Chapter 4

SETS OF REAL NUMBERS

4.1 Introduction

Modern set theory and the world it has opened to mathematics has its origins in a problem in analysis. A young Georg Cantor in 1870 began to attack a problem given to him by his senior colleague Edward Heine who worked at the same university. (We shall see Heine playing a key role in some ideas of this chapter too.)

The problem was to determine if the equation

$$\tfrac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) = 0 \qquad (1)$$

must imply that all the coefficients of the series, the $\{a_k\}$ and the $\{b_k\}$ are zero. Cantor solved this using the methods of his time. It was a good achievement, but not the one that was to make him famous. What he did next was to ask, as any good mathematician would, whether his result could be generalized. Suppose that the series (1) converges to zero for all x except possibly for those in a given set E. If this set E is very small then perhaps, still, the coefficients of the series should also have to be all zero.

The nature of these exceptional sets (nowadays called sets of uniqueness) required a language and techniques that were entirely new. Previously a number of authors had needed a language to describe sets that arose in various problems. What was used at the time was very limited and few interesting examples of sets were available. Cantor went beyond these, introducing a new collection of ideas that are now indispensable to analysis. We shall encounter in this chapter many of the notions that arose then: accumulation points, derived sets, countable sets, dense sets, nowhere dense sets.

Incidentally Cantor never did finish his problem of describing the sets of uniqueness as the development of the new set theory was more important and consumed his energies. In fact the problem remains unsolved, although much interesting information about the nature of sets of uniqueness has been discovered.

The theory of sets that Cantor initiated has proved to be fundamental to all of mathematics. Very quickly the most talented analysts of that time began applying his ideas to the theory of functions and by now this material is essential to an understanding of the subject. This chapter contains the most basic material. Later in Chapter 6 we will need some further concepts.

4.2 Points

In our studies of analysis we shall need very often to have a language that describes sets of points and the points that belong to them. That language did not develop until late in the nineteenth century and was a reason for many of the difficulties that the early mathematicians encountered in understanding some problems.

For example consider the set of solutions to an equation

$$f(x) = 0$$

where f is some well behaved function. In the simplest cases, e.g., if f is a polynomial function, the solution set could be empty or a finite number of points. There is no difficulty there. But in more general settings the solution set could be very complicated indeed. It may have points that are "isolated", points appearing in clusters, it may contain intervals or merely fragments of intervals. You can see that we even lack the words to describe the possibilities.

The ideas in this section are all very geometric. Try to draw mental images that depict all of these ideas to get a feel for the definition. The definitions themselves should be remembered, but may prove hard to remember without some associated picture.

The simplest types of sets are intervals. We call

$$[a, b] = \{x : a \leq x \leq b\}$$

a closed interval, and

$$(a, b) = \{x : a < x < b\}$$

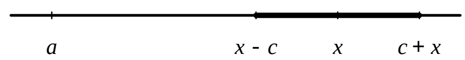

Figure 4.1: Every point in (a,b) is an interior point.

an open interval. The other sets that we often consider are the sets $\mathbb{N}$ of natural numbers, $\mathbb{Q}$ of rational numbers, and the set $\mathbb{R}$ of all real numbers. Use these in your pictures, as well as sets obtained by combining them in many ways.

4.2.1 Interior Points

Every point inside an open interval $I = (a,b)$ has the feature that there is a smaller open interval centered at that point that is also inside I. Thus if $x \in (a,b)$ then for any positive number c that is small enough

$$(x-c, x+c) \subset (a,b).$$

Indeed the arithmetic to show this is easy (and a picture makes it transparent). Let c be any positive number that is smaller than the shortest distance from x to either a or b. Then $(x-c, x+c) \subset (a,b)$. (See Figure 4.1.)

Note. Often one uses the following suggestive language. An open interval that contains a point x is said to be a *neighborhood* of x. Thus each point in (a,b) possesses a neighborhood, indeed many neighborhoods, that lie entirely inside the set I. On occasion the point x itself is excluded from the neighborhood: we say an interval (c,d) is a neighborhood of x if x belongs to the interval and we say that the set $(c,d) \setminus \{x\}$ is a *deleted neighborhood*. This is just the interval with the point x removed.

We can distinguish between points that are merely in a set and points that are more deeply inside the set. The word chosen to convey this image of "inside" is *interior*.

Definition 4.1 (Interior Point) Let E be a set of real numbers. Any point x that belongs to E is said to be *an interior point of* E provided some interval

$$(x - c, x + c) \subset E.$$

Thus an interior point of the set E is not merely *in the set* E; it is, so to speak, deep inside the set, at a positive distance c at least away from every point that does not belong to E.

Example 4.2 The following examples are immediate if a picture is sketched. In each case, though, one should try to find the interval $(x - c, x + c)$ inside or explain why there can be no such interval.

1. Every point x of an open interval (a, b) is an interior point.

2. Every point x of a closed interval $[a, b]$, except the two endpoints a and b, is an interior point.

3. The set of natural numbers $\mathbb{N}$ has no interior points whatsoever.

4. Every point of $\mathbb{R}$ is an interior point.

5. No point of the set of rational numbers $\mathbb{Q}$ is an interior point. (This is because any interval $(x - c, x + c)$ must contain both rational numbers and irrational numbers and, hence, can never be a subset of $\mathbb{Q}$.)

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4.2.2 Isolated Points

Most sets that we consider will have infinitely many points. Certainly any interval (a, b) or $[a, b]$ has infinitely many points. The set $\mathbb{N}$ of natural numbers also has infinitely many points, but as we look closely at any one of these points we see that each point is all alone, at a certain distance away from every other point in the set. We call these points *isolated points* of the set.

Definition 4.3 (Isolated Point) Let E be a set of real numbers. Any point x that belongs to E is said to be *an isolated point of* E provided for some interval $(x - c, x + c)$, that

$$(x - c, x + c) \cap E = \{x\}.$$

Thus an isolated point of the set E is in the set E but has no close neighbors who are also in E. It is at some positive distance c at least away from every other point that belongs to E.

Example 4.4 As before, the examples are immediate if a picture is sketched. In each case, though, one should try to find the interval $(x-c, x+c)$ that meets the set at no other point or show that there is none.

1. No point x of an open interval (a, b) is an isolated point.
2. No point x of a closed interval $[a, b]$ is an isolated point.
3. Every point belonging to the set of natural numbers $\mathbb{N}$ is an isolated point.
4. No point of $\mathbb{R}$ is isolated.
5. No point of $\mathbb{Q}$ is isolated.

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4.2.3 Points of Accumulation

Most sets that we consider will have infinitely many points. While the isolated points are of interest on occasion, more than likely we would be interested in points that are not isolated. These points have the property that every containing interval contains many points of the set. Indeed we are interested in any point x with the property that the intervals $(x-c, x+c)$ meet the set E at infinitely many points. This could happen even if x itself does not belong to E. We call these points *accumulation points* of the set. An accumulation point need not itself belong to the set.

Definition 4.5 (Accumulation Point) Let E be a set of real numbers. Any point x (not necessarily in E) is said to be *an accumulation point of E* provided for every $c > 0$ the interval $(x-c, x+c)$ contains points of the set E, in fact that the intersection

$$(x-c, x+c) \cap E$$

contains infinitely many points.

Thus an accumulation point of E is a point that may or may not itself belong to E and that has very many close neighbors who are in E.

Note. The definition requires that for all $c > 0$ the intersection

$$(x - c, x + c) \cap E$$

contains infinitely many points of E. In checking for an accumulation point it may be preferable to merely check that there is at least one point in this intersection (other than possibly x itself). If there is always at least one point then there must be also infinitely many (Exercise 4:2.18.)

Example 4.6 Yet again, the examples are immediate if a picture is sketched.

1. Every point of an open interval (a, b) is a an accumulation point of (a, b). Moreover the two endpoints a and b are also accumulation points of (a, b) (although they do not belong themselves to (a, b)).

2. Every point of a closed interval $[a, b]$ is an accumulation point of (a, b). No point outside can be.

3. No point at all is an accumulation point of the set of natural numbers $\mathbb{N}$.

4. Every point of $\mathbb{R}$ is an accumulation point.

5. Every point on the real line, both rational and irrational, is an accumulation point of the set $\mathbb{Q}$.

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4.2.4 Boundary Points

The intervals (a, b) and $[a, b]$ have what appears to be an "edge". The points a and b mark the boundaries between the inside of the set (i.e., the interior points) and the "outside" of the set. This inside/outside language with an idea of a boundary between them is most useful but needs a precise definition.

Definition 4.7 (Boundary Point) Let E be a set of real numbers. Any point x (not necessarily in E) is said to be *a boundary point of* E provided every interval $(x - c, x + c)$, contains at least one point of E and also at least one point that does not belong to E.

This definition is easy to apply to the intervals (a, b) and $[a, b]$ but harder to imagine for general sets. For these intervals the only points which are immediately seen to satisfy the definition are the two endpoints that we would have naturally said to be at the boundary.

Example 4.8 The examples are not all transparent, but require careful thinking about the definition.

1. The two endpoints a and b are the only boundary points of an open interval (a, b).

2. The two endpoints a and b are the only boundary points of a closed interval $[a, b]$.

3. Every point in the set $\mathbb{N}$ of natural numbers is a boundary point.

4. No point at all is boundary point of the set $\mathbb{R}$.

5. Every point on the real line, both rational and irrational is a boundary point of the set $\mathbb{Q}$. (Think for a while about this one!)

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Exercises

4:2.1 Determine the set of interior points, points of accumulation, isolated points and boundary points for each of the following sets:

(a) $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$.

(b) $\{0\} \cup \{1, 1/2, 1/3, 1/4, 1/5, \dots\}$.

(c) $(0, 1) \cup (1, 2) \cup (2, 3) \cup (3, 4) \cdots \cup (n, n + 1) \cup \dots$

(d) $(1/2, 1) \cup (1/4, 1/2) \cup (1/8, 1/4) \cup (1/16, 1/8) \cup \dots$

(e) $\{x : |x - \pi| < 1\}$.

(f) $\{x : x^2 < 2\}$

(g) $\mathbb{R} \setminus \mathbb{N}$.

(h) $\mathbb{R} \setminus \mathbb{Q}$.

4:2.2 Give an example of each of the following or explain why you think such a set could not exist.

(a) A nonempty set with no accumulation points and no isolated points.

(b) A nonempty set with no interior points and no isolated points.

(c) A nonempty set with no boundary points and no isolated points.

4:2.3 Show that every interior point of a set must also be an accumulation point of that set, but not conversely.

4:2.4 Show that no interior point of a set can be a boundary point, that it is possible for an accumulation point to be a boundary point, and that every isolated point must be a boundary point.

4:2.5 Let E be a nonempty set of real numbers that is bounded above but has no maximum. Let $x = \sup E$. Show that x is a point of accumulation of E. Is it possible for x to also be an interior point of E? Is x a boundary point of E?

4:2.6 State and solve the version of Exercise 4:2.5 that would use the infimum in place of the supremum.

4:2.7 Let A be a set and $B = \mathbb{R} \setminus A$. Show that every boundary point of A is also a boundary point of B.

4:2.8 Let A be a set and $B = \mathbb{R} \setminus A$. Show that every boundary point of A is a point of accumulation of A or else a point of accumulation of B, perhaps both.

4:2.9 Must every boundary point of a set be also an accumulation point of that set?

4:2.10 Show that every accumulation point of a set that does not itself belong to the set must be a boundary point of that set.

4:2.11 Show that a point x is not an interior point of a set E if and only if there is a sequence of points $\{x_n\}$ converging to x and no point $x_n \in E$.

4:2.12 Let A be a set and $B = \mathbb{R} \setminus A$. Show that every interior point of A is not an accumulation point of B.

4:2.13 Let A be a set and $B = \mathbb{R} \setminus A$. Show that every accumulation point of A is not an interior point of of B.

4:2.14 Give an example of a set that has the set $\mathbb{N}$ as its set of accumulation points.

4:2.15 Show that there is no set which has the interval $(0, 1)$ as its set of accumulation points.

4:2.16 Show that there is no set which has the set $\mathbb{Q}$ as its set of accumulation points.

4:2.17 Give an example of a set that has the set

$$E = \{0\} \cup \{1, 1/2, 1/3, 1/4, 1/5, \dots\}$$

as its set of accumulation points.

4:2.18 Show that a point x is an accumulation point of a set E if and only if for every $\varepsilon > 0$ there are at least two points belonging to the set $E \cap (x - \varepsilon, x + \varepsilon)$.

4:2.19 Suppose that $\{x_n\}$ is a convergent sequence converging to a number L and that $x_n \neq L$ for all n. Show that the set

$$\{x : x = x_n \text{ for some } n\}$$

has exactly one point of accumulation, namely L. Of what importance was the assumption that $x_n \neq L$ for all n for this exercise?

4:2.20 Let E be a set and $\{x_n\}$ a sequence of distinct elements of E. Suppose that $\lim_{n\to\infty} x_n = x$. Show that x is a point of accumulation of E.

4:2.21 Let E be a set and $\{x_n\}$ a sequence of points, not necessarily elements of E. Suppose that $\lim_{n\to\infty} x_n = x$ and that x is an interior point of E. Show that there is an integer N so that $x_n \in E$ for all $n \geq N$.

4:2.22 Let E be a set and $\{x_n\}$ a sequence of elements of E. Suppose that $\lim_{n\to\infty} x_n = x$ and that x is an isolated point of E. Show that there is an integer N so that $x_n = x$ for all $n \geq N$.

4:2.23 Let E be a set and $\{x_n\}$ a sequence of distinct points, not necessarily elements of E. Suppose that $\lim_{n\to\infty} x_n = x$ and that $x_{2n} \in E$ and $x_{2n+1} \notin E$ for all n. Show that x is a boundary point of E.

4:2.24 If E is a set of real numbers then E', called the *derived set* of E, denotes the set of all points of accumulation of E. Give an example of each of the following or explain why you think such a set could not exist.

(a) A nonempty set E such that $E' = E$.

(b) A nonempty set E such that $E' = \emptyset$.

(c) A nonempty set E such that $E' \neq \emptyset$ but $E'' = \emptyset$.

(d) A nonempty set E such that E', $E'' \neq \emptyset$ but $E''' = \emptyset$.

(e) A nonempty set E such that E', E'', E''', ... are all different.

(f) A nonempty set E such that $(E \cup E')' \neq (E \cup E')$.

4:2.25 Show that there is no set with uncountably many isolated points.

✂ **4:2.26** Cantor, in 1885, defined a set E to be *dense-in-itself* if $E \subset E'$. Develop some facts about such sets. Include illustrative examples.

4.3 Sets

We now begin a classification of sets of real numbers. Almost all of the concepts of analysis (limits, derivatives, integrals, etc.) can be better understood if a classification scheme for sets is in place. By far the most important notions are those of closed sets and open sets.

This is the basis for much advanced mathematics and leads to the subject known as topology which is fundamental to an understanding of many areas of mathematics. On the real line we can master open and closed sets and describe precisely what they are.

4.3.1 Closed Sets

In many parts of mathematics the word "closed" is used to indicate that some operation stays within a system. For example the set of natural numbers $\mathbb{N}$ is closed under addition and multiplication (any sum or product of two of them is yet another) but not closed under subtraction or division (2 and 3 are natural numbers, but $2-3$ and $3/2$ are not). This same word was employed originally to indicate sets of real numbers that are "closed" under the operation of taking points of accumulation. If all points of accumulation turn out to be in the set then the set is said to be closed. This terminology has survived and become, perhaps, the best known usage of the word "closed".

Definition 4.9 (Closed) Let E be a set of real numbers. The set E is said to be *closed* provided every accumulation point of E belongs to the set E.

Thus a set E is not closed if there is some accumulation point of E that does not belong to E. In particular a set with no accumulation points would have to be closed since there is no point that needs to be checked.

Example 4.10 The examples are immediate since we have already described all of the accumulation points of these sets.

1. The empty set $\emptyset$ is closed since it contains all of its accumulation points (there are none).

2. The open interval is (a, b) not closed because the two endpoints a and b are accumulation points of (a, b) and yet they do not belong to the set.

3. The closed interval $[a, b]$ is closed since only points that are already in the set are accumulation points.

4. The set of natural numbers $\mathbb{N}$ is closed because it has no points of accumulation.

5. The real line $\mathbb{R}$ is closed since it contains all of its accumulation points, namely every point.

6. The set of rational numbers $\mathbb{Q}$ is not closed. Every point on the real line, both rational and irrational, is an accumulation point of $\mathbb{Q}$, but the set fails to contain any irrationals..

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The closure of a set If a set is not closed it is because it neglects to contain points that "should" be there since they are accumulation points but not in the set. On occasions it is best to throw them in and consider a larger set composed of the original set together with the offending accumulation points that may not have belonged originally to the set.

Definition 4.11 (Closure) Let E be any set of real numbers and let E' denote the set of all accumulation points of E. Then the set

$$\overline{E} = E \cup E'$$

is called the *closure* of the set E.

For example $\overline{(a,b)} = [a,b]$, $\overline{[a,b]} = [a,b]$, $\overline{\mathbb{N}} = \mathbb{N}$, and $\overline{\mathbb{Q}} = \mathbb{R}$. Each of these is an easy observation since we know what the points of accumulation of these sets are.

4.3.2 Open Sets

Originally the word "open" was used to indicate a set that was not closed. In time it was realized that this is a waste of terminology, since the class of "not closed sets" is not of much general interest. Instead the word is now used to indicate a contrasting idea, an idea that is not quite an opposite—just at a different extreme. This may be a bit unfortunate since now a set that is not open need not be closed. Indeed some sets can be both open and closed, and some sets can be both not open and not closed.

Definition 4.12 (Open) Let E be a set of real numbers. Then E is said to be *open* if every point of E is also an interior point of E.

Thus every point of E is not merely a point *in the set* E; it is, so to speak, deep inside the set, at a fixed positive distance away from every point that does not belong to E. Note that this means that an open set cannot contain any of its boundary points.

Example 4.13 These examples are immediate since we have seen them before in the context of interior points in Section 4.2.1.

1. The empty set $\emptyset$ is open since it contains no points that are not interior points of the set. (This is the first example of a set that is both open and closed.)

2. The open interval (a, b) is open since, every point x of an open interval (a, b) is an interior point.

3. The closed interval $[a, b]$ is not open since there are points in the set (namely the two endpoints a and b) that are in the set and yet are not interior points.

4. The set of natural numbers $\mathbb{N}$ has no interior points and so this set is not open; all of its points fail to be interior points.

5. Every point of $\mathbb{R}$ is an interior point and so $\mathbb{R}$ is open. (Remember $\mathbb{R}$ is also closed so it is both open and closed. Note that $\mathbb{R}$ and $\emptyset$ are the only examples of sets that are both open and closed.)

6. No point of the set of rational numbers $\mathbb{Q}$ is an interior point and so $\mathbb{Q}$ definitely fails to be open.

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The interior of a set If a set is not open it is because it contains points that "shouldn't" be there since they are not interior. On occasions it is best to throw them away and consider a smaller set composed entirely of the interior points.

Definition 4.14 (Interior) Let E be any set of real numbers. Then the set

$$\text{int}(E)$$

denotes the set of all interior points of E and is called the *interior* of the set E.

For example $\text{int}((a, b)) = (a, b)$, $\text{int}([a, b]) = (a, b)$, $\text{int}(\mathbb{N}) = \emptyset$, and $\text{int}(\mathbb{Q}) = \emptyset$. Each of these is an easy observation since we know what the interior points of these sets are.

Component Intervals of Open Sets Think of the most general open set G that you can. A first feeble suggestion might be any open interval $G = (a, b)$. We can do a little better. How about the union of two of these

$$G = (a, b) \cup (c, d).$$

If these are disjoint then we would tend to think of G as having two "components". It is easy to see that every point is an interior point. We need not stop at two component intervals; any number would work

$$G = (a_1, b_1) \cup (a_2, b_2) \cup (a_3, b_3) \cup \cdots \cup (a_n, b_n).$$

The argument is the same and elementary. If x is a point in this set then x is an interior point. Indeed we can form the union of a sequence of such open intervals and it is clear that we shall obtain an open set. For a specific example consider

$$(-\infty, -3) \cup (1/2, 1) \cup (1/8, 1/4) \cup (1/32, 1/16) \cup (1/128, 1/64) \cup \ldots.$$

At this point our imagination stalls and it is hard to come up with any more examples that are not obtained by stringing together open intervals in exactly this way. This suggests that, perhaps, all open sets have this structure. They are either open intervals or else a union of a sequence of open intervals. This theorem characterizes all open sets of real numbers and reveals their exact structure.

Theorem 4.15 *Let G be a nonempty open set of real numbers. Then there is a unique sequence (finite or infinite) of disjoint, open intervals*

$$(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots, (a_n, b_n), \ldots$$

called the component intervals of G *such that*

$$G = (a_1, b_1) \cup (a_2, b_2) \cup (a_3, b_3) \cup \cdots \cup (a_n, b_n) \cup \ldots.$$

Proof. Take any point $x \in G$. We know that there must be some interval (a, b) containing the point x and contained in the set G. This is because G is open and so every point in G is an interior point. We need to take the largest such interval. The easiest way to describe this is to write

$$\alpha = \inf\{t : (t, x) \subset G\}$$

and

$$\beta = \sup\{t : (x, t) \subset G\}.$$

Note that $\alpha < x < \beta$. Then $I_x = (\alpha, \beta)$ is called the *component* of G containing the point x. (It is possible here for $\alpha = -\infty$ or $\beta = \infty$.)

One feature of components that we require is this: if x and y belong to the same component then $I_x = I_y$. If x and y do not belong to the same component then I_x and I_y have no points in common. This is easily checked (Exercise 4:3.21).

There remains the task of listing the components as the theorem requires. If the collection $\{I_x : x \in G\}$ is finite then this presents no difficulties. If it is infinite we need a clever strategy.

Let $r_1, r_2, r_3, \dots$ be a listing of all the rational numbers contained in the set G. We construct our list of components of G by writing for the first step

$$(a_1, b_1) = I_{r_1}.$$

The second component must be disjoint from this first component so we cannot simply choose I_{r_2} since if r_2 belongs to (a_1, b_1) then in fact $(a_1, b_1) = I_{r_1} = I_{r_2}$.

Instead we travel along the sequence $r_1, r_2, r_3, \dots$ until we reach the first one, say r_{m_2}, that does not already belong to the interval (a_1, b_1). This then serves to define our next interval

$$(a_2, b_2) = I_{r_{m_2}}.$$

If there is no such point then the process stops. This process is continued inductively resulting in a sequence of open intervals.

$$(a_1, b_1) \cup (a_2, b_2) \cup (a_3, b_3) \cup \cdots \cup (a_n, b_n) \cup \dots$$

which may be infinite or finite. At the kth stage a point r_{m_k} is selected so that r_{m_k} does not belong to any component thus far selected. If this cannot be done then the process stops and produces only a finite list of components.

The proof is completed by checking that (i) every point of G is in one of these intervals, (ii) every point in one of these interval belongs to G, (iii) the intervals in the sequence must be disjoint.

For (i) note that if $x \in G$ then there must be rational numbers in the component I_x. Indeed there is a first number r_k in the list that belongs to this component. But then $x \in I_{r_k}$ and so we must have chosen this interval I_{r_k} at some stage. Thus x does belong to one of these intervals.

For (ii) note that if x is in G then $I_x \subset G$. Thus every point in one of the intervals belongs to G.

For (iii) consider some pair of intervals in the sequence we have constructed. The later one chosen was required to have a point r_{m_k} that did not belong to any of the preceding choices. But that means then that the new component chosen is disjoint from all the previous ones.

This completes the checking of the details and so the proof is done. ■

Exercises

4:3.1 Is it true that a set, all of whose points are isolated, must be closed.

4:3.2 If a set has no isolated points must it be closed? Must it be open?

4:3.3 Some students, when asked, remember that a set is closed "if all its points are points of accumulation". Must such a set be closed?

4:3.4 Some students, when asked, incorrectly remember that a set is open "if it contains all of its interior points". Is there an example of a set that fails to have this property?

4:3.5 Determine which of the following sets are open, which closed and which are neither open nor closed.

(a) $(-\infty, 0) \cup (0, \infty)$.
(b) $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$.
(c) $\{0\} \cup \{1, 1/2, 1/3, 1/4, 1/5, \dots\}$.
(d) $(0,1) \cup (1,2) \cup (2,3) \cup (3,4) \cdots \cup (n, n+1) \cup \dots$
(e) $(1/2, 1) \cup (1/4, 1/2) \cup (1/8, 1/4) \cup (1/16, 1/8) \cup \dots$
(f) $\{x : |x - \pi| < 1\}$.
(g) $\{x : x^2 < 2\}$
(h) $\mathbb{R} \setminus \mathbb{N}$.
(i) $\mathbb{R} \setminus \mathbb{Q}$.

4:3.6 Show that the closure operation has the following properties:

(a) If $E_1 \subset E_2$ then $\overline{E_1} \subset \overline{E_2}$.
(b) $\overline{E_1 \cup E_2} = \overline{E_1} \cup \overline{E_2}$.
(c) $\overline{E_1 \cap E_2} \subset \overline{E_1} \cap \overline{E_2}$.
(d) Give an example of two sets E_1 and E_2 such that
$$\overline{E_1 \cap E_2} \neq \overline{E_1} \cap \overline{E_2}.$$
(e) $\overline{\overline{E}} = \overline{E}$.

4:3.7 Show that the interior operation has the following properties:

(a) If $E_1 \subset E_2$ then $\operatorname{int}(E_1) \subset \operatorname{int}(E_2)$.
(b) $\operatorname{int}(E_1 \cap E_2) = \operatorname{int}(E_1) \cap \operatorname{int}(E_2)$.
(c) $\operatorname{int}(E_1 \cup E_2) \supset \operatorname{int}(E_1) \cup \operatorname{int}(E_2)$.
(d) Give an example of two sets E_1 and E_2 such that
$$\operatorname{int}(E_1 \cup E_2) \neq \operatorname{int}(E_1) \cup \operatorname{int}(E_2).$$
(e) $\operatorname{int}(\operatorname{int}(E)) = \operatorname{int}(E)$.

4:3.8 Show that if the set E' of points of accumulation of E is empty then the set E must be closed.

4:3.9 Show the set E' of points of accumulation of any set E must be closed.

4:3.10 Show the set $\text{int}(E)$ of interior points of any set E must be open.

4:3.11 Show that a set E is closed if and only if $\overline{E} = E$.

4:3.12 Show that a set E is open if and only if $\text{int}(E) = E$.

4:3.13 If A is open and B is closed what can you say about the sets $A \setminus B$ and $B \setminus A$?

4:3.14 If A and B are both open or both closed what can you say about the sets $A \setminus B$ and $B \setminus A$?

4:3.15 If E is a nonempty bounded, closed set show that $\max\{E\}$ and $\min\{E\}$ both exist. If E is a bounded, open set show that neither $\max\{E\}$ and $\min\{E\}$ exist (although $\sup\{E\}$ and $\inf\{E\}$ do).

4:3.16 Show that if a set of real numbers E has at least one point of accumulation then for every $\varepsilon > 0$ there exist points x, $y \in E$ so that $0 < |x - y| < \varepsilon$.

4:3.17 Construct an example of a set of real numbers E that has no points of accumulation and yet has the property that for every $\varepsilon > 0$ there exist points x, $y \in E$ so that $0 < |x - y| < \varepsilon$.

4:3.18 Let $\{x_n\}$ be a sequence of real numbers. Let E denote the set of all numbers z that have the property that there exists a subsequence $\{x_{n_k}\}$ convergent to z. Show that E is closed.

4:3.19 Determine the components of the open set $\mathbb{R} \setminus \mathbb{N}$.

4:3.20 Let $F = \{0\} \cup \{1, 1/2, 1/3, 1/4, 1/5, \dots\}$. Show that F is closed and determine the components of the open set $\mathbb{R} \setminus F$.

4:3.21 In the proof of Theorem 4.15 show that if x and y belong to the same component then $I_x = I_y$, while if x and y do not belong to the same component then I_x and I_y have no points in common.

4:3.22 In the proof of Theorem 4.15 after obtaining the collection of components $\{I_x : x \in G\}$ there remained the task of listing them. In classroom discussions the following suggestions were made as to how the components might be listed:

(a) List the components from largest to smallest.

(b) List the components from smallest to largest.

(c) List the components from left to right.

(d) List the components from right to left.

For each of these give an example of an open set with infinitely many components for which this strategy would work and also an example where it would fail.

4:3.23 In searching for interesting examples of open sets the reader may have run out of ideas. Here is an example of a construction due to Cantor and which has become the source for many important examples in analysis. We describe the component intervals of an open set G inside the interval $(0, 1)$. At each "stage" n we shall describe 2^{n-1} components.

At the first stage, stage one, take $(1/3, 2/3)$ and at stage 2 take $(1/9, 2/9)$ and $(7/9, 8/9)$ and so on so that each stage we take all the middle third intervals of the intervals remaining inside $(0, 1)$. The set G is the open subset of $(0, 1)$ having these intervals as components.

(a) Describe exactly the collection of intervals forming the components of G.

(b) What are the endpoints of the components. How do they relate to ternary expansions of numbers in $[0, 1]$?

(c) What is the sum of the lengths of all components?

(d) Sketch a picture of the set G by illustrating the components at the first three stages.

(e) Show that if x, $y \in G$, $x < y$, but x and y are not in the same component then there are infinitely many components of G in the interval (x, y).

4:3.24 One of Cantor's early results in set theory is that for every closed set E there is a set S with $E = S'$. Attempt a proof.

4.4 Elementary Topology

The study of open and closed sets in any space is called *topology*. Our goal now is to find relations between these ideas and examine the properties of these sets. Much of this is a useful introduction to topology in any space; some is very specific to the real line where the topological ideas are easier to sort out.

The first theorem establishes the connection between the open sets and the closed sets. They are not quite opposites. They are better described as "complementary".

Theorem 4.16 (Open vs. closed) *Let A be a set of real numbers and $B = \mathbb{R} \setminus A$ its complement. Then A is open if and only if B is closed.*

Proof. If A is open and B fails to be closed then there is a point z that is a point of accumulation of B and yet is not in B. Thus z must be in A. But if z is a point in an open set it must be an interior point. Hence there is an interval $(z-\delta, z+\delta)$ contained entirely in A; such an interval contains no points of B. Hence z cannot be a point of accumulation of B. This is a contradiction and so we have proved that B must be closed if A is open.

Conversely if B is closed and A fails to be open then there is a point $z \in A$ that is not an interior point of A. Hence every interval $(z-\delta, z+\delta)$ must contain points outside of A, namely points in B. By definition this means that z is a point of accumulation of B. But B is closed and so z, which is a point in A, should really belong to B. This is a contradiction and so we have proved that A must be open if B is closed. ■

Theorem 4.17 (Properties of open sets) *The open subsets of the real numbers have the following properties:*

1. *The sets $\emptyset$ and $\mathbb{R}$ are open.*
2. *Any intersection of a finite number of open sets is open.*
3. *Any union of an arbitrary collection of open sets is open.*
4. *The complement of an open set is closed.*

Proof. The first assertion is immediate and the last we have already proved. The third is easy. Thus it is enough for us to prove the second assertion. Let us suppose that E_1 and E_2 are open. To show that $E_1 \cap E_2$ is also open we need to show that every point is an interior point. Let $z \in E_1 \cap E_2$. Then, since z is in both of the sets E_1 and E_2 and both are open there are intervals

$$(z-\delta_1, z+\delta_1) \subset E_1$$

and

$$(z-\delta_2, z+\delta_2) \subset E_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. We must then have

$$(z-\delta, z+\delta) \subset E_1 \cap E_2$$

which shows that z is an interior point of $E_1 \cap E_2$. Since z is any point this proves that $E_1 \cap E_2$ is open.

Having proved the theorem for two open sets, it now follows for three open sets since

$$E_1 \cap E_2 \cap E_3 = (E_1 \cap E_2) \cap E_3.$$

That any intersection of an arbitrary finite number of open closed sets is open now follows by induction. ■

Theorem 4.18 (Properties of closed sets) *The closed subsets of the real numbers have the following properties:*

1. *The sets $\emptyset$ and $\mathbb{R}$ are closed.*
2. *Any union of a finite number of closed sets is closed.*
3. *Any intersection of an arbitrary collection of closed sets is closed.*
4. *The complement of a closed set is open.*

Proof. Except for the second assertion these are easy or have already been proved. Let us prove the second one. Let us suppose that E_1 and E_2 are closed. To show that $E_1 \cup E_2$ is also closed we need to show that every accumulation point belongs to that set. Let z be an accumulation point of $E_1 \cup E_2$ that does not belong to the set. Since z is in neither of the closed sets E_1 and E_2 this point z cannot be a an accumulation point of either. Thus some interval $(z-\delta, z+\delta)$ contains no points of either E_1 or E_2. Consequently that interval contains no points of $E_1 \cup E_2$ and is not an accumulation point after all, contradicting our assumption. Since z is any accumulation point this proves that $E_1 \cup E_2$ is closed.

Having proved the theorem for two closed sets, it now follows for three closed sets since

$$E_1 \cup E_2 \cup E_3 = (E_1 \cup E_2) \cup E_3.$$

That any union of an arbitrary finite number of closed sets is closed now follows by induction. ■

Exercises

4:4.1 Explain why it is that the sets $\emptyset$ and $\mathbb{R}$ are open and also closed.

4:4.2 Show that a union of an arbitrary collection of open sets is open.

4:4.3 Show that an intersection of an arbitrary collection of closed sets is closed.

4:4.4 Give an example of a sequence of open sets G_1, G_2, G_3, ... whose intersection is neither open nor closed. Why does this not contradict Theorem 4.17?

4:4.5 Give an example of a sequence of closed sets F_1, F_2, F_3, ... whose union is neither open nor closed. Why does this not contradict Theorem 4.18?

4:4.6 Show that the set $\overline{E}$ can be described as the *smallest closed set that contains every point of* E.

4:4.7 Show that the set $\text{int}(E)$ can be described as the *largest open set that is contained inside* E.

4:4.8 A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *bounded at a point* x_0 provided there are positive numbers ε and M so that $|f(x)| < M$ for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$. Show that the set of points at which a function is bounded is open. Let E be an arbitrary closed set. Is it possible to construct a function $f : \mathbb{R} \to \mathbb{R}$ so that the set of points at which f is not bounded is precisely the set E?

4:4.9 This exercise continues Exercise 4:3.23. Define the *Cantor ternary set* K to be the complement of the open set G of Exercise 4:3.23 in the interval $[0, 1]$.

(a) If all the open intervals up to the nth stage in the construction of G are removed from the interval $[0, 1]$ there remains a closed set K_n that is the union of a finite number of closed intervals. How many intervals?

(b) What is the sum of the lengths of these closed intervals that make up K_n?

(c) Show that $K = \bigcap_{n=1}^{\infty} K_n$.

(d) Sketch a picture of the set K by illustrating the sets K_1, K_2, and K_3.

(e) Show that if $x, y \in K$, $x < y$, then there is an open subinterval $I \subset (x, y)$ containing no points of K.

(f) Give an example of a number $z \in K \cap (0, 1)$ that is not an endpoint of a component of G.

4:4.10 Express the closed interval $[0, 1]$ as an intersection of a sequence of open sets. Can it also be expressed as a union of a sequence of open sets?

4:4.11 Express the open interval $(0, 1)$ as a union of a sequence of closed sets. Can it also be expressed as an intersection of a sequence of closed sets?

4:4.12 Can the closed interval $[0, 1]$ be expressed as the union of a sequence of disjoint closed subintervals each of length smaller than 1?

4:4.13 In many applications of open sets and closed sets one wishes to work just inside some other set A. It is convenient to have a language for this. A set $E \subset A$ is said to be *open relative* to A if $E = A \cap G$ for some set $G \subset \mathbb{R}$ that is open. A set $E \subset A$ is said to be *closed relative* to A if $E = A \cap F$ for some set $F \subset \mathbb{R}$ that is closed. Answer the following questions.

(a) Let $A = [0,1]$ describe, if possible, sets that are open relative to A but not open as subsets of $\mathbb{R}$.

(b) Let $A = [0,1]$ describe, if possible, sets that are closed relative to A but not closed as subsets of $\mathbb{R}$.

(c) Let $A = (0,1)$ describe, if possible, sets that are open relative to A but not open as subsets of $\mathbb{R}$.

(d) Let $A = (0,1)$ describe, if possible, sets that are closed relative to A but not closed as subsets of $\mathbb{R}$.

(e) Let $A = \mathbb{Q}$. Give examples of sets that are neither open nor closed but are both relative to $\mathbb{Q}$.

(f) Show that all the subsets of $\mathbb{N}$ are both open and closed relative to $\mathbb{N}$.

4.5 Compactness Arguments

In analysis one frequently encounters the problem of arguing from a set of "local" assumptions to a "global" conclusion. Let us focus on just one problem of this type and see the kind of arguments that can be used.

Local Boundedness of a Function Suppose that a function f is *locally bounded* at each point of a set E. By this we mean that for every point $x \in E$ there is an interval $(x-\delta, x+\delta)$ and f is bounded on the points in E that belong to that interval. Can we conclude that f is bounded on the whole of the set E?

Thus we have been given a local condition at each point x in the set E. There must be numbers δ_x and M_x so that

$$|f(t)| \leq M_x \text{ for all } t \in E \text{ in the interval } (x - \delta_x, x + \delta_x).$$

The global condition we want, if possible, is to have some single number M that works for all $t \in E$, i.e.,

$$|f(t)| \leq M \text{ for all } t \in E.$$

Two examples show that this depends on the nature of the set E.

Example 4.19 The function $f(x) = 1/x$ is locally bounded at each point x in the set $(0,1)$ but is not bounded on the set $(0,1)$. It is clear that f cannot be bounded on $(0,1)$ since the statement

$$\frac{1}{t} \leq M \text{ for all } t \in (0,1)$$

cannot be true for any M. But this function is locally bounded at each point x here. Let $x \in (0,1)$. Take $\delta_x = x/2$ and $M_x = 2/x$. Then

$$f(t) = \frac{1}{t} \leq \frac{2}{x} = M_x$$

if

$$x/2 = x - \delta_x < t < x + \delta_x.$$

What is wrong here? What is there about this set $E = (0,1)$ that does not allow the conclusion? The point 0 is a point of accumulation of $(0,1)$ that does not belong to $(0,1)$, and so there is no assumption that f is bounded at that point. We could avoid this difficulty if we assume that E is closed. ◀

Example 4.20 The function $f(x) = x$ is locally bounded at each point x in the set $[0,\infty)$ but is not bounded on the set $[0,\infty)$. It is clear that f cannot be bounded on $[0,\infty)$ since the statement

$$f(t) = t \leq M \text{ for all } t \in [0,\infty)$$

cannot be true for any M. But this function is locally bounded at each point x here. Let $x \in [0,\infty)$. Take $\delta_x = 1$ and $M_x = x+1$. Then

$$f(t) = t \leq x + 1 = M_x$$

if $x - 1 < t < x + 1$.

What is wrong here? What is there about this set $E = [0,\infty)$ that does not allow the conclusion. This set is closed and so contains all of its accumulation points so that the difficulty we saw in the preceding example does not arise. The difficulty is that the set is too big, allowing larger and larger bounds as we move to the right. We could avoid this difficulty if we assume that E is bounded. ◀

Indeed, as we shall see, we have reached the correct hypotheses now for solving our problem. The version of the theorem we were searching for is this:

> **Theorem** *Suppose that a function f is locally bounded at each point of a closed and bounded set E. Then f is bounded on the whole of the set E.*

Arguments that exploit the special features of closed and bounded sets of real numbers are called *compactness arguments.* Most often they are used to prove that some local property has global implications, which is precisely the nature of our boundedness theorem. We now solve our problem using various different compactness arguments. Each of these arguments will become a formidable tool in proving theorems in analysis. Many situations will arise in which some local property must be proved to hold globally and compactness will play a huge role in these.

4.5.1 Bolzano–Weierstrass Property

A closed and bounded set has a special feature that can be used to design compactness arguments. This property is a essentially a repeat of a property about convergent subsequences that we saw already in Section 2.11.

Theorem 4.21 (Bolzano–Weierstrass Property) *A set of real numbers E is closed and bounded if and only if every sequence of points chosen from the set has a subsequence that converges to a point that belongs to E.*

Proof. Suppose that E is both closed and bounded and let $\{x_n\}$ be a sequence of points chosen from E. Since E is bounded this sequence $\{x_n\}$ must be bounded too. We apply the Bolzano–Weierstrass theorem for sequences (Theorem 2.40) to obtain a subsequence $\{x_{n_k}\}$ that converges. If $x_{n_k} \to z$ then since all the points of the subsequence belong to E either the sequence is constant after some term or else z is a point of accumulation of E. In either case we see that $z \in E$. This proves the theorem in one direction.

In the opposite direction we suppose that a set E, which we do not know in advance to be either closed or bounded, has the Bolzano–Weierstrass property. Then E cannot be unbounded. For example if E is unbounded above then there is a sequence of points $\{x_n\}$ of E with $x_n \to \infty$ and no subsequence of that sequence will converge, contradicting the assumption.

Also E must be closed. If not there is a point of accumulation z that is not in E. This means that there is a sequence of points $\{x_n\}$ in E converging to z. But any subsequence of $\{x_n\}$ would also converge to z and, since $z \notin E$ we again have a contradiction. ■

This theorem can also be interpreted as a statement about accumulation points.

Corollary 4.22 *A set of real numbers E is closed and bounded if and only if every infinite subset of E has a point of accumulation that belongs to E.*

Now we prove our theorem about local boundedness by using the Bolzano–Weierstrass property to frame an argument.

Theorem *Suppose that a function f is locally bounded at each point of a closed and bounded set E. Then f is bounded on the whole of the set E.*

Proof. (Bolzano–Weierstrass compactness argument). To use this argument we will need to construct a sequence of points in E that we can use. Our proof is a proof by contradiction. If f is not bounded on E there must be a sequence of points $\{x_n\}$ chosen from E so that $|f(x_n)| > n$. (If such a sequence could not be chosen then at some stage, N say, there are no more points with $|f(x_N)| > N$ and N is an upper bound.)

By compactness (i.e., by Theorem 4.21) there is a convergent subsequence $\{x_{n_k}\}$ converging to a point $z \in E$. By the local boundedness assumption there is an open interval $(z-\delta, z+\delta)$ and a number M_z so that $|f(t)| \leq M_z$ whenever t is in E and inside that interval. But for all sufficiently large values of k, the point x_{n_k} must belong to the interval $(z-\delta, z+\delta)$. The two statements

$$|f(x_{n_k})| > n_k \text{ and } |f(x_{n_k})| \leq M_z$$

cannot both be true for all large k and so we have reached a contradiction, proving the theorem. ■

4.5.2 Cantor's Intersection Property

A famous compactness argument, one that is used very often in analysis, involves the intersection of a *descending* sequence of sets, i.e., a sequence with $E_1 \supset E_2 \supset E_3 \supset E_4 \supset \dots$. What conditions on the sequence will imply that

$$\bigcap_{n=1}^{\infty} E_n \neq \emptyset?$$

Example 4.23 An example shows that some conditions are needed. Suppose for each $n \in \mathbb{N}$ we let $E_n = (0, 1/n)$. Then $E_1 \supset E_2 \supset \dots$, so $\{E_n\}$ is a descending sequence of sets with empty intersection. The same is true of the sequence $F_n = [n, \infty)$. Observe that the sets in the sequence $\{E_n\}$ are bounded (but not closed) while the sets in the sequence $\{F_n\}$ are closed (but not bounded). ◀

In a paper in 1879 Cantor described the following theorem and the role it plays in analysis. He pointed out that variants on this idea had been already used throughout most of that century, notably by Lagrange, Legendre, Dirichlet, Cauchy, Bolzano and Weierstrass.

Theorem 4.24 *Let $\{E_n\}$ be a sequence of nonempty closed and bounded subsets of real numbers such that $E_1 \supset E_2 \supset E_3 \supset \dots$. Let $E = \bigcap_{n=1}^{\infty} E_n$. Then E is not empty.*

Proof. For each $i \in \mathbb{N}$ choose $x_i \in E_i$. The sequence $\{x_i\}$ is bounded since every point lies inside the bounded set E_1. Therefore, because of Theorem 4.21, $\{x_i\}$ has a convergent subsequence $\{x_{i_k}\}$. Let z denote that limit. Fix an integer m. Because the sets are descending, $x_{i_k} \in E_m$ for all sufficiently large $k \in \mathbb{N}$. But E_m is closed, from which it follows that $z \in E_m$. This is true for all $m \in \mathbb{N}$, so $z \in E$. ■

Corollary 4.25 (Cantor Intersection Theorem) *Let $\{E_n\}$ be a sequence of nonempty closed subsets of real numbers such that $E_1 \supset E_2 \supset E_3 \supset \dots$. If*

$$\textit{diameter } E_n \to 0,$$

then the intersection

$$E = \bigcap_{n=1}^{\infty} E_n$$

consists of a single point.

Proof. Here the diameter of a nonempty, closed bounded set E would just be $\max E - \min E$ which exists and is finite for such a set (see Exercise 4:3.15). Since we are assuming that the diameters shrink to zero it follows that, at least for all sufficiently large n, E_n must be bounded.

That $E \neq \emptyset$ follows from Theorem 4.24. It remains to show that E contains only one point. Let $x \in E$ and let $y \in \mathbb{R}$, $y \neq x$. Since diameter $E_n \to 0$, there exists $i \in \mathbb{N}$ such that diameter $E_i < |x - y|$. Since $x \in E_i$, y cannot be in E_i. Thus $y \notin E$ and $E = \{x\}$ as required. ■

Now we prove our theorem about local boundedness by using the Cantor Intersection property to frame an argument.

> **Theorem** *Suppose that a function f is locally bounded at each point of a closed and bounded set E. Then f is bounded on the whole of the set E.*

Proof. (Cantor Intersection compactness argument). To use this argument we will need to construct a sequence of closed and bounded sets shrinking to a point. Our proof is again a proof by contradiction. Suppose that f is not bounded on E.

Since E is bounded we may assume that E is contained in some interval $[a, b]$. Divide that interval in half, forming two subintervals of the same length, namely $(b - a)/2$. At least one of these intervals contains points of E and f is unbounded on that interval. Call it $[a_1, b_1]$.

Now do the same to the interval $[a_1, b_1]$. Divide that interval in half, forming two subintervals of the same length, namely $(b - a)/4$. At least one of these intervals contains points of E and f is unbounded on that interval. Call it $[a_2, b_2]$. Continue this process inductively producing a descending sequence of intervals $\{[a_n, b_n]\}$ so that the nth interval $[a_n, b_n]$ has length $(b-a)/2^n$, contains points of E and f is unbounded on $E \cap [a_n, b_n]$.

By the Cantor Intersection property there is a single point $z \in E$ contained in all of these intervals. But by our local boundedness assumption there is an interval $(z - c, z + c)$ so that f is bounded on the points of E in that interval. For any large enough value of n, though, the interval $[a_n, b_n]$ would be contained inside the interval $(z - c, z + c)$. This would be impossible and so we have reached a contradiction, proving the theorem. ■

4.5.3 Cousin's Property

Another compactness argument dates back to Pierre Cousin in the last years of the nineteenth century. This exploits the order of the real line and considers how small intervals may be pieced together to give larger intervals. The larger interval $[a, b]$ is subdivided

$$a = x_0 < x_1 < \cdots < x_n = b$$

and then expressed as a finite union of nonoverlapping subintervals said to form a *partition*:

$$[a, b] = \bigcup_{i=1}^{n} [x_{i-1}, x_i].$$

This again provides us with a compactness argument since it allows a way to argue from the local to the global.

Lemma 4.26 (Cousin) *Let $\mathcal{C}$ be a collection of closed subintervals of $[a, b]$ with the property that for each $x \in [a, b]$ there exists $\delta = \delta(x) > 0$ such that $\mathcal{C}$ contains all intervals $[c, d] \subset [a, b]$ that contain x and have length smaller than δ. Then there exists a partition*

$$a = x_0 < x_1 < \cdots < x_n = b$$

of $[a, b]$ such that $[x_{i-1}, x_i] \in \mathcal{C}$ for $i = 1, \ldots, n$.

Observe that this lemma makes precise the statement that if a collection of closed intervals contains all "sufficiently small" ones for $[a,b]$, then it contains a partition of $[a,b]$. We shall frequently see the usefulness of such a partition. This is the most elementary of a collection of tools called *covering theorems.* Roughly a *cover* of a set is a family of intervals covering the set in the sense that each point in the set is contained in one or more of the intervals. We formalize the assumption in Cousin's lemma in this language:

Definition 4.27 (Full Cover) A collection $\mathcal{C}$ of closed intervals satisfying the hypothesis of Cousin's lemma is called a *full cover* of $[a,b]$.

the fact that z_0 is $\sup S$.

Proof. (Proof of Cousin's lemma.) Let us, in order to obtain a contradiction, suppose that $\mathcal{C}$ does not contain a partition of the interval $[a,b]$. Let c be the midpoint of that interval and consider the two subintervals $[a,c]$ and $[c,b]$. If $\mathcal{C}$ contains a partition of both intervals $[a,c]$ and $[c,b]$ then by putting those partitions together we can obtain a partition of $[a,b]$, which we have supposed is impossible.

Let $I_1 = [a,b]$ and let I_2 be either $[a,c]$ or $[c,b]$ chosen so that $\mathcal{C}$ contains no partition of I_2. Inductively we can continue in this fashion obtaining a shrinking sequence of intervals $I_1 \supset I_2 \supset I_3 \supset \dots$ so that the length of I_n is $(b-a)/2^{n-1}$ and $\mathcal{C}$ contains no partition of I_n.

By the Cantor intersection theorem (Theorem 4.25) there is a single point z in all of these intervals. The interval $(z-\delta(z), z+\delta(z))$ contains I_n for all sufficiently large n and so, by definition, $I_n \in \mathcal{C}$. In particular $\mathcal{C}$ does indeed contain a partition of that interval I_n since the single interval $\{I_n\}$ is itself a partition. But this contradicts the way in which the sequence was chosen and this contradiction completes our proof. ■

Now we reprove our theorem about local boundedness by using Cousin's property to frame an argument.

> **Theorem** *Suppose that a function f is locally bounded at each point of a closed and bounded set E. Then f is bounded on the whole of the set E.*

Proof. (Cousin compactness argument). The set E is bounded and so is contained in some interval $[a,b]$. Let us say that an interval $[c,d] \subset [a,b]$ is "black" if the following statement is true:

> There is a number M (which may depend on $[c,d]$) so that $|f(t)| \leq M$ for all $t \in E$ that are in the interval $[c,d]$.

The collection of all black intervals is a full cover of $[a,b]$. This is because of the local boundedness assumption on f. Consequently, by Cousin's Lemma, there is a partition of the interval $[a,b]$ consisting of black intervals. The function f is bounded in E on each of these finitely many black intervals and so, since there are only finitely many of them, f must be bounded on E in $[a,b]$. But $[a,b]$ includes all of E and so the proof is complete. ■

4.5.4 Heine–Borel Property ✂

Another famous compactness property involves covers too, as in the Cousin lemma, but this time covers consisting of open intervals. This theorem has wide applications, including again extensions of local properties to global ones. The student may find this compactness argument more difficult to work with than the others. On the real line all of the arguments here are equivalent and, in most cases, any one will do the job. Why not use the simpler ones then? The answer is that in more general spaces than the real line these other versions may be more useful. Time spent learning them now will pay off in later courses.

We begin with some definitions.

Definition 4.28 (Open Cover) Let $A \subset \mathbb{R}$ and let $\mathcal{U}$ be a family of open intervals. If for every $x \in A$ there exists at least one interval $U \in \mathcal{U}$ such that $x \in U$, then $\mathcal{U}$ is called an *open cover* of A.

Definition 4.29 (Heine-Borel Property) A set $A \subset \mathbb{R}$ is said to have the *Heine-Borel property* if every open cover of A can be reduced to a finite subcover. That is, if $\mathcal{U}$ is an open cover of A, then there exists a finite subset of $\mathcal{U}$, $\{U_1, U_2, \dots, U_n\}$ such that $A \subset U_1 \cup U_2 \cup \cdots \cup U_n$.

Example 4.30 Any finite set has the Heine-Borel property. Just take one interval from the cover for each element in the finite set. ◀

Example 4.31 The set $\mathbb{N}$ does not have the Heine-Borel property. Take, for example, the collection of open intervals

$$\{(0,n) : n = 1, 2, 3, \dot{\}}.$$

While this forms an open cover of $\mathbb{N}$ no finite subcollection could also be an open cover. ◀

Example 4.32 The set $A = \{1/n : n \in \mathbb{N}\}$ does not have the Heine-Borel property. Take, for example, the collection of open intervals

$$\{(1/n, 2) : n = 1, 2, 3, \dot{\}}.$$

While this forms an open cover of A no finite subcollection could also be an open cover. ◀

Observe in these examples that that $\mathbb{N}$ is closed (but not bounded) while A is bounded (but not closed). We shall prove, in Theorem 4.33, that a set A has the Heine-Borel property if and only if that set is both closed and bounded.

Theorem 4.33 (Heine-Borel) *A set $A \subset \mathbb{R}$ has the Heine-Borel property if and only if A is both closed and bounded.*

Proof. Suppose $A \subset \mathbb{R}$ is both closed and bounded, and $\mathcal{U}$ is an open cover for A. We may assume $A \neq \emptyset$, otherwise there is nothing to prove. Let $[a, b]$ be the smallest closed interval containing A: i.e. $a = \inf\{x : x \in A\}$ and $b = \sup\{x : x \in A\}$. Observe that $a \in A$ and $b \in A$. We shall apply Cousin's lemma to the interval $[a, b]$, so we need to first define an appropriate full cover of $[a, b]$.

For each $x \in A$, since $\mathcal{U}$ is an open cover of A, there exists an open interval $U_x \in \mathcal{U}$ such that $x \in U_x$. Since U_x is open, there exists $\delta(x) > 0$ for which $(x-t, x+t) \subset U_x$ for all $t \in (0, \delta(x))$. This defines $\delta(x)$ for points in A. Now consider points in $V = [a, b] \setminus A$. We must define $\delta(x)$ for points of V. Since A is closed and $\{a, b\} \subset A$, V is open (why?); thus for each $x \in V$ there exists $\delta(x) > 0$ such that $(x - t, x + t) \subset V$ for all $t \in (0, \delta(x))$. We can therefore obtain a full cover $\mathcal{C}$ of $[a, b]$ as follows: An interval $[c, d]$ is a member of $\mathcal{C}$ if there exists $x \in [a, b]$ such that either

(i) $x \in A$ and $x \in [c, d] \subset U_x$ or

(ii) $x \in V$ and $x \in [c, d] \subset V$.

Observe that an interval of type (i) can contain points of V, but an interval of type (ii) cannot contain points of A. Figure 4.2 illustrates examples of both types of intervals. In that figure $[c, d] \subset U_x$ is an interval of type (i) in $\mathcal{C}$; $[c', d'] \subset V$ is an interval of type (ii) in $\mathcal{C}$.

It is clear that $\mathcal{C}$ forms a full cover of $[a, b]$. From Cousin's lemma we infer the existence of a partition $a = x_0 < x_1 < \cdots < x_n = b$ with

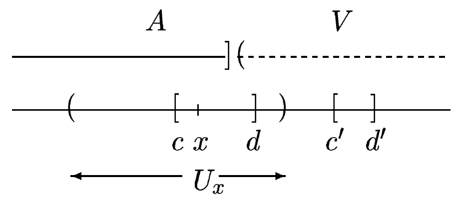

Figure 4.2: The two types of intervals.

$[x_{i-1}, x_i] \in \mathcal{C}$ for $i = 1, \ldots, n$. Each of the intervals $[x_{i-1}, x_i]$ is either contained in V (in which case it is disjoint from A), or is contained in some member $U_i \in \mathcal{U}$. We now 'throw away' from the partition those intervals which contain only points of V, and the union of the remaining closed intervals covers all of A. Each interval of this finite collection is contained in some open interval U from the cover $\mathcal{U}$. More precisely, let

$$S = \{i : 1 \leq i \leq n \text{ and } [x_{i-1}, x_i] \subset U_i\}.$$

Then

$$A \subset \bigcup_{i \in S} [x_{i-1}, x_i] \subset \bigcup_{i \in S} U_i,$$

so $\{U_i : i \in S\}$ is the required subcover of A.

To prove the converse, we must show that if A is not bounded, or is not closed then there exists an open cover of A with no finite subcover. Suppose first that A is not bounded. Then there must exist either an increasing sequence of points $\{x_n\}$ contained in A so that $x_n \to \infty$ or a decreasing sequence of points $\{x_n\}$ contained in A so that $x_n \to -\infty$. Let us suppose the former. For each $i \in \mathbb{N}$ let $U_1 = (-\infty, x_1)$, $U_{i+1} = (x_i, x_{i+1})$ and $V_i = (x_i - 1, x_{i+1})$. Finally let $\mathcal{U}$ be the collection of all the intervals U_i, V_i for $i = 1, 2, 3, \ldots$. Then $\mathcal{U}$ is an open cover of A. (Indeed it is an open cover of all of $\mathbb{R}$). But it is clear that $\mathcal{U}$ contains no finite subcover of A since, for any integer N, the totality of all the sets U_i, V_i for $i = 1, 2, 3, \ldots N$ cannot cover all of A since no point x_n with $n > N$ can belong to any of these intervals.

Now suppose A is not closed. Then there is a point of accumulation z of A that does not belong to A. Then there must exist either an increasing sequence of points $\{x_n\}$ contained in A so that $x_n \to z$ or a decreasing sequence of points $\{x_n\}$ contained in A so that $x_n \to z$. Suppose the former. For each $i \in \mathbb{N}$ let $U_1 = (-\infty, x_1)$,

$V_1 = (z, \infty)$, $U_{i+1} = (x_i, x_{i+1})$ and $V_i = (x_i - 1, x_{i+1})$. Then $\mathcal{U}$ is an open cover of A. (Indeed, as before, it is an open cover of all of $\mathbb{R}$). But it is clear that $\mathcal{U}$ contains no finite subcover of A since, for any integer N, the totality of all the sets U_i, V_i for $i = 1, 2, 3, \ldots N$ cannot cover all of A since no point x_n with $n > N$ can belong to any of these intervals.

■

Once again we return our sample theorem that shows how a local property can be used to prove a global condition, this time using a Heine–Borel compactness argument.

> **Theorem** *Suppose that a function f is locally bounded at each point of a closed and bounded set E. Then f is bounded on the whole of the set E.*

Proof. **(Heine–Borel compactness argument).** Since f is locally bounded at each point of E for every $x \in E$ there exists an open interval U_x containing x and a positive number M_x such that $|f(t)| < M_x$ for all $t \in U_x \cap E$. Let $\mathcal{U} = \{U_x : x \in E\}$. Then $\mathcal{U}$ is an open cover of E. By the Heine–Borel theorem there exists $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$ such that

$$E \subset U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}.$$

Let

$$M = \max\{M_{x_1}, M_{x_2}, \ldots, M_{x_n}\}.$$

Let $x \in E$. Then there exists i, $1 \le i \le n$, for which $x \in U_i$. Since

$$|f(x)| \le M_{x_i} \le M$$

we conclude that f is bounded on E. ■

Our ability to reduce $\mathcal{U}$ to a *finite* subcover in the proof of this theorem was crucial. The reader may wish to use the function $f(x) = 1/x$ on $(0, 1]$ to appreciate this statement.

4.5.5 Compact Sets

We have seen now a wide range of techniques called compactness arguments that can be applied to a set that is closed and bounded. We now introduce the modern terminology for such sets.

Definition 4.34 A set of real numbers E is said to be *compact* if it has any of the following equivalent properties:

1. E is closed and bounded.

2. E has the Bolzano–Weierstrass property.

3. E has the Heine–Borel property.

In spaces more general than the real line there may be analogues of the notions of closed, bounded, convergent sequences, and open covers. Thus there can also be analogues of closed and bounded sets, the Bolzano–Weierstrass property, and the Heine–Borel property. In these more general spaces the three properties are not always equivalent and it is the Heine–Borel property that is normally chosen as the definition of compact sets there. Even so a thorough understanding of compactness arguments on the real line is an excellent introduction to these advanced and very important ideas in other settings.

If we return to our sample theorem we see that now, perhaps, it should best be described in the language of compact sets:

> **Theorem** *Suppose that E is compact. Then every function $f : E \to \mathbb{R}$ that is locally bounded on E is bounded on the whole of the set E. Conversely if every function $f : E \to \mathbb{R}$ that is locally bounded on E is bounded on the whole of the set E, then E must be compact.*

In real analysis there are many theorems of this type. The concept of compact set captures exactly when many local conditions can have global implications.

Exercises

4:5.1 Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is not locally bounded at any point.

4:5.2 Show directly that the interval $[0, \infty)$ does not have the Bolzano–Weierstrass property.

4:5.3 Show directly that the interval $[0, \infty)$ does not have the Heine–Borel property. ✂

4:5.4 Show directly that the set $[0, 1] \cap \mathbb{Q}$ does not have the Heine–Borel property. ✂

4:5.5 Develop the properties of compact sets. For example, is the union of a pair of compact sets compact? The intersection. The union of a family of compact sets?

4:5.6 Show directly that the union of two sets with the Bolzano–Weierstrass property must have the Bolzano–Weierstrass property.

4:5.7 Show directly that the union of two sets with the Heine–Borel property must have the Heine–Borel property.

4:5.8 We defined an open cover of a set E to consist of open *intervals* covering E. Let us change that definition to allow an open cover to consist of any family of open *sets* covering E. What changes are needed in the proof of Theorem 4.33 so that it remains valid in this greater generality?

4:5.9 A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *locally increasing* at a point x_0 if there is a $\delta > 0$ so that

$$f(x) < f(x_0) < f(y)$$

whenever

$$x_0 - \delta < x < x_0 < y < x_0 + \delta.$$

Show that a function that is locally increasing at every point in $\mathbb{R}$ must be increasing, i.e., that $f(x) < f(y)$ for all $x < y$.

4:5.10 We have seen that the following four conditions on a set $A \subset \mathbb{R}$ are equivalent:

(i) A is closed and bounded

(ii) Every infinite subset of A has a limit point in A.

(iii) Every sequence of points from A has a subsequence converging to a point in A

(iv) Every open cover of A has a finite subcover.

Prove directly that (ii)⇒(iii), (ii)⇒(iv) and (iii)⇒(iv).

4:5.11 Prove the following variant of Lemma 4.26:

> Let $\mathcal{C}$ be a collection of closed subintervals of $[a, b]$ with the property that for each $x \in [a, b]$ there exists $\delta = \delta(x) > 0$ such that $\mathcal{C}$ contains all intervals $[c, d] \subset [a, b]$ that contain x and have length smaller than δ. Suppose that $\mathcal{C}$ has the property that if $[\alpha, \beta]$ and $[\beta, \gamma]$ both belong to $\mathcal{C}$ then so too does $[\alpha, \gamma]$. Then $[a, b]$ belongs to $\mathcal{C}$.

4:5.12 Use the version of Cousin's lemma given in Exercise 4:5.11 to give a rather simpler proof of the sample theorem on local boundedness.

4:5.13 Give an example of an open covering of the set $\mathbb{Q}$ of rational numbers that does not reduce to a finite subcover.

4:5.14 Suppose that E is closed and K is compact. Show that $E \cap K$ is compact. Do this in two ways (using the definition and using the Bolzano–Weierstrass property).

4:5.15 Prove that every function $f : E \to \mathbb{R}$ that is locally bounded on E is bounded on the whole of the set E only if the set E is compact, by supplying the following two constructions:

(a) Show that if E is not bounded then there is an unbounded function $f : E \to \mathbb{R}$ so that f is locally bounded on E.

(b) Show that if E is not closed then there is an unbounded function $f : E \to \mathbb{R}$ so that f is locally bounded on E.

.

4:5.16 Suppose that E is closed and K is compact. Show that $E \cap K$ is compact using the Heine–Borel property.

4:5.17 Suppose that E is compact. Is the set of boundary points of E also compact?

4:5.18 Prove Lindelöff's covering theorem:

> *Let $\mathcal{C}$ be a collection of open intervals such that every point of a set E belongs to at least one of the intervals. Then there is a sequence of intervals $I_1, I_2, I_3, \ldots$ chosen from $\mathcal{C}$ that also covers E.*

4:5.19 Describe briefly the distinction between the covering theorem of Lindelöff (Exercise 4:5.18) and that of Heine-Borel.

4:5.20 We have seen that the following four conditions on a set $A \subset \mathbb{R}$ are equivalent:

(a) A is closed and bounded

(b) Every infinite subset of A has a limit point in A.

(c) Every sequence of points from A has a subsequence converging to a point in A

(d) Every open cover of A has a finite subcover.

Prove directly that (b)⇒(c), (b)⇒(d) and (c)⇒(d).

4:5.21 Let f be a function that is locally bounded on a compact interval $[a, b]$. Let

$$S = \{a < x \leq b : \ f \text{ is bounded on } [a, x]\}.$$

(a) Show that $S \neq \emptyset$.

(b) Show that if $z = \sup S$ then $a < z \leq b$.

(c) Show that $z \in S$.

(d) Show that $z = b$ by showing that $z < b$ is impossible.

Using these steps construct a proof of the sample theorem on local boundedness.

4.6 Countable sets

As part of our discussion of properties of sets in this chapter let us review a special property of sets that relates, not to their topological properties, but to their size. We can divide sets into finite sets and infinite sets. How do we divide infinite sets into "large" and "larger" infinite sets?

We did this in our discussion of sequences in Section 2.3. (If the reader skipped over that section now is a good time to go back.) If an infinite set E has the property that the elements of E can be written as a list (i.e., as a sequence)

$$e_1, e_2, e_3, \dots, e_n \dots$$

then that set is said to be *countable*. Note that this property has nothing particularly to do with the other properties of sets encountered in this chapter. It is yet another and different way of classifying sets.

The following properties review our understanding of countable sets. Remember that the empty set, any finite set and any infinite set that can be listed are all said to be countable. An infinite set that cannot be listed is said to be *uncountable*.

Theorem 4.35 *Countable sets have the following properties:*

1. *Any subset of a countable set is countable.*
2. *Any union of a sequence of countable sets is countable.*
3. *No interval is countable.*

Exercises

4:6.1 Give examples of closed sets that are countable and closed sets that are uncountable.

4:6.2 Is there a nonempty open set that is countable?

4:6.3 If a set is countable what can you say about its complement?

4:6.4 Is the intersection of two uncountable sets uncountable?

4:6.5 Show that the Cantor set of Exercise 4:3.23 is infinite and uncountable.

4:6.6 Give (if possible) an example of a set with

(a) Countably many points of accumulation.

(b) Uncountably many points of accumulation.

(c) Countably many boundary points.

(d) Uncountably many boundary points.

(e) Countably many interior points.

(f) Uncountably many interior points.

4:6.7 A set is said to be co-countable if it has a countable complement. Show that the intersection of finitely many co-countable sets is itself co-countable.

4:6.8 Let E be a set and $f : \mathbb{R} \to \mathbb{R}$ be an *increasing* function (i.e., if $x < y$ then $f(x) < f(y)$). Show that E is countable if and only if the image set $f(E)$ is countable. (What property other than "increasing" would work here?)

4:6.9 Show that every uncountable set of real numbers has a point of accumulation.

4:6.10 Let $\mathcal{F}$ be a family of (nondegenerate) intervals, i.e., each member of $\mathcal{F}$ is an interval (open, closed or neither) but is not a single point. Suppose that any two intervals I and J in the family have no point in common. Show that the family $\mathcal{F}$ can be arranged in a sequence $I_1, I_2, \ldots$.

4.7 Additional Problems for Chapter 4

4:7.1 Introduce for any set $E \subset \mathbb{R}$ the notation

$$\partial E = \{x : \ x \text{ is a boundary point of } E\}.$$

(a) Show for any set E that $\partial E = \overline{E} \cap \overline{(\mathbb{R} \setminus E)}$.

(b) Show that for any set E the set ∂E is closed.

(c) For what sets E is it true that $\partial E = \emptyset$?

(d) Show that $\partial E \subset E$ for any closed set E.

(e) If E is closed show that $\partial E = E$ if and only if E has no interior points.

(f) If E is open show that ∂E can contain no interval.

4:7.2 Let E be a nonempty set of real numbers and define the function

$$f(x) = \inf\{|x - e| : e \in E\}.$$

(a) Show that $f(x) = 0$ for all $x \in E$.

(b) Show that $f(x) = 0$ if and only if $x \in \overline{E}$.

(c) Show for any nonempty closed set E that

$$\{x \in \mathbb{R} : f(x) > 0\} = (\mathbb{R} \setminus E).$$

4:7.3 Let $f : \mathbb{R} \to \mathbb{R}$ have this property: for every $x_0 \in \mathbb{R}$ there is a $\delta > 0$ so that

$$|f(x) - f(x_0)| < |x - x_0|$$

whenever $0 < |x - x_0| < \delta$. Show that for all $x, y \in \mathbb{R}$, $x \neq y$,

$$|f(x) - f(y)| < |x - y|.$$

4:7.4 Let $f : E \to \mathbb{R}$ have this property: for every $e \in E$ there is an $\varepsilon > 0$ so that

$$f(x) > \varepsilon \text{ if } x \in E \cap (e - \varepsilon, e + \varepsilon).$$

Show that if the set E is compact then there is some positive number c so that

$$f(e) > c$$

for all $e \in E$. Show that if E is not closed or is not bounded then this conclusion may not be valid.

4:7.5 (Separation of Compact Sets) Let A and B be nonempty sets of real numbers and let

$$\delta(A, B) = \inf\{|a - b| : a \in A, b \in B\}.$$

$\delta(A, B)$ is often called the "distance" between the sets A and B.

(a) Prove $\delta(A, B) = 0$ if $A \cap B \neq \emptyset$.

(b) Give an example of two closed, disjoint sets in $\mathbb{R}$ for which $\delta(A, B) = 0$.

(c) Prove that if A is compact, B is closed and $A \cap B = \emptyset$, then $\delta(A, B) > 0$.

4:7.6 Show that every closed set can be expressed as the intersection of a sequence of open sets.

4:7.7 Show that every open set can be expressed as the union of a sequence of closed sets.

4:7.8 A collection of sets $\{S_\alpha : \alpha \in A\}$ is said to have the *finite intersection property* if every finite subfamily has a nonempty intersection.

(a) Show that if $\{S_\alpha : \alpha \in A\}$ is a family of compact sets that has the finite intersection property then

$$\bigcap_{\alpha \in A} S_\alpha \neq \emptyset.$$

(b) Give an example of a collection of closed sets $\{S_\alpha : \alpha \in A\}$ that has the finite intersection property and yet

$$\bigcap_{\alpha \in A} S_\alpha = \emptyset.$$

4:7.9 A set $S \subset \mathbb{R}$ is said to be *disconnected* if there exist two disjoint open sets U and V each containing a point of S so that $S \subset U \cup V$. A set that is not disconnected is said to be *connected.*

(a) Give an example of a disconnected set.

(b) Show that every compact interval $[a, b]$ is connected.

(c) Show that $\mathbb{R}$ is connected.

(d) Show that every nonempty connected set is an interval.

4:7.10 Show that the only subsets of $\mathbb{R}$ that are both open and closed are $\emptyset$ and $\mathbb{R}$.

4:7.11 Given any uncountable set of real numbers E show that it is possible to extract a sequence $\{a_k\}$ of distinct terms of E so that the series $\sum_{k=1}^{\infty} a_k/k$ diverges.