Chapter 5

CONTINUOUS FUNCTIONS

5.1 Introduction to Limits

The definition of the limit of a function

$$\lim_{x \to x_0} f(x)$$

is given in calculus courses, but in many classes it is not explored to any great depth. Computation of limits is interesting and offers its challenges, but for a course in real analysis we must master the definition itself and derive its consequences.

Our viewpoint is larger than that in most calculus treatments. There it is common to insist, in order for a limit to be defined, that the function f must be defined at least in some interval $(x_0 - \delta, x_0 + \delta)$ that contains the point x_0 (with the possible exception of x_0 itself). Here we must allow a function f that is defined only on some set E and study limits for points x_0 that are not too remote from E. We do not insist that x_0 be in the domain of f but we do require that it be "close". This requirement is expressed using our language from Chapter 4. We must have x_0 a point of accumulation of E.

Except for this detail about the domain of the function the definition we use is the usual ε, δ definition from the calculus. Readers familiar with the sequence limit definitions of Chapter 2 will have no trouble handling this definition. It is nearly the same in general form as the ε, N definition for sequences and many of the proofs use very similar ideas. After all, a sequence is a just a function on $\mathbb{N}$ and so sequence limits and function limits must be closely related.