Then claim the formula is now proved for all n by induction.

A:8.9 The induction step requires us to show that if the statement for n is true then so is the statement for $n + 1$. This step must be true if $n = 1$ and if $n = 2$ and if $n = 3 \ldots$, in short, for all n. Check the induction step for $n = 3$ and you will find that it does work; there is no flaw. Does it work for all n?