Chapter 9

SEQUENCES AND SERIES OF FUNCTIONS

9.1 Introduction

We have seen that a function f that is the sum of two or more functions will share certain desirable properties with those functions. For example, our study of continuity, differentiation and integration allows us to state that if

$$f = f_1 + f_2 + \cdots + f_n$$

on an interval $I = [a, b]$, then:

(1) If $f_1, f_2, \ldots, f_n$ are continuous on I, so is f.

(2) If $f_1, f_2, \ldots, f_n$ are differentiable on I, so is f, and

$$f' = f_1' + f_2' + \cdots + f_n'.$$

(3) If $f_1, f_2, \ldots, f_n$ are integrable on I, so is f, and

$$\int_a^b f(x)\, dx = \int_a^b f_1(x)\, dx + \int_a^b f_2(x)\, dx + \cdots + \int_a^b f_n(x)\, dx.$$

It is natural to ask whether the corresponding results hold when f is the sum of an *infinite* series of functions,

$$f = \sum_{k=0}^{\infty} f_k.$$

If each term of the series is continuous, is the sum function also continuous? Can the derivative be obtained by summing the derivatives? Can the integral be obtained by summing the integrals? We study such questions in this chapter.

These problems are of considerable practical importance. For example if we are allowed to take limits, integrate and differentiate freely then the computations in the following example would all be valid.

Example 9.1 From the formula for the sum of a geometric series we know that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots \tag{1}$$

on the interval $(-1, 1)$. Differentiation of both sides of (1) leads immediately to

$$\frac{-1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 - 5x^4 + \dots.$$

Repeated differentiation would give formulas for $(1 + x)^{-n}$ for all positive integers n.

On the other hand integration of both sides of (1) from 0 to t leads immediately to

$$\ln(1+t) = t - \frac{1}{2}t + \frac{1}{3}t^2 - \frac{1}{4}t^3 + \frac{1}{5}t^4 - \dots$$

Taking limits as $t \to 1$ in the latter yields the intriguing formula for the sum of the alternating harmonic series:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

◀

The conclusions in the example are all true and useful. But have we used illegitimate means to find them? If we use such methods freely might we find situations where our conclusions are very wrong?

We first formulate our questions in the language of sequences of functions (rather than series). We do this in Section 9.2, where we see that the answer to our questions is "not necessarily". Then in Sections 9.3–9.6 we see that if we require a bit more of convergence, the answer to each of our questions is "yes".

9.2 Pointwise Limits

Suppose $f_1, f_2, f_3, \dots$ is a sequence of functions, each of which is defined on a common domain D. What should we mean by the sum $f = \sum_{k=0}^{\infty} f_k$? Perhaps the simplest notion for the sum is to extend the definition of finite sum using our familiar interpretation of convergence of an infinite series of numbers as a limit of the sequence of partial sums. We consider this idea first.

Definition 9.2 For each x in D and $n \in \mathbb{N}$ let

$$S_n(x) = f_1(x) + \cdots + f_n(x).$$

If $\lim_{n\to\infty} S_n(x)$ exists (as a real number), we say the series $\sum_1^\infty f_k$ *converges at* x and we write $\sum_1^\infty f_k(x)$ for $\lim_{n\to\infty} S_n(x)$. If the series $\sum_1^\infty f_k(x)$ converges for all $x \in D$, we say the series *converges pointwise* on D to the function $f = \sum_1^\infty f_k$ defined by

$$f(x) = \sum_1^\infty f_k(x) \quad (= \lim_{n\to\infty} \sum_{k=1}^n f_k(x)).$$

We would like such infinite sums of functions to behave like finite sums of functions (as our three questions in Section 9.1 suggest): If $f = \sum_1^\infty f_k$ on an interval $I = [a, b]$, is it true that

(1) If f_k is continuous on I for all $k \in \mathbb{N}$, then so is f?

(2) If f_k is differentiable on I for all $k \in \mathbb{N}$, then so is f and

$$f'(x) = \sum_1^\infty f_k'(x)?$$

(3) If f_k is integrable on I for all $k \in \mathbb{N}$, then so is f, and

$$\int_a^b f(x)\, dx = \sum_1^\infty \int_a^b f_k(x)\, dx?$$

Let us reformulate our questions in the language of sequences.

Definition 9.3 Let $\{f_n\}$ be a sequence of functions defined on a common domain D. If $\lim_{n\to\infty} f_n(x)$ exists (as a real number) for all $x \in D$, we say that the sequence $\{f_n\}$ *converges pointwise* on D. This limit defines a function f on D by the equation

$$f(x) = \lim_n f_n(x).$$

We write $\lim_n f_n = f$ or $f_n \to f$.

Our questions then become (for D an interval $I = [a, b]$): Is it true that

1. if f_n is continuous on I for all n, then f is continuous on I ?

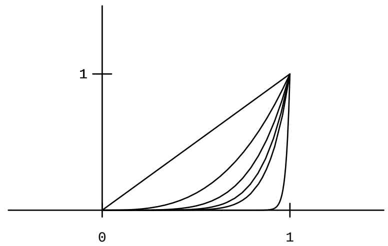

Figure 9.1: Graphs of x^n on $[0,1]$ for $n = 1, 2, 3, 4$.

2. if f_n is differentiable on I for all n, then f is differentiable on I and $f' = \lim_n f_n'$?

3. if f_n is integrable on I for all n, then f is integrable on I and $\int_a^b f(x)\ dx = \lim_n \int_a^b f_n(x)\ dx$?

These questions have *negative* answers in general, as the three examples that follow show.

Example 9.4 (A discontinuous limit of continuous functions) For each $n \in \mathbb{N}$ and $x \in [0,1]$, let $f_n(x) = x^n$. Each of the functions is continuous on $[0,1]$. Notice, however, that for each $x \in (0,1)$, $\lim_n f_n(x) = 0$ and yet $\lim_n f_n(1) = 0$. This is easy to see, but it is instructive to check the details since we can use them later to see what is going wrong in this example. At the right hand endpoint it is clear that, for $x = 1$, $\lim_n f_n(x) = 1$. For $0 < x_0 < 1$ and $\varepsilon > 0$, let $N \geq \ln \varepsilon / \ln x_0$. Then $(x_0)^N \leq \varepsilon$, so for $n \geq N$

$$|f_n(x_0) - 0| = (x_0)^n < (x_0)^N \leq \varepsilon.$$

Thus

$$f(x) = \lim_n f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1, \end{cases}$$

so the pointwise limit f of the sequence of continuous functions $\{f_n\}$ is discontinuous at $x = 1$. (Figure 9.1 shows the graphs of the first few functions of the sequence.) ◀

Example 9.5 (The derivative of the limit is not the limit of the derivative.) Let $f_n(x) = x^n/n$. Then $f_n \to 0$ on $[0,1]$. Now $f_n'(x) = x^{n-1}$, so by the previous example, Example 9.4,

$$\lim_n f_n'(x) = x^{n-1} = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1, \end{cases}$$

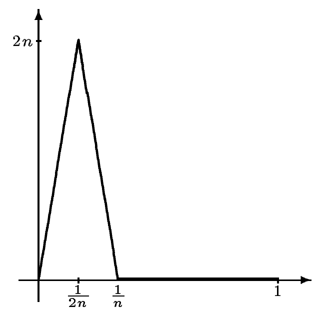

Figure 9.2: Graph of $f_n(x)$ on $[0,1]$.

while the derivative of the limit function, $f \equiv 0$, equals zero on $[0,1]$. Thus

$$\lim_{n\to\infty} \frac{d}{dx}(f_n(x)) \neq \frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right)$$

at $x = 1$. ◀

Example 9.6 (The integral of the limit is not the limit of the integrals.) In this example we consider a sequence of continuous functions, each of which has the same integral over the domain. For each $n \in \mathbb{N}$ let f_n be defined on $[0,1]$ as follows: $f_n(0) = 0$, $f_n(1/2) = 2n$, $f_n(1/n) = 0$, f_n is linear on $[0, 1/(2n)]$ and on $[1/(2n), 1/n]$, and $f_n = 0$ on $[1/n, 1]$. (See Figure 9.2.)

It is easy to verify that $f_n \to 0$ on $[0,1]$. (For $x = 0$, $f_n(x) = 0$ for all n, while for each x, $0 < x \leq 1$, there exists N such that $1/N < x$, so $f_n(x) = 0$ for all $n \geq N$.)

Now, for each $n \in \mathbb{N}$, $\int_0^1 f_n(x)\ dx = 1$. But

$$\int_0^1 (\lim_n f_n(x))\, dx = \int_0^1 0\, dx = 0.$$

Thus

$$\lim_n \int_0^1 f_n(x)\ dx \neq \int_0^1 \lim_n f_n(x)\, dx.$$

◀

These examples show that the answer to each of our three questions is negative, in general. We present some additional examples that illustrate similar phenomena in the exercises.

We shall see in the next few sections that by replacing pointwise convergence with a stronger form of convergence, the answers to our questions become affirmative. The form of convergence in question is called *uniform convergence.*

Interchange of Limit Operations Before turning to uniform convergence, let us first try to get an insight into a difficulty one must overcome if one wishes affirmative answers to our questions.

To say f is continuous at x_0 means that $\lim_{x\to x_0} f(x) = f(x_0)$. If $f = \lim_n f_n$, continuity of f at x_0 means

$$\lim_{x\to x_0}\big(\lim_{n\to\infty} f_n(x)\big) = \lim_{n\to\infty}\big(\lim_{x\to x_0} f_n(x)\big).$$

Thus, two limit operations are required, and to assert that f is continuous requires us to know that the order of passing to the limits is immaterial.

The reader will remember situations in which two limit operations are involved and the order of taking the limit does not affect the result. For example, in elementary calculus one finds conditions under which the value of a double integral can be obtained by iterating "single integrals" in either order. By way of contrast, we present an example in the setting of double sequences in which the order of taking limits *is* important.

Example 9.7 In this example we illustrate that an interchange of limit operations may not give a correct result. Let

$$S_{mn} = \begin{cases} 0, & \text{if } m \le n \\ 1, & \text{if } m > n. \end{cases}$$

Viewed as a matrix,

$$S_{mn} = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

For each row m, we have $\lim_{n\to\infty} S_{mn} = 0$, so

$$\lim_{m\to\infty}\big(\lim_{n\to\infty} S_{mn}\big) = 0.$$

On the other hand, for each column n, $\lim_{m\to\infty} S_{mn} = 1$, so

$$\lim_{n\to\infty}\big(\lim_{m\to\infty} S_{mn}\big) = 1.$$

◀

Exercises

9:2.1 Examine the pointwise limiting behavior of the sequence of functions

$$f_n(x) = \frac{x^n}{1+x^n}.$$

9:2.2 Show that the logarithm function can be expressed as the pointwise limit of a sequence of "simpler" functions,

$$\ln x = \lim_{n\to\infty} n\left(\sqrt[n]{x} - 1\right)$$

for every point in its domain. If the answer to our three questions for this particular limit is affirmative what can you say about the continuity of the logarithm function? What would be its derivative? What would be $\int_1^2 \ln x\, dx$?

9:2.3◇ Let $x_1, x_2, \ldots$ be an enumeration of $\mathbb{Q}$, let

$$f_n(x) = \begin{cases} 1, & \text{if } x \in \{x_1, \ldots, x_n\} \\ 0, & \text{otherwise,} \end{cases}$$

and let

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Show that $f_n \to f$ pointwise on $[0,1]$, but $\int_0^1 f_n(x)\, dx = 0$ for all $n \in \mathbb{N}$, while f is not integrable on $[0,1]$.

9:2.4 Let $f_n(x) = \sin nx/\sqrt{n}$. Show that $\lim_n f_n = 0$ but $\lim_n f_n'(0) = \infty$.

9:2.5 Each of Examples 9.4, 9.5 and 9.6 can be interpreted as a statement that the order of taking the limit operation does matter. Verify this.

9:2.6 Refer to Example 9.7. What should one mean by the statement that a "double sequence" $\{t_{mn}\}$ converges, i.e., that

$$\lim_{m\to\infty, n\to\infty} t_{mn}$$

exists)? Does the double sequence $\{S_{mn}\}$ of Example 9.7 converge? If so, what is its limit?

9:2.7 Let $f_n \to f$ pointwise at every point in the interval $[a,b]$. We have seen that even if each f_n is continuous it does not follow that f is continuous. Are any of the following statements true?

(a) If each f_n is increasing on $[a,b]$ then so is f.

(b) If each f_n is nondecreasing on $[a,b]$ then so is f.

(c) If each f_n is bounded on $[a,b]$ then so is f.

(d) If each f_n is everywhere discontinuous on $[a,b]$ then so is f.

(e) If each f_n is constant on $[a,b]$ then so is f.

(f) If each f_n is positive on $[a,b]$ then so is f.

(g) If each f_n is linear on $[a,b]$ then so is f.

(h) If each f_n is convex on $[a,b]$ then so is f.

9:2.8 If $f_n \to f$ pointwise at every real number then prove that

$$\{x : f(x) > \alpha\} = \bigcup_{m=1}^{\infty}\bigcup_{r=1}^{\infty}\bigcap_{n=r}^{\infty}\{x : f_n(x) \geq \alpha + 1/m\}.$$

9:2.9 Let $\{f_n\}$ be a sequence of real functions. Show that the set E of points of convergence of the sequence can be written in the form

$$E = \bigcap_{k=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}\bigcap_{m=N}^{\infty}\left\{x : |f_n(x) - f_m(x)| \leq \tfrac{1}{k}\right\}.$$

9.3 Uniform Limits

Pointwise limits do not allow the interchange of limit operations. Generally uniform limits will. To see how the definition of a uniform limit needs to be formulated let us return to the sequence of Example 9.4. That sequence illustrated the fact that a pointwise limit of continuous functions need not be continuous. The difficulty there was that

$$\lim_{x\to 1-}\left(\lim_{n\to\infty} f_n(x)\right) \neq \lim_{n\to\infty}\left(\lim_{x\to 1-} f_n(x)\right).$$

A closer look at the limits involved here shows what went wrong and suggests what we need to look for in order to allow an interchange of limits.

Example 9.8 Consider once again the sequence $\{f_n\}$ of functions $f_n(x) = x^n$. We saw that $f_n \to 0$ pointwise on $[0,1)$, and that for every fixed $x_0 \in (0,1)$ and $\varepsilon > 0$,

$$|x_0|^n < \varepsilon \text{ if and only if } n \geq \ln\varepsilon/\ln x_0.$$

Now fix ε but let the point x_0 vary. Observe that when x_0 is relatively small in comparison with ε, the number $\ln x_0$ is large in absolute value compared with $\ln\varepsilon$, so relatively small values of n suffice for the inequality $|x_0|^n < \varepsilon$. On the other hand, when x_0 is near 1, $\ln x_0$ is very small in absolute value, so $\ln\varepsilon/\ln x_0$ will be very large. In fact,

$$\lim_{x_0\to 1-}\frac{\ln\varepsilon}{\ln x_0} = \infty. \qquad (2)$$

The table below illustrates how large n must be before $|x_0^n| < \varepsilon$ for $\varepsilon = .1$.

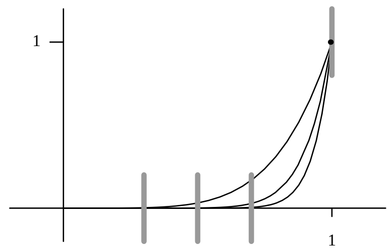

Figure 9.3: The sequence $\{x^n\}$ converges infinitely slowly on $[0,1]$.

x_0	n
.1	1
.5	4
.9	22
.99	230
.999	2,302
.9999	23,025

Note that For $\varepsilon = .1$, there is no single value of N such that $|x_0|^n < \varepsilon$ for every value of $x_0 \in (0,1)$ and $n > N$. (Figure 9.3 illustrates this.) ◀

Some 19th century mathematicians would have described the varying rates of convergence in the example by saying[1] that

"the sequence $\{x^n\}$ converges *infinitely slowly* on $(0,1)$".

Today we would say that this sequence, which does converge pointwise, does *not* converge uniformly. Our definition is formulated precisely to avoid this possibility of infinitely slow convergence.

Definition 9.9 Let $\{f_n\}$ be a sequence of functions defined on a common domain D. We say that $\{f_n\}$ *converges uniformly* to a function f on D if, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } n \geq N \text{ and } x \in D.$$

We write

$$f_n \to f \text{ [unif] on } D \text{ or } \lim_n f_n = f \text{ [unif] on } D$$

to indicate that the sequence $\{f_n\}$ converges uniformly to f on D. If the domain D is understood from the context, we may delete explicit reference to D and write

[1] T. Hawkins, *Lebesgue's Theory of Integration*, Chelsea Publishing Co.,(1979), pp 21–23.

$$f_n \to f \text{ [unif] or } \lim_n f_n = f \text{ [unif]}.$$

Uniform convergence plays an important role in many parts of analysis. In particular, it figures in questions involving the interchanging of limit processes such as those we discussed in Section 9.2. This was not apparent to mathematicians in the early part of the 19th century. As late as 1823, Cauchy believed a convergent series of continuous functions could be integrated term-by-term. Similarly, Cauchy believed that a convergent series of continuous functions has a continuous sum. Abel provided a counterexample in 1826. It may have been Weierstrass who first recognized the importance of uniform convergence in the middle of the 19th century.[2]

Example 9.10 Let $f_n(x) = x^n$, $D = [0, \eta]$, $0 < \eta < 1$. We observed that the sequence $\{f_n\}$ converges pointwise, but not uniformly, on $(0, 1)$ (or on $[0, 1]$). We realized that the difficulty arises from the fact that the convergence near 1 is very 'slow'. But for any fixed η with $0 < \eta < 1$, the convergence *is* uniform on $[0, \eta]$.

To see this, observe that for $0 \leq x_0 < \eta$, $0 \leq (x_0)^n < \eta^n$. Let $\varepsilon > 0$. Since $\lim_n \eta^n = 0$, there exists N such that if $n \geq N$, then $0 < \eta^n < \varepsilon$. Thus, if $n \geq N$, we have

$$0 \leq x_0^n < \eta^n < \varepsilon,$$

so the same N that works for $x = \eta$, also works for all $x \in [0, \eta)$.
◀

Suppose that $f_n \to f$ on $[0, 1]$. It follows easily from the definition that the convergence is uniform on any *finite* subset D of $[0, 1]$ (Exercise 9:3.3). Thus given any $\varepsilon > 0$ and any finite set $x_1, x_2, \ldots, x_m$ in [0,1], we can find $n \in \mathbb{N}$ such that

$$|f_n(x_i) - f(x_i)| < \varepsilon$$

for all $n \geq N$ and all $i = 1, 2, \ldots, m$. (Figure 9.4 illustrates this.)

The vertical line segments over the points $x_1, \ldots, x_m$ are centered on the graph of f, and are of length 2ε. In simple geometric language, we can go sufficiently far out in the sequence to guarantee that the graphs of all the functions f_n intersect all of these finitely many vertical segments.

In contrast, uniform convergence on $[0, 1]$ requires that we can go sufficiently far out in the sequence to guarantee that the graphs of the functions go through such vertical segments at *all* points of

[2] More on the history of uniform convergence can be found in Thomas Hawkins' interesting historical book *Lebesgue's Theory of Integration*, Chelsea Publishing Co.,(1975).

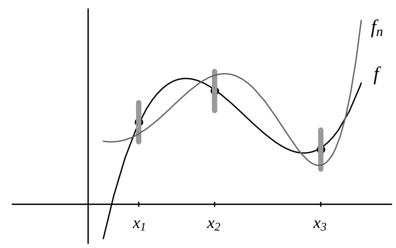

Figure 9.4: Uniform convergence on the finite set $\{x_1, x_2, x_3\}$.

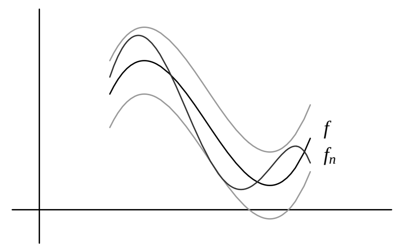

Figure 9.5: Uniform convergence on the whole interval.

[0,1]; that is, that the graph of f_n for n sufficiently large lies in the "ε-band" centered on the graph of f. (See Figure 9.5.)

9.3.1 The Cauchy Criterion

Suppose now that we are given a sequence of functions $\{f_n\}$ on an interval I, and we wish to know whether it converges uniformly to some function on I. We are not told what that limit function might be. The problem is similar to one we faced for a sequence of *numbers* $\{a_n\}$ in our study of sequences. There we saw that $\{a_n\}$ converges if and only if it is a Cauchy sequence. We can formulate a similar criterion for uniform convergence of a sequence of functions.

Definition 9.11 Let $\{f_n\}$ be a sequence of functions defined on a set D. The sequence is said to be *uniformly Cauchy* on D if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ and $m \geq N$ then $|f_m(x) - f_n(x)| < \varepsilon$ for all $x \in D$.

Theorem 9.12 (Cauchy Criterion) *Let $\{f_n\}$ be a sequence of functions defined on a set D. Then there exists a function f defined on*

D such that $f_n \to f$ uniformly on D if and only if $\{f_n\}$ is uniformly Cauchy.

Proof. We leave the proof of Theorem 9.12 as Exercise 9:3.15. ■

Example 9.13 In Example 9.10 we showed that the sequence $f_n(x) = x^n$ converges uniformly on any interval $[0, \eta]$, for $0 < \eta < 1$. Let us prove this again, but using the Cauchy criterion.

Fix $n \geq m$ and compute

$$\sup_{x \in [0,\eta]} |x^n - x^m| \leq \eta^m. \tag{3}$$

Let $\varepsilon > 0$ and choose an integer N so that $\eta^N < \varepsilon$ (i.e., so that $N > \ln \varepsilon / \ln \eta$). Then it follows from (3) for all $n \geq m \geq N$ and all $x \in [0, \eta]$ that

$$|x^n - x^m| \leq \eta^m < \varepsilon.$$

We conclude, by the Cauchy criterion, that the sequence $f_n(x) = x^n$ converges uniformly on any interval $[0, \eta]$, for $0 < \eta < 1$. Here there was no computational advantage over the argument in Example 9.10. Frequently, though, one does not know the limit function and *must* use the Cauchy criterion rather than the definition. ◀

Cauchy Criterion for Series The Cauchy criterion can be expressed for uniformly convergent series too. We say that a series $\sum_1^\infty f_k$ converges *uniformly* to the function f on D if the sequence $\{S_n\} = \{\sum_{k=1}^n f_k\}$ of partial sums converges uniformly to f on D.

Theorem 9.14 (Cauchy Criterion) *Let $\{f_n\}$ be a sequence of functions defined on a set D. Then the series $\sum_1^\infty f_k$ converges uniformly to some function f on D if and only if for every $\varepsilon > 0$ there is an integer N so that*

$$\left| \sum_{j=m}^{n} f_j(x) \right| < \varepsilon$$

for all $n \geq m \geq N$ and all $x \in D$.

Proof. This follows immediately from Theorem 9.12. ■

Example 9.15 Let us show that the series

$$1 + x + x^2 + x^3 + x^4 + \ldots$$

converges uniformly on any interval $[0, \eta]$, for $0 < \eta < 1$. Our computations could be based on the fact that the sum of this series is known to us; it is $(1-x)^{-1}$. We could prove the uniform convergence directly from the definition. Instead let us use the Cauchy criterion.

Fix $n \geq m$ and compute

$$\sup_{x\in[0,\eta]}\left|\sum_{j=m}^{n} x^j\right| \leq \sup_{x\in[0,\eta]}\left|\frac{x^m}{1-x}\right| \leq \frac{\eta^m}{1-\eta}. \tag{4}$$

Let $\varepsilon > 0$. Since $\eta^m(1-\eta)^{-1} \to 0$ as $m \to \infty$ we may choose an integer N so that

$$\eta^N(1-\eta)^{-1} < \varepsilon.$$

Then it follows from (4) for all $n \geq m \geq N$ and all $x \in [0,\eta]$ that

$$\left|x^m + x^{m+1} + x^{m+2} + x^n\right| \leq \frac{\eta^m}{1-\eta} < \varepsilon.$$

It follows now, by the Cauchy criterion, that the series converges uniformly on any interval $[0,\eta]$, for $0 < \eta < 1$. Observe, however, that the series does not converge uniformly on $(-1,1)$, though it does converge pointwise there. (See Exercise 9:3.16.) ◀

9.3.2 Weierstrass M–Test

It is not always easy to determine whether a sequence of functions is uniformly convergent. In the settings of *series* of functions, a certain simple test is often useful. This will certainly become one of the most frequently used tools in your study of uniform convergence.

Theorem 9.16 (M–Test) *Let $\{f_k\}$ be a sequence of functions defined on a set D and let $\{M_k\}$ be a sequence of positive constants. If*

$$\sum_{0}^{\infty} M_k < \infty$$

and if

$$|f_k(x)| \leq M_k$$

for each $x \in D$ and $k = 0, 1, 2, \ldots$, then the series $\sum_0^\infty f_k$ converges uniformly on D.

Proof. Let $S_n(x) = \sum_{k=0}^{n} f_k(x)$. We show that $\{S_n\}$ is uniformly Cauchy on D. Let $\varepsilon > 0$. For $m < n$ we have

$$S_n(x) - S_m(x) = f_{m+1}(x) + \cdots + f_n(x),$$

so

$$|S_n(x) - S_m(x)| \leq M_{m+1} + \cdots + M_n.$$

Since the series of constants $\sum_{k=0}^{\infty} M_k$ converges by hypothesis, there exists an integer N such that if $n > m \geq N$,

$$M_{m+1} + \cdots + M_n < \varepsilon.$$

This implies that for $n > m \geq N$,

$$|S_n(x) - S_m(x)| < \varepsilon$$

for all $x \in D$. Thus the sequence $\{S_n\}$ is uniformly convergent on D; i.e. $\sum_0^{\infty} f_k$ is uniformly convergent on D. ■

Example 9.17 Consider again the geometric series $1 + x + x^2 + \ldots$ on the interval $[-a, a]$, for any $0 < a < 1$ (as we did in Example 9.15). Then $|x^k| \leq a^k$ for every $k = 0, 1, 2 \ldots$ and $x \in [-a, a]$. Since $\sum_0^{\infty} a^k$ converges, the M–test implies that the series $\sum_0^{\infty} x^k$ converges uniformly on $[-a, a]$. ◀

Example 9.18 Let us investigate the uniform convergence of the series

$$\sum_{k=1}^{\infty} \frac{\sin k\theta}{k^p}$$

for values of $p > 0$. The crudest estimate on the size of the terms in this series is obtained just by using the fact that the sin function never exceeds 1 in absolute value. Thus

$$\left| \frac{\sin k\theta}{k^p} \right| \leq \frac{1}{k^p} \quad \text{for all } \theta \in \mathbb{R}.$$

Since the series $\sum_{k=1}^{\infty} 1/k^p$ converges for $p > 1$ we obtain immediately by the M–test that our series converges uniformly (and absolutely) for all real θ provided $p > 1$. In particular, as we shall see in subsequent sections, this series represents a continuous function, one which could be integrated term by term in any bounded interval.

We seem to have been particularly successful here but a closer look also reveals a limitation in the method. The series is also pointwise convergent for $0 < p \leq 1$ (use the Dirichlet test) for all values of θ, but it converges nonabsolutely. The M–test can not be of any help in this situation since it can address only absolutely convergent series. ◀

Because of the remark at the end of this example it is perhaps best to conclude, when using the M–test, that the series tested "converges absolutely and uniformly" on the set given. This serves, too, to remind us to use a different method for checking uniform convergence of nonabsolutely convergent series (see the next section).

✂ 9.3.3 Abel's Test for Uniform Convergence

The M–test is a highly useful tool for checking the uniform convergence of a series. By its nature, though, it clearly applies only to absolutely convergent series. For a more delicate test that will apply to nonabsolutely convergent series we should search through our methods in Chapter 3 for tests that handled nonabsolute convergence. Two of these, the Dirichlet test and Abel's test, can be modified so as to give uniform convergence.

A number of nineteenth century authors (including Abel, Dirichlet, Dedekind, and du Bois–Reymond) arrived at similar tests for uniform convergence. We recall that Abel's test for convergence of a series $\sum_{k=1}^{\infty} a_k b_k$ required the sequence $\{b_k\}$ to be convergent and monotone and for the series $\sum_{k=1}^{\infty} a_k$ to converge. Dirichlet's variant weakened the latter requirement so that $\sum_{k=1}^{\infty} a_k$ had bounded partial sums but required of the sequence $\{b_k\}$ that it converge monotonically to zero. Here we seek similar conditions on a series

$$\sum_{k=1}^{\infty} a_k(x) b_k(x)$$

of functions in order to obtain uniform convergence. The next theorem is one variant; others can be found in the Exercises.

Theorem 9.19 (Abel) *Let $\{a_k\}$ and $\{b_k\}$ be sequences of functions on a set $E \subset \mathbb{R}$. Suppose that there is a number M so that*

$$-M \le s_N(x) = \sum_{k=1}^{N} a_k(x) \le M$$

for all $x \in E$ and every $N \in \mathbb{N}$. Suppose that the sequence of functions $\{b_k\} \to 0$ converges monotonically to zero at each point and that this convergence is uniform on E. Then the series

$$\sum_{k=1}^{\infty} a_k b_k$$

converges uniformly on E.

Proof. We will use the Cauchy criterion applied to the series to obtain uniform convergence. We may assume that the $b_k(x)$ are nonnegative and decrease to zero. Let $\varepsilon > 0$. We need to estimate the sum

$$\left| \sum_{k=m}^{n} a_k(x) b_k(x) \right| \tag{5}$$

for large n and m and all $x \in E$. Since the sequence of functions $\{b_k\}$ converges uniformly to zero on E we can find an integer N so that

$$0 \leq b_k(x) \leq \frac{\varepsilon}{2M}$$

for all $k \geq N$ and all $x \in E$.

The key to estimating the sum (5), now, is the summation by parts formula that we have used earlier (see Section 3.2). This is just the elementary identity

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n} (s_k - s_{k-1}) b_k$$

$$= s_m(b_m - b_{m+1}) + s_{m+1}(b_{m+1} - b_{m+2}) \cdots + s_{n-1}(b_{n-1} - b_n) + s_n b_n.$$

This provides us with

$$\left| \sum_{k=m}^{n} a_k(x) b_k(x) \right| \leq 2M \left(\sup_{x \in E} |b_m(x)| \right) < \varepsilon$$

for all $n \geq m \geq N$ and all $x \in E$ which is exactly the Cauchy criterion for the series and proves the theorem. ■

It is worth pointing out that in many applications of this theorem the sequence $\{b_k\}$ can be taken as a sequence of numbers, in which case the statement and the conditions that need to be checked are rather simpler.

Corollary 9.20 *Let $\{a_k\}$ be a sequence of functions on a set $E \subset \mathbb{R}$. Suppose that there is a number M so that*

$$\left| \sum_{k=1}^{N} a_k(x) \right| \leq M$$

for all $x \in E$ and every integer N. Suppose that the sequence of real numbers $\{b_k\}$ converges monotonically to zero. Then the series

$$\sum_{k=1}^{\infty} b_k a_k$$

converges uniformly on E.

Proof. Consider that $\{b_k\}$ is a sequence of constant functions on E and then apply the theorem. ■

In the exercises there are several other variants of Theorem 9.19, all with similar proofs and all of which have similar applications.

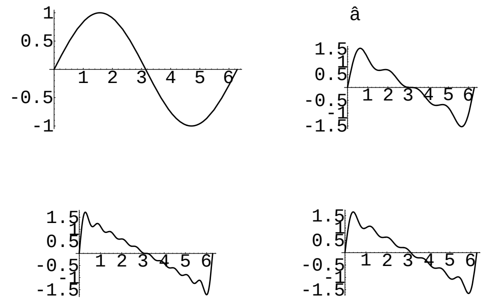

Figure 9.6: Graph of $\sum_{k=1}^{n} (\sin k\theta)/k$ for $n = 1$, 4, 10 and 7.

Example 9.21 As an interesting application of Theorem 9.19 consider a series which arises in Fourier analysis:

$$\sum_{k=1}^{\infty} \frac{\sin k\theta}{k}.$$

It is possible by using Dirichlet's test (see Section 3.6.13) to prove that this series converges for all θ.

Questions about the uniform convergence of this series are intriguing. In Figure 9.6 we have given a graph of some of the partial sums of the series.

The behavior near $\theta = 0$ is most curious. Apparently if we can avoid that point, more precisely if we can stay a small distance away from that point, we should be able to obtain uniform convergence. Theorem 9.19 will provide a proof. We apply that theorem with $b_k(\theta) = 1/k$ and $a_k(\theta) = \sin k\theta$. All that is required is to obtain an estimate for the sums

$$\left| \sum_{k=1}^{n} \sin k\theta \right|$$

for all n and all θ in an appropriate set. Let $0 < \eta < \pi/2$ and consider making this estimate on the interval $[\eta, 2\pi - \eta]$. From Exercise 3:2.11 we obtain the formula

$$\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta + \cdots + \sin n\theta = \frac{\cos\theta/2 - \cos(2n+1)\theta/2}{2\sin\theta/2}$$

and using this we can see that

$$\left|\sum_{k=1}^{n} \sin k\theta\right| \leq \frac{1}{\sin(\eta/2)}.$$

Now Theorem 9.19 immediately shows that

$$\sum_{k=1}^{\infty} \frac{\sin k\theta}{k}$$

converges uniformly on $[\eta, 2\pi - \eta]$.

Figure 9.6 illustrates graphically why the convergence cannot be expected to be uniform near to 0. A computation here is instructive. To check the Cauchy criterion on $[0, \pi]$ we need to show that the sums

$$\sup_{\theta \in [0,\pi]} \left|\sum_{k=m}^{n} \frac{\sin k\theta}{k}\right|$$

are small for large m, n. But in fact

$$\sup_{\theta \in [0,\pi]} \left|\sum_{k=m}^{2m} \frac{\sin k\theta}{k}\right| \geq \sum_{k=m}^{2m} \frac{\sin(k/2m)}{k} \geq \sum_{k=m}^{2m} \frac{\sin 1/2}{2m} > \frac{\sin 1/2}{2},$$

obtained by checking the value at points $\theta = 1/2m$. Since this is not arbitrarily small the series cannot converge uniformly on $[0, \pi]$. ◀

Exercises

9:3.1 Examine the uniform limiting behavior of the sequence of functions

$$f_n(x) = \frac{x^n}{1 + x^n}.$$

On what sets can you determine uniform convergence?

9:3.2 Examine the uniform limiting behavior of the sequence of functions

$$f_n(x) = x^2 e^{-nx}.$$

On what sets can you determine uniform convergence? On what sets can you determine uniform convergence for the sequence of functions $n^2 f_n(x)$?

9:3.3 Prove that if $f_n \to f$ pointwise on a finite set D, then the convergence is uniform.

9:3.4 Prove that if $f_n \to f$ uniformly on a set E_1 and also on a set E_2 then $f_n \to f$ uniformly on $E_1 \cup E_2$.

9:3.5 Prove or disprove that if $f_n \to f$ uniformly on each set E_1, E_2, E_3, ... then $f_n \to f$ uniformly on the union of all these sets $\bigcup_{k=1}^{\infty} E_k$.

9:3.6 Prove that if $f_n \to f$ uniformly on a set E then $f_n \to f$ uniformly on every subset of E.

9:3.7 Prove or disprove that if $f_n \to f$ uniformly on each set $E \cap [a, b]$ for every interval $[a, b]$ then $f_n \to f$ uniformly on E.

9:3.8 Prove or disprove that if $f_n \to f$ uniformly on each closed interval $[a, b] \subset (c, d)$ then $f_n \to f$ uniformly on (c, d).

9:3.9 Prove that if $\{f_n\}$ and $\{g_n\}$ both converge uniformly on a set D then so too does the sequence $\{f_n + g_n\}$.

9:3.10 Prove or disprove that if $\{f_n\}$ and $\{g_n\}$ both converge uniformly on a set D then so too does the sequence $\{f_n g_n\}$.

9:3.11 Prove or disprove that if f is a continuous function on $(-\infty, \infty)$ then

$$f(x + 1/n) \to f(x)$$

uniformly on $(-\infty, \infty)$. (What extra condition, stronger than continuity, would work if not?)

9:3.12 Prove that $f_n \to f$ converges uniformly on D if and only if

$$\lim_n \sup_{x \in D} |f_n(x) - f(x)| = 0.$$

9:3.13 Show that a sequence of functions $\{f_n\}$ fails to converge to a function f uniformly on a set E if and only if there is some positive ε_0 so that a sequence $\{x_k\}$ of points in E and a subsequence $\{f_{n_k}\}$ can be found such that

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0.$$

9:3.14 Apply the criterion in the preceding exercise to show that the sequence $f_n(x) = x^n$ does not converge uniformly to zero on $(0, 1)$.

9:3.15 Prove Theorem 9.12.

9:3.16 Verify that the geometric series $\sum_{k=0}^{\infty} x^k$ which converges pointwise on $(-1, 1)$ does not converge uniformly there.

9:3.17 Do the same for the series obtained by differentiating the series in Exercise 9:3.16; i.e., show that $\sum_1^{\infty} kx^{k-1}$ converges pointwise but not uniformly on (-1,1). Show that this series does converges uniformly on every closed interval $[a, b]$ contained in $(-1, 1)$.

9:3.18 Verify that the series

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2}$$

converges uniformly on all of $\mathbb{R}$.

9:3.19 If $\{f_n\}$ is a sequence of functions converging uniformly on a set E to a function f what conditions on the function g would allow you to conclude that $g \circ f_n$ converges uniformly on E to $g \circ f$?

9:3.20 Show that if $f_n \to f$ uniformly on $[a,b]$ and each f_n is continuous then the sequence of functions

$$F_n(x) = \int_a^x f_n(t)\,dt$$

also converges uniformly on $[a,b]$.

9:3.21 Show that if $f_n \to f$ uniformly on $[a,b]$ and each f_n is continuous then

$$\lim_{n\to\infty} \int_a^b \left(\int_a^x f_n(t)\,dt\right) dx = \int_a^b \left(\int_a^x f(t)\,dt\right) dx.$$

9:3.22 A sequence of functions $\{f_n\}$ is said to be *uniformly bounded* on an interval $[a,b]$ if there is a number M so that

$$|f_n(x)| \leq M$$

for every n and also for every $x \in [a,b]$. Show that a uniformly convergent sequence $\{f_n\}$ of continuous functions on $[a,b]$ must be uniformly bounded. Show that the same statement would not be true for pointwise convergence.

9:3.23 Suppose that $f_n \to f$ on $(-\infty,+\infty)$. What conditions would allow you to compute that

$$\lim_{n\to\infty} f_n(x+1/n) = f(x)?$$

9:3.24 Suppose that $\{f_n\}$ is a sequence of continuous functions on the interval $[0,1]$ and that you know that $f_n \to f$ uniformly on the set of rational numbers inside $[0,1]$. Can you conclude that $f_n \to f$ uniformly on $[0,1]$? (Would this be true without the continuity assertion?)

9:3.25 Prove the following variant of the Weierstrass M–test: Let $\{f_k\}$ and $\{g_k\}$ be sequences of functions on a set $E \subset \mathbb{R}$. Suppose that $|f_k(x)| \leq g_k(x)$ for all k and $x \in E$ and that $\sum_{k=1}^\infty g_k$ converges uniformly on E. Then the series

$$\sum_{k=1}^\infty f_k$$

converges uniformly on E.

9:3.26 Prove the following variant on Theorem 9.19: Let $\{a_k\}$ and $\{b_k\}$ be sequences of functions on a set $E \subset \mathbb{R}$. Suppose that $\sum_{k=1}^\infty a_k(x)$ converges uniformly on E. Suppose that $\{b_k\}$ is monotone for each $x \in E$ and uniformly bounded on E. Then the series

$$\sum_{k=1}^\infty a_k b_k$$

converges uniformly on E.

9:3.27 Prove the following variant on Theorem 9.19: Let $\{a_k\}$ and $\{b_k\}$ be sequences of functions on a set $E \subset \mathbb{R}$. Suppose that there is a number M so that

$$\left|\sum_{k=1}^{N} a_k(x)\right| \leq M$$

for all $x \in E$ and every integer N. Suppose that

$$\sum_{k=1}^{\infty} |b_k - b_{k+1}|$$

converges uniformly on E and that $b_k \to 0$ uniformly on E. Then the series

$$\sum_{k=1}^{\infty} a_k b_k$$

converges uniformly on E.

9:3.28 Prove the following variant on Theorem 9.19: Let $\{a_k\}$ and $\{b_k\}$ be sequences of functions on a set $E \subset \mathbb{R}$. Suppose that $\sum_{k=1}^{\infty} a_k$ converges uniformly on E. Suppose that the series

$$\sum_{k=1}^{\infty} |b_k - b_{k+1}|$$

has uniformly bounded partial sums on E. Suppose that the sequence of functions $\{b_k\}$ is uniformly bounded on E. Then the series

$$\sum_{k=1}^{\infty} a_k b_k$$

converges uniformly on E.

9:3.29 Suppose that $\{f_n\}$ is a sequence of continuous functions on an interval $[a, b]$ converging uniformly to a function f on the open interval (a, b). If f is also continuous on $[a, b]$ show that the convergence is uniform on $[a, b]$.

9:3.30 Suppose that $\{f_n\}$ is a sequence of functions converging uniformly to zero on an interval $[a, b]$. Show that

$$\lim_{n\to\infty} f_n(x_n) = 0$$

for every convergent sequence $\{x_n\}$ of points in $[a, b]$. Give an example to show that this statement may be false if $f_n \to 0$ merely pointwise.

9:3.31 Suppose that $\{f_n\}$ is a sequence of functions on an interval $[a, b]$ with the property that

$$\lim_{n\to\infty} f_n(x_n) = 0$$

for every convergent sequence $\{x_n\}$ of points in $[a, b]$. Show that $\{f_n\}$ converges uniformly to zero on $[a, b]$.

9.4 Uniform Convergence and Continuity

We can now address the questions we asked at the beginning of this chapter. We begin with continuity. We know that the pointwise limit of a sequence of continuous functions need not be continuous. We now show that the *uniform* limit of a sequence of continuous functions must be continuous.

Theorem 9.22 *Let $\{f_n\}$ be a sequence of functions defined on an interval I, and let $x_0 \in I$. If the sequence $\{f_n\}$ converges uniformly to some function f on I and if each of the functions f_n is continuous at x_0, then the function f is also continuous at x_0. In particular, if each of the functions f_n is continuous on I, then so too is f.*

Proof. Let $\varepsilon > 0$. We must show there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ if $|x - x_0| < \delta$, $x \in I$. For each $x \in I$ we have

$$f(x) - f(x_0) = (f(x) - f_n(x)) + (f_n(x) - f_n(x_0)) + (f_n(x_0) - f(x_0)),$$

so

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|. \tag{6}$$

Since $f_n \to f$ *uniformly*, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \tag{7}$$

for all $x \in I$ and all $n \geq N$. We infer from inequalities (6) and (7) that

$$|f(x) - f(x_0)| < \frac{\varepsilon}{3} + |f_N(x) - f_N(x_0)| + \frac{\varepsilon}{3} = |f_N(x) - f_N(x_0)| + \frac{2}{3}\varepsilon. \tag{8}$$

We now use the continuity of the function f_N. We choose $\delta > 0$ such that if $x \in I$ and $|x - x_0| < \delta$, then

$$|f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3}. \tag{9}$$

Combining (8) and (9) we have

$$|f(x) - f(x_0)| < \frac{\varepsilon}{3} + \frac{2}{3}\varepsilon = \varepsilon,$$

for each $x \in I$ for which $|x - x_0| < \delta$, as was to be shown. ■

Note. Let us look a bit more closely at the proof of Theorem 9.22. We first obtained $N \in \mathbb{N}$ such that the function f_N approximated f closely (within $\varepsilon/3$) on all of I. This function f_N served as a "stepping stone" towards verifying the continuity of f at x_0. There are three small "steps" involved:

1. $|f_N(x) - f(x)|$ is small (*for all* $x \in I$) because of uniform convergence.
2. $|f_N(x) - f_N(x_0)|$ is small (for all x near x_0) because of the continuity of f_N.
3. $|f_N(x_0) - f(x_0)|$ is small because $\{f_n(x_0)\} \to f(x_0)$.

If we tried to imitate the proof under the assumption of pointwise convergence, the first of these steps would fail. The reader may wish to observe the failure by working Example 9.4

Theorem 9.22 can be stated in terms of series. Recall that a series $\sum_1^\infty f_k$ converges uniformly to the function f on D if the sequence $\{S_n\} = \{\sum_{k=1}^n f_k\}$ of partial sums converges uniformly to f on D.

Corollary 9.23 *If $\sum_1^\infty f_k$ converges uniformly to f on an interval I and if each of the functions f_k is continuous on I, then f is continuous on I.*

Proof. This follows immediately from Theorem 9.22. ■

✂ 9.4.1 Dini's Theorem

Observe that Theorem 9.22 provides a *sufficient* condition for continuity of the limit function f. The condition is not necessary. (The sequence in Example 9.6 converges to the zero function, which is continuous, even though the convergence is not uniform.)

Under certain circumstances, however, uniform convergence *is* necessary as Theorem 9.24 below shows. (See also Exercise 9:4.6.) This theorem is due to Ulysses Dini and gives a condition under which pointwise convergence of a sequence of continuous functions to a continuous function must be uniform.

Theorem 9.24 (Dini) *Let $\{f_n\}$ be a sequence of continuous functions on an interval $[a,b]$. Suppose for each $x \in [a,b]$ and for all $n \in \mathbb{N}$, $f_n(x) \geq f_{n+1}(x)$. Suppose in addition that for all $x \in [a,b]$ $f(x) = \lim_n f_n(x)$. If f is continuous, then the convergence is uniform.*

Proof. Suppose the convergence were *not* uniform. Then

$$\max_{x\in[a,b]}(f_n(x) - f(x))$$

does not approach zero as $n \to \infty$ (see Exercise 9:3.12). Hence there exists $c > 0$ such that for infinitely many $n \in \mathbb{N}$,

$$\max_{x\in[a,b]}(f_n(x) - f(x)) > c > 0.$$

Now, for each $n \in \mathbb{N}$, $f_n - f$ is continuous, so it achieves a maximum value at a point $x_n \in [a, b]$. By the Bolzano-Weierstrass theorem we can thus choose a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $\{x_{n_k}\}$ converges to a point $x_0 \in [a, b]$. Note that we must have

$$f_{n_k}(x_{n_k}) - f(x_{n_k}) > c$$

for all $k \in \mathbb{N}$.

Because of our assumption that $f_n(x) \geq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in [a, b]$, we infer

$$f_n(x_{n_k}) - f(x_{n_k}) > c \text{ for all } n \leq n_k.$$

Now fix n and let $k \to \infty$. Using the continuity of the functions $f_n - f$ we obtain $f_n(x_0) - f(x_0) \geq c$ for all $n \in \mathbb{N}$. But this is impossible since $f_n(x_0) \to f(x_0)$ by hypothesis. Thus our assumption that the convergence is not uniform has led to a contradiction. ■

Example 9.25 The sequence of continuous functions $f_n(x) = x^n$ is converging monotonically to a function f on the interval $[0, 1]$. But that function f is (as we have seen before) discontinuous at $x = 1$, so immediately we know that the convergence cannot be uniform. Dini's theorem implies that the convergence is uniform on $[0, b]$ for any $0 < b < 1$ since the function f is continuous there. ◀

Exercises

9:4.1 Can a sequence of discontinuous functions converge uniformly on an interval to a continuous function?

9:4.2 Let $f_n(x) = x^n$, $0 \leq x \leq 1$. Try to imitate the proof of Theorem 9.22 for $x_0 = 1$ and observe where the proof breaks down.

9:4.3 Let $\{f_n\}$ be a sequence of functions each of which is uniformly continuous on an open interval (a, b). If $f_n \to f$ uniformly on (a, b) can you conclude that f is also uniformly continuous on (a, b)?

9:4.4 Give an example of a sequence of continuous functions $\{f_n\}$ on the interval $(0, 1)$ that is monotonic decreasing and converges pointwise to a continuous function f on $(0, 1)$ but for which the convergence is not uniform. Why does this not contradict Theorem 9.24?

9:4.5 Give an example of a sequence of continuous functions $\{f_n\}$ on the interval $[0, \infty)$ that is monotonic decreasing and converges pointwise to a continuous function f on $[0, \infty)$ but for which the convergence is not uniform. Why does this not contradict Theorem 9.24?

9:4.6 Let $\{f_n\}$ be a sequence of continuous nondecreasing functions defined on an interval $[a, b]$. Suppose $f_n \to f$ pointwise on $[a, b]$.

Prove that if f is continuous on $[a, b]$, then the convergence is uniform. Observe that in this exercise, the *functions* are assumed monotonic, whereas in Theorem 9.24 it is the *sequence* that we assume monotonic.

9:4.7 The proof of Theorem 9.24 depends on the compactness of the interval $[a, b]$. The compactness argument used here relied on the Bolzano-Weierstrass theorem. Attempt another proof using one of our other strategies from Section 4.5.

9:4.8 Prove this variant on Dini's theorem (Theorem 9.24). Let $\{f_n\}$ be a sequence of continuous functions on an interval $[a, b]$. Suppose for each $x \in [a, b]$ and for all $n \in \mathbb{N}$, $f_n(x) \leq f_{n+1}(x)$. Suppose in addition that for all $x \in [a, b]$ $\lim_n f_n(x) = \infty$. Show that for all $M > 0$ there is an integer N so that

$$f_n(x) > M$$

for all $x \in [a, b]$ and all $n \geq N$. Show that this conclusion would not be valid without the monotonicity assumption.

9:4.9 Show that if, in Exercise 9:4.8 the interval $[a, b]$ is replaced by the unbounded interval $[0, \infty)$ or the non-closed interval $(0, 1)$ that the conclusion need not be valid.

9:4.10 Let $\{f_n\}$ be a sequence of Lipschitz functions on $[a, b]$ with common Lipschitz constant M. (This means that $|f_n(x) - f_n(y)| \leq M|x - y|$ for all $n \in \mathbb{N}$, $x, y \in [a, b]$.)

(a) If $f = \lim_n f_n$ pointwise, then f is continuous and, in fact, satisfies a Lipschitz condition with constant M,

(b) If $f = \lim_n f_n$ pointwise the convergence is uniform.

(c) Show by example that the results in (a) and (b) fail if we weaken our hypotheses by requiring only that each function is a Lipschitz function. (Here, the constant M may depend on n.)

9:4.11 Give an example to show that the analogue of Theorem 9.24 fails if $[a, b]$ is replaced with an interval that is not closed or is not bounded.

9:4.12 **(Continuous convergence vs. uniform convergence.)** A sequence of functions $\{f_n\}$ defined on an interval I is said to *converge continuously* to the function f if $f_n(x_n) \to f(x_0)$ whenever $\{x_n\}$ is a sequence of points in the interval I that converges to a point x_0 in I. Prove the following theorem:

> *Let $\{f_n\}$ be a sequence of continuous functions on an interval $[a, b]$. Then $\{f_n\}$ converges continuously on $[a, b]$ if and only if $\{f_n\}$ converges to f uniformly on $[a, b]$.*

Does the theorem remain true if the interval $[a, b]$ is replaced with (a, b) or $[a, \infty)$?

9:4.13 Show that the sequence $f_n(x) = x^n/n$ converges uniformly on [0,1]:

(a) by direct computation using the definition of uniform convergence.

(b) by using Theorem 9.24

(c) by using Exercise 9:4.6.

(d) by using Exercise 9:4.12.

9.5 Uniform Convergence and the Integral

Calculus students often learn the following simple computation. The geometric series

$$\frac{1}{1-t} = \sum_0^\infty t^k \tag{10}$$

is valid on the interval $(-1, 1)$. An integration of both sides for t in the interval $[0, x]$, and any choice of $x < 1$ will yield

$$-\log(1-x) = \int_0^x \frac{1}{1-t}\,dt = \sum_0^\infty \frac{x^{k+1}}{k+1}.$$

Indeed this identity is valid and provides a series expansion for the logarithm function. But can this really be justified?

In general do we know that if $f(x) = \sum_0^\infty f_n(x)$ on an interval $[a, b]$, then

$$\int_a^b f(x)\,dx = \sum_0^\infty \int_a^b f_n(x)\,dx?$$

In fact, we already observed in Section 9.3 that during the early part of the 19th century, some prominent mathematicians took for granted the permissibility of term-by-term integration of convergent infinite series of functions. This was true of Fourier, Cauchy and Gauss. Example 9.6, cast in the setting of sequences of integrable functions, shows that these mathematicians were mistaken.

9.5.1 Sequences of Continuous Functions

Around the middle of the 19th century, Weierstrass showed that term-by-term integration is permissible when the series of integrable functions converges *uniformly*. Let us first verify this result for sequences of continuous functions.

Theorem 9.26 *Suppose* $f(x) = \lim_{n\to\infty} f_n(x)$ *for all* $x \in [a,b]$, *that each function* f_n *is continuous on* $[a,b]$, *and that the convergence is uniform. Then*

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \int_a^b f_n(x)\ dx.$$

Proof. By Theorem 9.22, f is continuous, so $\int_a^b f(x)\ dx$ exists. We must show that $\int_a^b f_n(x)\ dx \to \int_a^b f(x)\ dx$.

Let $\varepsilon > 0$. We wish to obtain $N \in \mathbb{N}$ such that

$$\left| \int_a^b f_n(x)\ dx - \int_a^b f(x)\ dx \right| < \varepsilon \text{ for all } n \geq N.$$

We calculate that for any $n \in \mathbb{N}$

$$\left| \int_a^b f_n(x)\ dx - \int_a^b f(x)\ dx \right| = \left| \int_a^b [f_n(x) - f(x)]\, dx \right|$$

$$\leq \int_a^b |f_n(x) - f(x)|\, dx \leq \int_a^b \sup_x |f_n(x) - f(x)|\, dx$$

$$\leq (b-a) \left(\sup_x |f_n(x) - f(x)| \right).$$

Since $f_n \to f$ uniformly on $[a,b]$, there exists $N \in \mathbb{N}$ such that

$$\sup_x |f_n(x) - f(x)| < \frac{\varepsilon}{b-a} \text{ for all } n \geq N.$$

Thus, for $n \geq N$, we have

$$\left| \int_a^b f_n(x)\, dx - \int_a^b f(x)\ dx \right| \leq (b-a) \frac{\varepsilon}{b-a} = \varepsilon$$

as was to be shown. ■

Applying the theorem to the partial sums S_n of a series allows us to express this result for series.

Corollary 9.27 *If an infinite series of continuous functions* $\sum_0^\infty f_k$ *converges uniformly to a function* f *on an interval* $[a,b]$, *then* f *is also continuous and*

$$\int_a^b f(x)\ dx = \sum_0^\infty \int_a^b f_k(x)\ dx.$$

Example 9.28 Let us justify the computations that we made in our introduction to this topic. The geometric series

$$\frac{1}{1-t} = \sum_0^\infty t^k \tag{11}$$

converges pointwise on the interval $(-1,1)$. Let $0 < x < 1$. By the M–test we see that this series converges uniformly on $[0,x]$. Each of the terms in the sum is continuous. As a result we may apply our theorem to integrate term by term just as we might have seen in a calculus course. Thus

$$\int_0^x \frac{1}{1-t}\,dt = \sum_0^\infty \frac{x^{k+1}}{k+1}.$$

◀

9.5.2 Sequences of Riemann Integrable Functions

In Theorem 9.26 we required that the functions f_n be continuous. Suppose we now weaken our hypotheses for these functions by requiring only that they be integrable, but still requiring the sequence $\{f_n\}$ to converge uniformly to f. We note that in all respects the proof is the same. Thus, if a uniformly convergent sequence of integrable functions converges to an integrable function, we can integrate the sequence term-by-term. Our next theorem shows that a uniform limit of integrable functions must be integrable and so we have the following extension of Theorem 9.26.

Theorem 9.29 *Let $\{f_n\}$ be a sequence of functions Riemann integrable on an interval $[a,b]$. If $f_n \to f$ uniformly on $[a,b]$, then f is Riemann integrable on $[a,b]$ and*

$$\int_a^b f(x)\,dx = \lim_n \int_a^b f_n(x)\,dx.$$

Proof. Because of the preceding development, we need only show that the limit function f is integrable on $[a,b]$.

One proof (see Exercise 9:5.7) would be to show that f is bounded and continuous everywhere except at a set of measure zero. It follows by Theorem 8.17 that f is Riemann integrable.

We can also give a proof by constructing, for any $\varepsilon > 0$, step functions having the property of Exercise 8:6.5. Since this proof is one that was available to nineteenth century mathematicians who would not have known about sets of measure zero this is worth presenting, if only for historical reasons.

Let $\varepsilon > 0$. We wish to find step functions L and U such that $L(x) \le f(x) \le U(x)$ for all $x \in [a,b]$, with $\int_a^b [U(x) - L(x)]\,dx < \varepsilon$. We shall obtain the functions L and U in three natural steps:

1. We approximate f by one of the functions f_N.

2. We obtain corresponding step functions L_N and U_N approximating f_N.

3. We modify L_N and U_N to obtain L and U.

We proceed according to the plan above.
(i) Since $f_n \to f$ *uniformly*, there exists $N \in \mathbb{N}$ such that

$$|f_N(x) - f(x)| \leq \frac{\varepsilon}{4(b-a)} \text{ for all } x \in [a,b].$$

(ii) Since f_N is integrable by hypothesis, there exist step functions L_N and U_N such that $L_N(x) \leq f_N(x) \leq U_N(x)$ for all $x \in [a,b]$ and

$$\int_a^b [U_N(x) - L_N(x)]\, dx < \frac{\varepsilon}{2}.$$

(iii) Let us define the step functions U and L by

$$L(x) = L_N(x) - \frac{\varepsilon}{4(b-a)} \ , \ U(x) = U_N(x) + \frac{\varepsilon}{4(b-a)}$$

for all $x \in [a,b]$.

We then have

$$L(x) < L_N(x) + |f(x) - f_N(x)| \leq f(x) \leq U_N(x) + |f(x) - f_N(x)| < U(x)$$

and

$$\begin{aligned} \int_a^b [U(x) - L(x)]\, dx &= \int_a^b \left\{ [U_N(x) - L_N(x)] + \frac{\varepsilon}{2(b-a)} \right\} dx \\ &= \int_a^b [U_N(x) - L_N(x)]\, dx + \int_a^b \frac{\varepsilon}{2(b-a)}\, dx \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

as was to be shown. ■

Corollary 9.30 *If an infinite series of integrable functions* $\sum_0^\infty f_k$ *converges uniformly to a function* f *on an interval* $[a,b]$*, then* f *is also integrable and*

$$\int_a^b f(x)\, dx = \sum_0^\infty \int_a^b f_k(x)\, dx.$$

Example 9.31 Let $f_n(x) = e^{-nx^2}$. Then for each $x \in [1,2]$ and for every $n \in \mathbb{N}$, $0 < e^{-nx^2} \leq e^{-n}$ and $e^{-n} \to 0$, so $f_n \to 0$ uniformly on [1,2]. It follows that

$$\lim_n \int_1^2 e^{-nx^2}\, dx = \int_1^2 0\, dx = 0.$$

◀

Note. We end this section with a short note that considers whether our main theorem would be true under weaker hypotheses than uniform convergence.

It is possible for a sequence $\{f_n\}$ of functions to converge pointwise (but *not* uniformly) to a function f on $[a, b]$ and still have

$$\lim_n \int_a^b f_n(x)\ dx = \int_a^b f(x)\ dx.$$

For example, suppose we modify the functions of Example 9.6 so that $f_n(1/(2n)) = 1$ instead of $f_n(1/(2n)) = 2n$. We still have $f_n \to 0$ pointwise (but not uniformly), but now $\int_0^1 f_n(x)\ dx \to 0$.

These functions form a ***uniformly bounded*** sequence of functions: that is, there exists a constant M ($M = 1$ in this case) such that $|f_n(x)| \le M$ for all $n \in \mathbb{N}$ and all $x \in [0, 1]$. A theorem (whose proof is beyond the scope of this chapter) asserts that if a uniformly bounded sequence of integrable functions $\{f_n\}$ converges pointwise to an integrable function f on $[a, b]$, then

$$\lim_n \int_a^b f_n(x)\ dx = \int_a^b f(x)\ dx.$$

One cannot drop the hypothesis of integrability of f in this theorem. If, for example, $\{r_n\}$ is an enumeration of the rationals in [0,1] and

$$f_n(x) = \begin{cases} 1, & \text{if } x = r_1, r_2, \dots, r_n \\ 0, & \text{otherwise,} \end{cases}$$

then

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0, & \text{if } x \in [0, 1] \setminus \mathbb{Q} \end{cases}$$

and f is not integrable on [0,1] by Exercise 9:2.3.

9.5.3 Sequences of Improper Integrals

Thus far we have studied limits of ordinary integrals, either of continuous functions on a finite interval $[a, b]$ or Riemann integrable functions on such an interval. What if the integrals are of unbounded functions so that they must be taken in the Cauchy (improper) sense? What if the integrals are to be taken on an infinite interval?

More narrowly let us just ask for the validity of the formulas:

$$\lim_{n\to\infty} \int_a^\infty f_n(t)\, dt = \int_a^\infty f(t)\, dt$$

in case $f_n \to f$ or

$$\sum_{k=1}^{\infty} \int_a^{\infty} g_k(t)\,dt = \int_a^{\infty} f(t)\,dt$$

in case $f = \sum_{k=1}^{\infty} g_k$. A fast and glib answer would be that we hardly expect these to be true for pointwise convergence but certainly uniform convergence will suffice.

But these integrals involve an extra limit operation and we therefore need extra caution. Indeed the following example shows that uniform convergence is far from enough. It is not just the "smoothness" of the convergence that is an issue here.

Example 9.32 Let $f_n(x)$ be defined as $1/n$ for all values of $x \in [0, n]$ but as zero for $x > n$. Then the sequence $\{f_n\}$ converges to zero uniformly on the interval $[0, \infty)$. But the integrals do not converge to zero (as we would have hoped) since

$$\int_0^{\infty} f_n(t)\,dt = 1$$

for all n. ◄

What further condition can we impose so that, together with uniform convergence we will be able to take the limit operation inside the integral

$$\lim_{n\to\infty} \int_0^{\infty} f_n(t)\,dt?$$

The condition we impose in the theorem just requires that all the functions are controlled or dominated by some function that is itself integrable. In our example above note that there is no possibility of an integrable function g on $[0, \infty)$ such that $f_n(x) \leq g(x)$ for all n and x. Theorems of this kind are called *dominated convergence theorems.*

Theorem 9.33 *Suppose that $\{f_n\}$ is a sequence of continuous functions on the interval $[a, \infty)$ such that $f_n \to f$ uniformly on any interval $[a, b]$. If there is a continuous function g on $[a, \infty)$ such that*

$$|f_n(x)| \leq g(x)$$

for all $a \leq x$ and such that the integral

$$\int_a^{\infty} g(x)\,dx$$

exists, then

$$\lim_{n\to\infty} \int_a^{\infty} f_n(t)\,dt = \int_a^{\infty} f(t)\,dt.$$

Proof. As a first step let us show that f is integrable on $[a, \infty)$. Certainly f is continuous since it is a uniform limit of a sequence of continuous functions. Since each $|f_n(x)| \leq g(x)$ it follows that $|f(x)| \leq g(x)$. We check then

$$\left| \int_c^d f(t)\,dt \right| \leq \int_c^d |f(t)|\,dt \leq \int_c^d g(t)\,dt.$$

Since g is integrable it follows by the Cauchy criterion for improper integrals (see Exercise 8:5.11) that the integral $\int_c^d g(t)\,dt$ can be made arbitrarily small for large c and d. But then so also is the integral $\int_c^d f(t)\,dt$ and a further application of the Cauchy criterion for improper integrals shows that f is integrable. (Indeed this argument shows that f is absolutely integrable in fact.)

Now let $\varepsilon > 0$. Choose L_0 so large that

$$\int_{L_0}^{\infty} g(t)\,dt < \varepsilon/4.$$

Choose N so large that

$$|f_n(t) - f(t)| < \frac{\varepsilon}{2(L_0 - a)}$$

if $n \geq N$ and $t \in [a, L_0]$. This is possible because $f_n \to f$ uniformly on $[a, L_0]$. Then we have

$$\left| \int_a^{\infty} f_n(t)\,dt - \int_a^{\infty} f(t)\,dt \right| \leq \int_a^{L_0} |f_n(t) - f(t)|\,dt + \int_{L_0}^{\infty} 2g(t)\,dt$$

$$\leq \frac{\varepsilon}{2(L_0 - a)}(L_0 - a) + \frac{2\varepsilon}{4} = \varepsilon$$

for all $n \geq N$. This proves the assertion of the theorem. ■

Exercises

9:5.1 Prove that $\lim_n \int_{\frac{\pi}{2}}^{\pi} \frac{\sin nx}{nx}\,dx = 0.$

9:5.2 Prove that the series $\sum_0^\infty \frac{x^k}{k}$ converges uniformly on $[0, b]$ for every $b \in [0, 1)$, but does not converge uniformly on $[0, 1)$.

9:5.3 Prove that $\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}\,dx = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^3}$

9:5.4 Prove that if $\sum_0^\infty f_k$ converges uniformly on a set D, then the sequence of terms $\{f_k\}$ converges uniformly on D.

9:5.5 Show that the series $\sum_0^\infty \frac{x^k}{k!}$ converges uniformly on $[-a, a]$ for every $a \in \mathbb{R}$, but does not converge uniformly on all of the real line. (Does it converge pointwise on the real line?) Obtain a series representation for $\int_{-a}^{a} \sum_0^\infty \frac{x^k}{k!}\,dx$.

9:5.6 Let $\{f_n\}$ be a sequence of continuous functions on an interval $[a, b]$ that converges uniformly to a function f. What conditions on g would allow you to conclude that

$$\lim_{n\to\infty} \int_a^b f_n(t)g(t)\,dt = \int_a^b f(t)g(t)\,dt?$$

9:5.7 Let $\{f_n\}$ be a sequence of bounded functions each continuous on an interval $[a, b]$ except at a set of measure zero. Show that if $f_n \to f$ uniformly on $[a, b]$ the the function f is also bounded and continuous on $[a, b]$ except at a set of measure zero. Conclude that a uniformly convergent sequence of Riemann integrable functions must converge to a function that is also Riemann integrable.

9:5.8 Let $p > -1$. Show that

$$\lim_{n\to\infty} \int_1^n \left(1 - \frac{t}{n}\right)^n t^p\,dt = \int_1^\infty e^{-t}t^p\,dt.$$

9:5.9 Formulate and prove a version of the dominated convergence theorem (Theorem 9.33) that would apply to improper integrals on an interval $[a, b]$ where the point of unboundedness is at the endpoint a.

9:5.10 Compute

$$\lim_{n\to\infty} \int_0^1 \frac{e^{-nt}}{\sqrt{t}}\,dt$$

where the integrals must be interpreted as improper integrals.

9.6 Uniform Convergence and Derivatives

We saw in Section 9.5 that a uniformly convergent sequence (or series) of continuous functions can be integrated term-by-term. This allows an easy proof of a theorem on term-by-term differentiation.

Theorem 9.34 *Let $\{f_n\}$ be a sequence of functions each with a continuous derivative on an interval $[a, b]$. If the sequence $\{f'_n\}$ of derivatives converges uniformly to a function on $[a, b]$ and the sequence $\{f_n\}$ converges pointwise to a function f, then f is differentiable on $[a, b]$ and*

$$f'(x) = \lim_n f'_n(x) \quad \textit{for all } x \in [a, b].$$

Proof. Let $g = \lim_n f_n'$. Since each of the functions f_n' is assumed continuous and the convergence is uniform, the function g is also continuous (Theorem 9.22). From Theorem 9.29 we infer

$$\int_a^x g(t)\, dt = \lim_n \int_a^x f_n'(t)\, dt \text{ for all } x \in [a,b]. \tag{12}$$

Applying the fundamental theorem of calculus (Theorem 8.9), we see that

$$\int_a^x f_n'(t)\, dt = f_n(x) - f_n(a) \text{ for all } x \in [a,b] \tag{13}$$

for all $n \in \mathbb{N}$.

But $f_n(x) \to f(x)$ for all $x \in [a,b]$ by hypothesis, so letting $n \to \infty$ in equation (13) and noting (12) we obtain

$$\int_a^x g(t)\, dt = f(x) - f(a)$$

or

$$f(x) = \int_a^x g(t)\, dt + f(a).$$

It follows from the continuity of g and the other half of the fundamental theorem of calculus (Theorem 8.8), that f is differentiable and that $f'(x) = g(x)$ for all $x \in [a,b]$. ■

For series, the theorem takes the following form:

Corollary 9.35 *Let $\{f_k\}$ be a sequence of functions each with a continuous derivative on $[a,b]$ and suppose $f = \sum_0^\infty f_k$ on $[a,b]$. If the series $\sum_0^\infty f_k'$ converges uniformly on $[a,b]$, then $f' = \sum_0^\infty f_k'$ on $[a,b]$.*

Example 9.36 Starting with the geometric series

$$\frac{1}{1-x} = \sum_0^\infty x^k \quad \text{on } (-1,1) \tag{14}$$

we obtain from Corollary 9.35 that

$$\frac{1}{(1-x)^2} = \sum_1^\infty kx^{k-1} \quad \text{on } (-1,1) \tag{15}$$

To justify (15) we observe first that the series (14) converges pointwise on $(-1,1)$. Next we note (Exercise 9:3.17) that the series (15) converges pointwise on $(-1,1)$ and uniformly on any closed interval $[a,b] \subset (-1,1)$. Thus, if $x \in (-1,1)$ and $-1 < a < x < b < 1$, then (15) converges uniformly on $[a,b]$, so (15) holds at x. ◄

9.6.1 Limits of Discontinuous Derivatives

The hypotheses of Theorem 9.34 are somewhat more restrictive than necessary for the conclusion to hold. We need not assume that $\{f_n\}$ converges on all of $[a,b]$; convergence at a single point suffices. Nor need we assume that each of the derivatives f_n' is continuous. (One cannot, however, replace uniform convergence of the sequence $\{f'_n\}$ with pointwise convergence as Example 9.5 shows.) The theorem that follows applies in a number of cases in which Theorem 9.34 does not apply.

Theorem 9.37 *Let $\{f_n\}$ be a sequence of continuous functions defined on an interval $[a,b]$ and suppose that $f_n'(x)$ exists for each n and each $x \in [a,b]$. Suppose that the sequence $\{f'_n\}$ of derivatives converges uniformly on $[a,b]$ and that there exists a point $x_0 \in [a,b]$ such that the sequence of numbers $\{f_n(x_0)\}$ converges. Then the sequence $\{f_n\}$ converges uniformly to a function f on the interval $[a,b]$, f is differentiable and*

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

at each point $x \in [a,b]$.

Proof. Let $\varepsilon > 0$. Since the sequence of derivatives converges uniformly on $[a,b]$ there is an integer N_1 so that

$$|f_n'(x) - f_m'(x)| < \varepsilon$$

for all n, $m \geq N_1$ and all $x \in [a,b]$. Also since the sequence of numbers $\{f_n(x_0)\}$ converges there is an integer $N > N_1$ so that

$$|f_n(x_0) - f_m(x_0)| < \varepsilon$$

for all n, $m \geq N$. Let us, for any $x \in [a,b]$, $x \neq x_0$, apply the mean-value theorem to the function $f_n - f_m$ on the interval $[x_0, x]$ (or on the interval $[x, x_0]$ if $x < x_0$). This gives us the existence of some point ξ between x and x_0 so that

$$f_n(x) - f_m(x) - [f_n(x_0) - f_m(x_0)] = (x - x_0)[f_n'(\xi) - f_m'(\xi)]. \quad (16)$$

From this we deduce that

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x_0) - f_m(x_0)| + |(x - x_0)(f_n'(\xi) - f_m'(\xi)| \\ &< \varepsilon(1 + (b - a)) \end{aligned}$$

for any n, $m \geq N$. Since this N depends only on ε this assertion is true for all $x \in [a,b]$ and we have verified that the sequence of continuous functions $\{f_n\}$ is uniformly Cauchy on $[a,b]$ and hence converges uniformly to a continuous function f on $[a,b]$.

Let us now show that $f'(x_0)$ is the limit of the derivatives $f_n'(x_0)$. Again, for any $\varepsilon > 0$, equation (16) implies that

$$|f_n(x) - f_m(x) - [f_n(x_0) - f_m(x_0)]| \leq |x - x_0|\varepsilon \tag{17}$$

for all $n, m \geq N$ and any $x \neq x_0$ in the interval $[a, b]$. In this inequality let $m \to \infty$ and, remembering that $f_m(x) \to f(x)$ and $f_m(x_0) \to f(x_0)$, we obtain

$$|f_n(x) - f_n(x_0) - [f(x) - f(x_0)]| \leq |x - x_0|\varepsilon \tag{18}$$

if $n \geq N$. Let C be the limit of the sequence of numbers $\{f_n'(x_0)\}$. Thus there exists $M > N$ such that

$$|f_M'(x_0) - C| < \varepsilon. \tag{19}$$

Since the function f_M is differentiable at x_0 there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then

$$\left| \frac{f_M(x) - f_M(x_0)}{x - x_0} - f_M'(x_0) \right| < \varepsilon. \tag{20}$$

From Equation (18) and the fact that $M > N$ we have that

$$\left| \frac{f_M(x) - f_M(x_0)}{x - x_0} - \frac{f(x) - f(x_0)}{x - x_0} \right| < \varepsilon.$$

This, together with the inequalities (19) and (20) shows that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - C \right| < 3\varepsilon$$

for $0 < |x - x_0| < \delta$. This proves that $f'(x_0)$ exists and is the number C, which we recall is $\lim_{n\to\infty} f_n'(x_0)$.

In this argument x_0 may be taken as any point inside the interval $[a, b]$ and so the theorem is proved. ■

For infinite series Theorem 9.37 takes the following form:

Corollary 9.38 *Let $\{f_k\}$ be a sequence of differentiable functions on an interval $[a, b]$. Suppose that the series $\sum_0^\infty f_k'$ converges uniformly on $[a, b]$. Suppose also that there exists $x_0 \in [a, b]$ such that the series $\sum_0^\infty f_k(x_0)$ converges. Then the series $\sum_{k=0}^\infty f_k(x)$ converges uniformly on $[a, b]$ to a function F, F is differentiable and*

$$F'(x) = \sum_{k=0}^{\infty} f_k'(x)$$

for all $a \leq x \leq b$.

Note. In the statement of Theorem 9.37 we hypothesized the existence of a single point x_0 at which the sequence $\{f_n(x_0)\}$ converges. It then followed that the sequence $\{f_n\}$ converges on all of the interval I. If we drop that requirement, but retain the requirement that the sequence $\{f'_n\}$ converges uniformly to a function g on I , we cannot conclude that $\{f_n\}$ converges on I (e.g. let $f_n(x) \equiv n$), but we can still conclude that there exists f such that $f' = g = \lim_n f'_n$ on I. (To see this, fix $x_0 \in I$, let $F_n = f_n - f_n(x_0)$ and apply Theorem 9.37 to the sequence $\{F_n\}$.) Thus, the uniform limit of a sequence of derivatives $\{f'_n\}$ is a derivative even if the sequence of primitives $\{f_n\}$ does not converge.

Exercises

9:6.1 Can the sequence of functions $f_n(x) = \dfrac{\sin nx}{n^3}$ be differentiated term-by-term? How about the series $\sum_1^\infty \dfrac{\sin kx}{k^3}$?

9:6.2 Verify that the function

$$y(x) = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots$$

is a solution of the differential equation $y' = 2xy$ on $(-\infty, \infty)$ without first finding an explicit formula for $y(x)$.

✂ 9.7 Pompeiu's Function

By the end of the nineteenth century analysts had developed enough tools to begin constructing examples of functions that challenged the then prevailing views. One famous mathematician, Henri Poincaré, complained that

> Before when one would invent a new function it was to some practical end; today they are invented to demonstrate the errors in the reasoning of our fathers ... .

Many mathematicians were both shocked and appalled that functions could be constructed which possessed, to them, bizarre and unnatural properties. The beautiful and elegant theories of the nineteenth century were being torn to pieces by pathological examples.

Perhaps the earliest shock was the construction by Weierstrass and others of continuous functions that had derivatives at no points. This did indeed demonstrate some earlier errors because not a few mathematicians thought they had succeeded in proving that continuous functions could not be like this. Another famous example is due to Vito Volterra who produced a differentiable function F with a bounded derivative F' that was not Riemann integrable.

In this section we present an example due to D. Pompeiu in 1906. This function h is differentiable and has the remarkable property that h' is discontinuous on a dense set and h' is zero on another dense set. In particular then h is a differentiable function which, like Volterra's example, has a derivative that is not Riemann integrable. In fact it is integrable on *no* interval while Volterra's example is integrable on many subintervals.

The example makes use of many theorems that we have established to this point and so offers an excellent review of our techniques. We present the example in a series of steps, each of which is left as a relatively easy exercise for the reader. (Exercise 9:7.4 is plausible but messy to verify, and the reader may prefer not to check the details.)

To begin the example we observe that the function

$$f(x) = \sqrt[3]{x-a}$$

has an infinite derivative at $x = a$ and a finite derivative elsewhere. Let $q_1, q_2, q_3, \ldots$ be an enumeration of $\mathbb{Q} \cap [0,1]$. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{\sqrt[3]{x-q_k}}{10^k}.$$

The Pompeiu function is the inverse of this function, $h = f^{-1}$.

The details appear in the exercises. Note especially that our main goal is to prove that h is differentiable, h' is bounded, $h' = 0$ on a dense set and h' is positive and discontinuous on another dense set, and h' is not Riemann integrable.

For comparisons let us recall that in Exercise 7:4.2, we provided an example of a differentiable function g with g' bounded but discontinuous on a nowhere-dense perfect set P. Because of Section 8.6.3 we know that if P does not have measure zero, such a function g' will not be integrable, so we cannot write

$$g(x) - g(a) = \int_a^x g'(t)\, dt,$$

i.e., the Fundamental Theorem of Calculus does not hold for the function g and its derivative g'. This is essentially how Volterra constructed his example, by ensuring that the set P does not have measure zero.

We also mentioned in Section 7.4 that it is possible for a differentiable function f to have f' discontinuous on a dense set and so Pompeiu's function justifies this comment.

Exercises

9:7.1 Show that the function $f(x) = (x-a)^{\frac{1}{3}}$ has an infinite derivative at $x = a$ and a finite derivative elsewhere.

9:7.2 Let $q_1, q_2, q_3, \ldots$ be an enumeration of $\mathbb{Q} \cap [0,1]$. For each $k \in \mathbb{N}$ let $f_k(x) = (x - q_k)^{\frac{1}{3}}$. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{f_k(x)}{10^k}.$$

Show that the series defining f converges uniformly.

9:7.3 Show that f is continuous on [0,1].

9:7.4 Check that, for all $x \in \mathbb{R}$,

$$f'(x) = \sum_{1}^{\infty} \frac{f_k'(x)}{10^k} = \sum_{1}^{\infty} \frac{(x-q_k)^{-\frac{2}{3}}}{3 \times 10^n}.$$

(This part is messy to prove. Indicate why it is that we cannot simply apply Corollary 9.38 and differentiate term by term.)

9:7.5 Show that $f'(x) = \infty$ for all $x \in \mathbb{Q} \cap [0,1]$. (There are also other points at which f' is infinite; see Exercise 9:7.17.)

9:7.6 Show that $f([0,1])$ is an interval. Call it $[a,b]$.

9:7.7 Let $S = f(\mathbb{Q} \cap [0,1])$. Show that S is dense in $[a,b]$.

9:7.8 Show that f has an inverse.

9:7.9 Let $h = f^{-1}$. Show that h is continuous and strictly increasing on $[a,b]$.

9:7.10 Show that $h' = 0$ on the dense set S.

9:7.11 Show that there exists $\gamma > 0$ such that $f'(x) \geq \gamma$ for all $x \in [0,1]$.

9:7.12 Show that h is differentiable, and h' is bounded.

9:7.13 Show that $h' > 0$ on a dense subset of $[a,b]$.

9:7.14 Show that h' is discontinuous on a dense subset of $[a,b]$.

9:7.15 Thus far we know that h is differentiable, has a bounded derivative, $h' = 0$ on a dense set and h' is positive and discontinuous on another dense set. Show that h' is not Riemann integrable.

9:7.16 Show that $\{x : h'(x) \neq 0\}$ does not have measure zero.

9:7.17 Show that there exists $x \notin S$ such that $h'(x) = 0$ and that there exists $t \notin \mathbb{Q}$ such that $f'(t) = \infty$.

9.8 Continuity and Pointwise Limits

Much of this chapter focused on the concept of uniform convergence because of its role in providing affirmative answers to the questions we raised in Section 9.1. In particular, we saw in Section 9.2 that a pointwise limit of a sequence of continuous functions need not be continuous. On the other hand, these problems will not occur if the convergence is uniform.

There are, however, many situations in which pointwise convergence arises naturally, but uniform convergence doesn't. Consider, for example, a function F that is differentiable on $\mathbb{R}$. Then for $x \in \mathbb{R}$,

$$F'(x) = \lim_{n\to\infty} \frac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{n}}.$$

If we define functions f_n by

$$f_n(x) = \frac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{n}},$$

then each of the functions f_n is continuous on $\mathbb{R}$ and $f_n \to F'$ pointwise.

Now, we have seen examples of derivatives that are discontinuous at many points. For example, the function h' in Section 9.7 is discontinuous on a set that is dense in [0,1] and does not have measure zero. Similarly, Exercise 7:4.2 provides an example of a differentiable function g whose derivative g' is discontinuous at every point of a Cantor set that does not have measure zero. One might ask the question, "Can the derivative of a differentiable function be discontinuous everywhere?" We shall see that the answer is "no". In fact, the set of points of continuity must be large in the sense of category—this set must be dense and of type $\mathcal{G}_\delta$, therefore residual (Theorem 6.17).

We actually prove a more general theorem.

Theorem 9.39 *Let $\{g_n\}$ be a sequence of continuous functions defined on an interval I and converging pointwise to a function g on I. Then the set of points of continuity of g forms a dense set of type $\mathcal{G}_\delta$ in I.*

Proof. Let us first outline the idea of the proof, leaving the formal proof for a moment. In Section 6.7 we defined the oscillation $\omega_f(x_0)$ of a funcion f at a point x_0 and showed (Theorem 6.25) that f is continuous at x_0 if and only if $\omega_f(x_0) = 0$. We now show that under the hypotheses of Theorem 9.39, $\omega_g(x)$ will be zero on a dense set. That will imply that g is continuous on a dense set. This set must be of type $\mathcal{G}_\delta$ (by Theorem 6.28).

We will argue by contradiction. We suppose that g is discontinuous at every point of some subinterval J. We will then use the Baire Category theorem (Theorem 6.11) to show that there exists $n \in \mathbb{N}$ and an interval $H \subset J$ such that $\omega_g(x) \geq 1/n$ at every point of H. (This argument is valid for any function discontinuous at every point of an interval J). We then use our hypotheses on g to show this is impossible. We do this by applying the Baire Category theorem once again to obtain a subinterval K of H which g maps onto a set of diameter less than $1/n$. This imples that $\omega_f(x) < 1/n$ for every $x \in K$, a contradiction.

Now we can begin a formal proof of Theorem 9.39.

In order to obtain a contradiction we suppose that g is discontinuous everywhere on some interval $J \subset I$. For each $n \in \mathbb{N}$, let

$$E_n = \{x \in J : \omega_g(x) \geq 1/n\}.$$

Each of the sets E_n is closed (by Theorem 6.27)and $J = \bigcup_{n=1}^\infty E_n$.

By the Baire Category theorem there exists $n \in \mathbb{N}$ and an interval $H \subset J$ such that E_n is dense in H. The interval H has the property that g maps every subinterval of H onto a set of diameter at least $1/n$. We now show this not possible for g, a pointwise limit of continuous functions.

Let $\{I_k = (a_k, b_k)\}$ be a sequence of intervals, each of length less than $1/n$, such that

$$g(H) \subset \bigcup_{k=1}^\infty I_k.$$

For each k, let $H_k = g^{-1}(I_k) \cap H$. Then $H = \bigcup_{k=1}^\infty H_k$, but none of the sets H_k can contain an interval (since each H_k has length less than $1/n$, but $\omega_g(x) \geq 1/n$ for all $x \in H$).

Now

$$H_k = \{x : g(x) < b_k\} \cap \{x : g(x) > a_k\}.$$

By Exercise 9:8.4, each of these sets is of type $\mathcal{F}_\sigma$, thus $H_k = \bigcup_{j=1}^\infty H_{kj}$, with each of the sets H_{kj} closed. It follows that

$$H = \bigcup_{k=1}^\infty H_k = \bigcup_{k=1}^\infty \bigcup_{j=1}^\infty H_{kj}.$$

We have expressed the interval H as a countable union of closed sets. It follows from the Baire Category theorem that at least one of these sets, say H_{ij}, is dense in some interval $K \subset H$. Since H_{ij} is closed, $H_{ij} \supset K$. But this implies that $H_i \supset K$, which we have seen is not possible (since each of the sets H_k contains no intervals). This contradiction completes the proof. ■

Corollary 9.40 *Let f be differentiable on an interval (a,b). Then f' is continuous on a residual subset of (a,b). Thus the set of points of continuity of f' must be dense in (a,b).*

Note. Theorem 9.39 and Exercise 9:8.4 describe two important properties of functions that are pointwise limits of sequences of continuous functions. Each such function f is continuous on a residual set, and every set of the form $\{x : f(x) > a\}$ or $\{x : f(x) < a\}$ is of type $\mathcal{F}_\sigma$.

Theorem 9.39 can be generalized. If P is a nonempty closed subset of the domain of f, then the function $f|P$ is continuous on a dense $\mathcal{G}_\delta$ subset of P.

The converses are also true[3]: A function f is a pointwise limit of a sequence of continuous functions on an interval I if and only if for every closed set $P \subset I$, $f|P$ is continuous on a dense $\mathcal{G}_\delta$ in P, and this happens if and only if every set of the form $\{x : f(x) > a\}$ or $\{x : f(x) < a\}$ is of type $\mathcal{F}_\sigma$.

These theorems have many applications. Functions that are pointwise limits of sequences of continuous functions are called *Baire* 1 functions. We have seen that this class of functions contains the class of derivatives. It also contains several other classes of functions we have encountered—the monotonic functions and the semi-continuous functions.

The exercises below may be instructive. You may need to use one of the unproved statements in this section to work some of these exercises.

Exercises

9:8.1 Give an example of a function F that is differentiable on $\mathbb{R}$ such that the sequence

$$f_n(x) = n(F(x + 1/n) - F(x)),$$

converges pointwise but not uniformly to F'.

9:8.2 Give an example of a function f that is Baire 1 and a real number a so that the sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are not open. Show that, for your example, these sets are of type $\mathcal{F}_\sigma$.

[3]Proofs of these statements and many others can be found in Natanson, *Theory of Functions of a Real Variable*, vol II, Chapter XV, Ungar (English translation).

9:8.3 Give an example of a function f that is Baire 1 and a real number a so that the sets $\{x : f(x) \geq a\}$ and $\{x : f(x) \leq a\}$ are not closed. Show that, for your example, these sets are of type $\mathcal{G}_\delta$.

9:8.4 Show that for any f that is Baire 1 and any real number a the sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are of type $\mathcal{F}_\sigma$.

9:8.5 If f has only countably many discontinuities on an interval I, then f is a Baire 1 function. In particular, this is true for every monotonic function.

9:8.6 Let K be the Cantor set in [0,1]. Define

$$f(x) = \begin{cases} 1, & \text{if } x \in K \\ 0, & \text{elsewhere;} \end{cases}$$

and

$$g(x) = \begin{cases} 1, & \text{if } x \text{ is a one-sided limit point of } K \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Show that f and g have exactly the same set of continuity points.

(b) Show that f is a Baire 1 function, but g is not.

9:8.7 Let $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$.

(a) Show that f is not a Baire 1 function.

(b) Show that f is a pointwise limit of a sequence of Baire 1 functions. (Such functions are called functions of Baire class 2.)

9.9 Additional Problems for Chapter 9

9:9.1 Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions converging pointwise to a function f. If the convergence is uniform, prove that there is a finite number M so that $|f_n(x)| < M$ for all n and all $x \in [0,1]$. Even if the convergence is not uniform, show that there must be a subinterval $[a,b] \subset [0,1]$ and a finite number M so that $|f_n(x)| < M$ for all n and all $x \in [a,b]$.

9:9.2 Let E be a set of real numbers, fixed throughout this exercise. For any function f defined on E write

$$\|f\|_\infty = \sup_{x \in E} |f(x)|.$$

Show that

(a) $\|f\|_\infty = 0$ if and only if f is identically zero on E.

(b) $\|cf\|_\infty = |c|\|f\|_\infty$ for any real number c.

(c) $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$ for any functions f and g.

(d) $f_n \to f$ uniformly on E if and only if $\|f - f_n\|_\infty \to 0$ as $n \to \infty$.

(e) f_n converges uniformly on E if and only if $\|f_m - f_n\|_\infty \to 0$ as $n, m \to \infty$.

(f) Using $E = (0,1)$ and $f_n(x) = x^n$ compute $\|f_n\|_\infty$ and, hence, show that $\{f_n\}$ is not converging uniformly to zero on $(0,1)$.