Chapter 2

SEQUENCES

2.1 Introduction

Let us start our discussion with a method for solving equations that originated with Newton in 1669. To solve an equation $f(x) = 0$ the method proposes the introduction of a new function

$$F(x) = x - \frac{f(x)}{f'(x)}.$$

One then begins with a guess at a solution of $f(x) = 0$, say x_1 and computes $x_2 = F(x_1)$ in the hopes that x_2 is closer to a solution than x_1 was. The process is repeated so that $x_3 = F(x_2)$, $x_4 = F(x_3)$, $x_5 = F(x_4), \ldots$ and so on until the desired accuracy is reached. Processes of this type have been known for at least 3500 years although not in such a modern notation.

We illustrate by finding an approximate value for $\sqrt{2}$ this way. We solve the equation $f(x) = x^2 - 2 = 0$ by computing the function

$$F(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 2}{2x}$$

and using it to improve our guess. A first (very crude) guess of $x_1 = 1$ will produce the following list of values for our subsequent steps in the procedure. We have retained 60 digits in the decimal expansions to show how this is working:

$$
\begin{aligned}
x_1 &= 1.00000000000000000000000000000000000000000000000000000000000\\
x_2 &= 1.50000000000000000000000000000000000000000000000000000000000\\
x_3 &= 1.41666666666666666666666666666666666666666666666666666666667\\
x_4 &= 1.41421568627450980392156862745098039215686274509803921568628\\
x_5 &= 1.41421356237468991062629557889013491011655962211574404458490\\
x_6 &= 1.41421356237309504880168962350253024361498192577619742849829\\
x_7 &= 1.41421356237309504880168872420969807856967187537723400156101
\end{aligned}
$$

To compare, here is the value of the true solution $\sqrt{2}$, computed in a different fashion to the same number of digits:

$$\sqrt{2} = 1.41421356237309504880168872420969807856967187537694807317668$$

Note that after only four steps the procedure gives a value differing from the true value only in the sixth decimal place, and all subsequent values remain this close. A convenient way of expressing this is to write that

$$|x_n - \sqrt{2}| < 10^{-5} \text{ for all } n \geq 4.$$

By the seventh step, things are going even better and we can claim that

$$|x_n - \sqrt{2}| < 10^{-47} \text{ for all } n \geq 7.$$

It is inconceivable that anyone would require any further accuracy for any practical considerations. The error after the sixth step cannot exceed 10^{-47} which is a very tiny number. Even so, as mathematicians we can ask what may seem an entirely impractical sort of question. Can this accuracy of approximation continue forever? Is it possible that, if we wait long enough, we can find an approximation to $\sqrt{2}$ with *any* degree of accuracy?

Expressed more formally, if we are given a positive number ε (we call it epsilon to suggest that it measures an *error*) no matter how small, can we find a stage in this procedure so that that value computed and all subsequent values are closer to $\sqrt{2}$ than ε? In symbols, is there an integer n_0 (which will depend on just how small ε is) that is large enough so that

$$|x_n - \sqrt{2}| < \varepsilon \text{ for all } n \geq n_0?$$

If this is true then this sequence has a remarkable property. It is not merely in its first few terms a convenient way of computing

$\sqrt{2}$ to some accuracy; the sequence truly represents the number $\sqrt{2}$ itself, and it cannot represent any other number. We shall say that the sequence converges to $\sqrt{2}$ and write

$$\lim_{n\to\infty} x_n = \sqrt{2}.$$

This is the beginning of the theory of convergence that is central to analysis. If mathematicians had never considered the ultimate behavior of such sequences and had contented themselves with using only the first few terms for practical computations, there would have been no subject known as analysis. These ideas lead, as you might imagine, to an ideal world of infinite precision, where sequences are not merely useful gadgets for getting good computations but are precise tools in discussing real numbers. From the theory of sequences and their convergence properties has developed a vast world of beautiful and useful mathematics.

For the student approaching this material for the first time this is a critical test. All of analysis, both pure and applied, rests on an understanding of limits. What you learn in this chapter will offer a foundation for all the rest that you will have to learn later on.

2.2 Sequences

A sequence (of real numbers, of sets, of functions, of anything) is simply a list. There is a first element in the list, a second element, a third element and so on continuing in an order forever. In mathematics a finite list is not called a sequence; a sequence must continue without interruption.

For a more formal definition notice that the natural numbers are playing a key role here. Every item in the sequence (the list) can be labelled by its position; label the first item with a "1", the second with a "2", and so on. Seen this way a sequence is merely then a function mapping the natural numbers $\mathbb{N}$ into some set. We state this as a definition. Since this chapter is exclusively about sequences of real numbers the definition considers just this situation.

Definition 2.1 By a sequence of real numbers we mean a function $f : \mathbb{N} \to \mathbb{R}$.

Thus the sequence *is* the function. Even so we usually return to the list idea and write out the sequence f as

$$f(1), f(2), f(3), \ldots, f(n), \ldots$$

with the ellipsis (i.e., the three dots) indicating that the list is to continue in this fashion. The function values $f(1)$, $f(2)$, $f(3)$, ... are called the *terms* of the sequence. If we need to return to the formality of functions we do, but try to keep the intuitive notion of a sequence as an unending list in mind. While computer scientists much prefer the function notation, mathematicians have become more accustomed to a subscript notation and would rather have the terms of the sequence above rendered as

$$f_1, f_2, f_3, \ldots, f_n, \ldots.$$

In this chapter we study sequences of real numbers. Later on we will encounter the same word applied to other lists of objects, e.g., sequences of intervals, sequences of sets, sequences of functions. In all cases the word sequence simply indicates a list of objects.

2.2.1 Sequence Examples

In order to specify some sequence we need to communicate what every term in the sequence is. For example the sequence of even integers

$$2, 4, 6, 8, 10, \ldots$$

could be communicated in precisely that way: "consider the sequence of even integers". Perhaps more direct would be to give a formula for all of the terms in the sequence: "consider the sequence whose nth term is $x_n = 2n$". Or we could note that the sequence starts with 2 and then all the rest of the terms are obtained by adding 2 to the previous term: "consider the sequence whose first term is 2 and whose nth term is 2 added to the $(n-1)$st term", i.e.,

$$x_n = 2 + x_{n-1}.$$

Often an explicit formula is best. Frequently though, a formula relating the nth term to some preceding term is preferable. Such formulas are called *recursion formulas* and would usually be more efficient if a computer is used to generate the terms.

Arithmetic progressions The simplest types of sequences are those in which each term is obtained from the preceding by adding a fixed amount. These are called *arithmetic progressions.* The sequence

$$c, c+d, c+2d, c+3d, c+4d, \ldots, c+(n-1)d, \ldots$$

is the most general arithmetic progression. The number d is called the *common difference.*

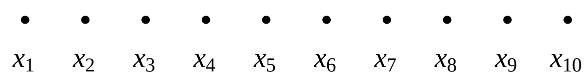

Figure 2.1: An arithmetic progression.

Every arithmetic progression could be given by a formula

$$x_n = c + (n-1)d$$

or a recursion formula

$$x_1 = c \qquad x_n = x_{n-1} + d.$$

Note that the explicit formula is of the form $x_n = f(n)$ where f is a linear function, $f(x) = dx + b$ for some b. If you plot the points (n, x_n) of an arithmetic progression you will find that they all lie on a straight line with slope d. (See Figure 2.1.)

Geometric progressions. A variant on the arithmetic progression is obtained by replacing the addition of a fixed amount by the multiplication by a fixed amount. These sequences are called *geometric progressions*. The sequence

$$c, cr, cr^2, cr^3, cr^4, \dots, cr^n, \dots$$

is the most general geometric progression. The number r is called the *common ratio*.

Every geometric progression could be given by a formula

$$x_n = cr^{n-1}$$

or a recursion formula

$$x_1 = c \qquad x_n = rx_{n-1}.$$

Note that the explicit formula is of the form $x_n = f(n)$ where f is an exponential function $f(x) = br^x$ for some b. If you plot the points (n, x_n) of a geometric progression you will find that they all lie on the graph of an exponential function. If $c > 0$ and the common ratio r is larger than 1 the terms increase in size becoming extremely large. If $0 < r < 1$ the terms decrease in size getting smaller and smaller. (See Figure 2.2.)

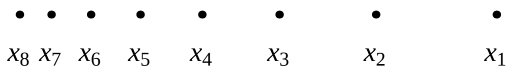

Figure 2.2: A geometric progression.

Iteration The examples of an arithmetic progression and a geometric progression are special cases of a process called *iteration*. So too is the sequence generated by Newton's method in the introduction to this chapter.

Let f be some function. Start the sequence $\{x_n\}$ by assigning some value in the domain of f, say $x_1 = c$. All subsequent values are now obtained by feeding these values through the function repeatedly:

$$c, f(c), f(f(c)), f(f(f(c))), f(f(f(f(c)))), \ldots .$$

As long as all these values remain in the domain of the function f the process can continue indefinitely and defines a sequence. If f is a linear function then the result is an arithmetic progression. If f is an exponential function then the result is a geometric progression.

A recursion formula best expresses this process and would offer the best way of writing a computer program to compute the sequence:

$$x_1 = c \qquad x_n = f(x_{n-1}).$$

Sequence of partial sums. If a sequence

$$x_1, x_2, x_3, x_4, \ldots$$

is given one can construct a new sequence by adding the terms of the old one:

$$\begin{aligned} s_1 &= x_1 \\ s_2 &= x_1 + x_2 \\ s_3 &= x_1 + x_2 + x_3 \\ s_4 &= x_1 + x_2 + x_3 + x_4 \end{aligned}$$

and continuing in this way. The process can also be described by a recursion formula:

$$s_1 = x_1 \qquad s_n = s_{n-1} + x_n.$$

The new sequence is called the *sequence of partial sums* of the old sequence $\{x_n\}$. We shall study such sequences in considerable depth in the next chapter.

For a particular example we could use $x_n = 1/n$ and the sequence of partial sums could be written as

$$s_n = 1 + 1/2 + 1/3 + \cdots + 1/n.$$

Is there a more attractive and simpler formula for s_n? The answer is no.

Example 2.2 These examples, given so far, are of a general nature and describe many sequences that we will encounter in analysis. But a sequence is just a list of numbers and need not be defined in any manner quite so systematic. For example consider the sequence defined by $a_n = 1$ if n is divisible by three, $a_n = n$ if n is one more than a multiple of three and $a_n = -2^n$ if n is two more than a multiple of three. The first few terms are evidently

$$1, 2, -8, 1, 5, -64, \ldots.$$

What would be the next three terms? ◀

Exercises

2:2.1 Let a sequence be defined by the phrase "consider the sequence of prime numbers $2, 3, 5, 7, 11, 13 \ldots$". Are you sure that this defines a sequence?

2:2.2 On I.Q. tests one frequently encounters statements such as "what is the next term in the sequence 3, 1, 4, 1, 5, ... ?". In terms of our definition of a sequence is this correct usage? (By the way what do you suppose the next term in the sequence might be?)

2:2.3 Give two different formulas (for two different sequences) that generate a sequence whose first four terms are 2, 4, 6, 8.

2:2.4 Give a formula that generates a sequence whose first five terms are 2, 4, 6, 8, π.

2:2.5 The examples listed here are the first few terms of a sequence that is either an arithmetic progression or a geometric progression. What is the next term in the sequence? Give a general formula for the sequence.

(a) 7, 4, 1, ...

(b) .1, .01, .001, ...

(c) 2, $\sqrt{2}$, 1, ...

2:2.6 Consider the sequence defined recursively by

$$x_1 = \sqrt{2} \quad x_n = \sqrt{2} + x_{n-1}.$$

Find an explicit formula for the nth term.

2:2.7 Consider the sequence defined recursively by

$$x_1 = \sqrt{2} \quad x_n = \sqrt{2}x_{n-1}.$$

Find an explicit formula for the nth term.

2:2.8 Consider the sequence defined recursively by

$$x_1 = \sqrt{2} \quad x_n = \sqrt{2 + x_{n-1}}.$$

Show, by induction, that $x_n < 2$ for all n.

2:2.9 Consider the sequence defined recursively by

$$x_1 = \sqrt{2} \quad x_n = \sqrt{2 + x_{n-1}}.$$

Show, by induction, that $x_n < x_{n+1}$ for all n.

2:2.10 The sequence defined recursively by ✂

$$f_1 = 1,\ f_2 = 1 \quad f_{n+2} = f_n + f_{n+1}$$

is called the *Fibonacci sequence*. It is possible to find an explicit formula for this sequence. Give it a try.

2.3 Countable Sets ✂

A sequence of real numbers, formally, is a function whose domain is the set $\mathbb{N}$ of natural numbers and whose range is a subset of the reals $\mathbb{R}$. What sets might be the range of some sequence? To put it another way, what sets can have their elements arranged into an unending list? Are there sets that cannot be arranged into a list?

The arrangement of a collection of objects into a list is sometimes called an *enumeration*. Thus another way of phrasing this question is to ask what sets of real numbers can be *enumerated*?

The set of natural numbers is already arranged into a list in its natural order. The set of integers (including 0 and the negative integers) is not usually presented in the form of a list but can easily be so presented as the following scheme suggests:

$$0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, 6, -6, 7, -7, \ldots.$$

Example 2.3 The rational numbers can also be listed but this is quite remarkable, for at first sight no reasonable way of ordering them into a sequence seems likely to be possible. The usual order of the rationals in the reals is of little help.

To find such a scheme define the "rank" of a rational number m/n in its lowest terms (with $n \geq 1$) to be $|m| + n$. Now begin making a finite list of all the rational numbers at a various rank; list these from smallest to largest. At rank 1 we would have only the rational number $0/1$. At rank 2 we would have only the rational numbers $-1/1$, $1/1$. At rank 3 we would have only the rational numbers $-2/1$, $-1/2$, $1/2$, $2/1$. Carry on in this fashion through all the ranks. Now construct the final list by concatenating these shorter lists in order of the ranks:

$$0/1, -1/1, 1/1, -2/1, -1/2, 1/2, 2/1, \ldots .$$

The range of this sequence is the set of all rational numbers. ◀

One's first impression might be that very few sets would be able to be the range of a sequence. But having seen in Example 2.3 that even the set of rational numbers $\mathbb{Q}$ that is seemingly so large can be listed it might then appear that all sets can be so listed. After all can you conceive of a set that is "larger" than the rationals in some way that would stop it being listed? The remarkable fact that there are sets which cannot be arranged to form the elements of some sequence was proved by G. Cantor (1845–1918) This proof is essentially his original proof. (Note that this requires some familiarity with infinite decimal expansions.)

Theorem 2.4 (Cantor) *No interval (a, b) of real numbers can be the range of some sequence.*

Proof. It is enough to prove this for the interval $(0, 1)$ since there is nothing special about it (see Exercise 2:3.1). The proof is a proof by contradiction. We suppose that the theorem is false and that there is a sequence $\{s_n\}$ so that every number in the interval $(0, 1)$ appears at least once in the sequence. We obtain a contradiction by showing that this cannot be so. We shall use the sequence $\{s_n\}$ to find a number c in the interval $(0, 1)$ so that $s_n \neq c$ for all n.

Each of the points s_1, s_2, s_3 ... in our sequence is a number between 0 and 1 and so can be written as a decimal fraction. If we write this sequence out in decimal notation it might look like

$$s_1 = 0.x_{11}x_{12}x_{13}x_{14}x_{15}x_{16}\ldots$$

$$s_2 = 0.x_{21}x_{22}x_{23}x_{24}x_{25}x_{26}\ldots$$

$$s_3 = 0.x_{31}x_{32}x_{33}x_{34}x_{35}x_{36}\ldots$$

etc. Now it is easy to find a number that is not in the list. Construct

$$c = 0.c_1c_2c_3c_4c_5c_6\ldots$$

by choosing c_i to be either 5 or 6 whichever is different from x_{ii}. This number cannot be equal to any of the listed numbers s_1, s_2, s_3 ... since c and s_i differ in the ith position of their decimal expansions. This gives us our contradiction and so proves the theorem. ■

Definition 2.5 (Countable) A set S of real numbers is said to be countable if there is a sequence of real numbers whose range contains the set S.

In the language of this definition then we can see that (i) any finite set is countable (ii) the natural numbers and the integers are countable (iii) the rational numbers are countable (iv) no interval of real numbers is countable.

Exercises

2:3.1 Show that, once it is known that the interval $(0, 1)$ cannot be expressed as the range of some sequence, it follows that any interval (a, b), $[a, b)$, $(a, b]$, or $[a, b]$ has the same property.

2:3.2 Some novices, on reading the proof of Cantor's theorem say "Why can't you just put the number c that you found at the front of the list." What is your rejoinder?

2:3.3 A set (any set of objects) is said to be countable if it is either finite or there is an enumeration (list) of the set. Show that the following properties hold for arbitrary countable sets:

(a) All subsets of countable sets are countable.

(b) Any union of a pair of countable sets is countable.

(c) All finite sets are countable.

2:3.4 Show that the following property holds for countable sets: if S_1, S_2, ... is a sequence of countable sets of real numbers then the set S formed by taking all elements that belong to at least one of the sets S_i is also a countable set.

2:3.5 Show that if a nonempty set is contained in the range of some sequence of real numbers then there is a sequence whose range is precisely that set.

2:3.6 In Cantor's proof presented in this section we took for granted material about infinite decimal expansions. This is entirely justified by the theory of sequences studied later on. Explain what it is that we need to prove about infinite decimal expansions to be sure that this proof is valid.

2:3.7 Define a relation on the family of subsets of $\mathbb{R}$ as follows. Say that $A \sim B$ where A and B are subsets of $\mathbb{R}$ if there is a function $f : A \to B$ that is one-one and onto. (If $A \sim B$ we would say that A and B are "cardinally equivalent".) Show that this is an *equivalence relation*, i.e. show that

(a) $A \sim A$ for any set A.

(b) If $A \sim B$ then $B \sim A$.

(c) If $A \sim B$ and $B \sim C$ then $A \sim C$.

.

2:3.8 Let A and B be finite sets. Under what conditions are these sets cardinally equivalent (in the language of Exercise 2:3.7)?

2:3.9 Show that an infinite set of real numbers that is countable is cardinally equivalent (in the language of Exercise 2:3.7) to the set $\mathbb{N}$. Give an example of an infinite set that is not cardinally equivalent to $\mathbb{N}$.

2:3.10 We define a real number to be *algebraic* if it is a solution of some polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where all the coefficients are integers. Thus $\sqrt{2}$ is algebraic because it is a solution of $x^2 - 2 = 0$. The number π is not algebraic because no such polynomial equation can ever be found (although this is hard to prove). Show that the set of algebraic numbers is countable.

2.4 Convergence

The sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$$

is getting closer and closer to the number 0. We say that this sequence *converges* to 0 or that the limit of the sequence is the number 0. How should this idea be properly defined?

The study of convergent sequences was undertaken and developed in the eighteenth century without any precise definition. The closest one might find to a definition in the early literature would have been something like

A sequence $\{s_n\}$ converges to a number L if the terms of the sequence get closer and closer to L.

Apart from being too vague to be used as anything but a rough guide for the intuition, this is misleading in other respects. What about the sequence

$$.1, .01, .02, .001, .002, .0001, .0002, .00001, .00002, \ldots?$$

Surely this should converge to 0 but the terms do not get steadily "closer and closer" but back off a bit at each second step. Also the sequence

$$.1, .11, .111, .1111, .11111, .111111, \ldots$$

is getting "closer and closer" to .2 but we would not say the sequence converges to .2. A smaller number (1/9 which it is also getting closer and closer to) is the correct limit. We want not merely "closer and closer" but somehow a notion of "arbitrarily close".

The definition which captured the idea in the best way was given by Cauchy in the 1820's. He found a formulation that expressed the idea of "arbitrarily close" using inequalities. In this way the notion of limit, involving apparently infinite ideas, is reduced to a straightforward mathematical statement about inequalities.

Definition 2.6 (Limit of a Sequence) Let $\{s_n\}$ be a sequence of real numbers. We say that $\{s_n\}$ *converges* to a number L and write

$$\lim_{n\to\infty} s_n = L$$

or

$$s_n \to L \text{ as } n \to \infty$$

provided that for every number $\varepsilon > 0$ there is an integer N so that

$$|s_n - L| < \varepsilon$$

whenever $n \geq N$.

A sequence that converges is said to be *convergent.* A sequence that fails to converge is said to *diverge.* We are equally interested in both convergent and divergent sequences.

Note. In the definition the N depends on ε. If ε is particularly small then N might have to be chosen very large. In fact then N is really a function of ε. Sometimes it is best to emphasize this and write $N(\varepsilon)$ rather than N.

Note, too, that if an N is found, then any larger N would also be able to be used. Thus the definition requires us to find some N but not necessarily the smallest N that would work.

While the definition does not say this, the real force of the definition is that the N can be determined *no matter how small a number ε is chosen.* If ε is given as rather large there may be no trouble finding the N value. If you find an N that works for $\varepsilon = .1$ that same N would work for all larger values of ε.

Example 2.7 Let us use the definition to prove that

$$\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2}.$$

It is by no means clear from the definition how to obtain the number $L = \frac{1}{2}$. Indeed the definition is not intended as a method of finding limits. It assigns a precise meaning to the statement about the limit but offers no way of computing that limit. Fortunately most of us remember some calculus devices that can be used to first obtain the limit before attempting a proof of its validity.

$$\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \lim_{n\to\infty} \frac{1}{2+1/n^2} = \frac{1}{\lim_{n\to\infty}(2+1/n^2)} =$$
$$\frac{1}{2+\lim_{n\to\infty}(1/n^2)} = \frac{1}{2}.$$

Indeed this would be a proof that the limit is $1/2$ provided that we can prove the validity of each of these steps. Later on we will prove this and so can avoid the ε, N arguments that we now use.

Let any positive ε be given. We need to find a number N (or $N(\varepsilon)$ if you prefer) so that every term in the sequence on and after the Nth term is closer to $1/2$ than ε, i.e., so that

$$\left| \frac{n^2}{2n^2+1} - \frac{1}{2} \right| < \varepsilon$$

for $n = N$, $n = N+1$, $n = N+2$, $\ldots$. It is easiest to work backwards and discover just how large n should be for this. A little work shows that this will happen if

$$\frac{1}{2(2n^2+1)} < \varepsilon$$

or

$$4n^2 + 2 > \frac{1}{\varepsilon}.$$

The smallest n for which this statement is true could be our N. Thus we could use any integer N with

$$N^2 > \frac{1}{4}\left(\frac{1}{\varepsilon} - 2\right).$$

There is no obligation to find the smallest N that works and so, perhaps, the most convenient one here might be a bit larger, say take any integer N larger than

$$N > \frac{1}{2\sqrt{\varepsilon}}.$$

◀

The real lesson of the example, perhaps, is that one wishes never to have to use the definition to check any limit computation. The definition offers a rigorous way to develop a theory of limits but a very impractical method of computation of limits and a clumsy method of verification. Only very rarely does one have to do a computation of this sort to verify a limit.

Uniqueness of Sequence Limits Let us take the first step in developing a theory of limits. This is to ensure that our definition has defined *limit* unambiguously. Is it possible that the definition allows for a sequence to converge to two different limits? If we have established that $s_n \to L$ is it possible that $s_n \to L_1$ for a different number L_1?

Theorem 2.8 (Uniqueness of Limits) *Suppose that*

$$\lim_{n\to\infty} s_n = L_1 \textit{ and } lim_{n\to\infty} s_n = L_2$$

are both true. Then $L_1 = L_2$.

Proof. Let ε be any positive number. Then, by definition, we must be able to find a number N_1 so that

$$|s_n - L_1| < \varepsilon$$

whenever $n \geq N_1$. We must also be able to find a number N_2 so that

$$|s_n - L_2| < \varepsilon$$

whenever $n \geq N_2$. Take m to be the maximum of N_1 and N_2. Then both assertions

$$|s_m - L_1| < \varepsilon$$

and

$$|s_m - L_2| < \varepsilon$$

are true.

This allows us to conclude that

$$|L_1 - L_2| \leq |L_1 - s_m| + |s_m - L_2| < 2\varepsilon$$

so that

$$|L_1 - L_2| < 2\varepsilon.$$

But ε can be any positive number whatsoever. This could only be true if $L_1 = L_2$ which is what we wished to show. ■

Exercises

2:4.1 Give a precise ε, N argument to prove that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

2:4.2 Give a precise ε, N argument to prove that

$$\lim_{n\to\infty} \frac{2n+3}{3n+4}$$

exists.

2:4.3 Show that a sequence $\{s_n\}$ converges to a limit L if and only if the sequence $\{s_n - L\}$ converges to zero.

2:4.4 Show that a sequence $\{s_n\}$ converges to a limit L if and only if the sequence $\{-s_n\}$ converges to $-L$.

2:4.5 Show that Definition 2.6 is equivalent to the following slight modification:

> We write $\lim_{n\to\infty} s_n = L$ provided that for every positive integer m there is a real number N so that $|s_n - L| < 1/m$ whenever $n \geq N$.

2:4.6 Compute the limit

$$\lim_{n\to\infty} \frac{1+2+3+\cdots+n}{n^2}$$

and verify it by the definition.

2:4.7 Compute the limit

$$\lim_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}.$$

2:4.8 Suppose that $\{s_n\}$ is a convergent sequence. Prove that $\lim_{n\to\infty} 2s_n$ exists.

2:4.9 Prove that $\lim_{n\to\infty} n$ does not exist.

2:4.10 Prove that $\lim_{n\to\infty}(-1)^n$ does not exist.

2:4.11 The sequence $s_n = (-1)^n$ does not converge. For what values of $\varepsilon > 0$ is it nonetheless true that there is an integer N so that

$$|s_n - 1| < \varepsilon$$

whenever $n \geq N$. For what values of $\varepsilon > 0$ is it nonetheless true that there is an integer N so that

$$|s_n - 0| < \varepsilon$$

whenever $n \geq N$.

2:4.12 Let $\{s_n\}$ be a sequence that assumes only integer values. Under what conditions can such a sequence converge?

2:4.13 Let $\{s_n\}$ be a sequence and obtain a new sequence (sometimes called the "tail" of the sequence) by writing

$$t_n = s_{M+n} \quad \text{for } n = 1, 2, 3, \dots$$

where M is some integer (perhaps large). Show that $\{s_n\}$ converges if and only if $\{t_n\}$ converges.

2:4.14 Show that the the statement "$\{s_n\}$ converges to L" is false if and only if there is a positive number c so that the inequality

$$|s_n - L| > c$$

holds for infinitely many values of n.

2:4.15 If $\{s_n\}$ be a sequence of positive numbers converging to 0 show that $\{\sqrt{s_n}\}$ also converges to zero.

2:4.16 If $\{s_n\}$ be a sequence of positive numbers converging to a positive number L show that $\{\sqrt{s_n}\}$ converges to $\sqrt{L}$.

2.5 Divergence

A sequence that fails to converge is said to *diverge*. Some sequences diverge in a particularly interesting way and it is worthwhile having a language for this.

The sequence $s_n = n^2$ diverges because the terms get larger and larger. One is tempted to write

$$n^2 \to \infty \quad \text{or} \quad \lim_{n\to\infty} n^2 = \infty.$$

This conflicts with our definition of limit and so needs its own definition. We do not say that this sequence "converges to ∞" but rather that it "diverges to ∞".

Definition 2.9 (Divergence to ∞) Let $\{s_n\}$ be a sequence of real numbers. We say that $\{s_n\}$ diverges to ∞ and write

$$\lim_{n\to\infty} s_n = \infty$$

or

$$s_n \to \infty \text{ as } n \to \infty$$

provided that for every number M there is an integer N so that

$$s_n > M$$

whenever $n \geq N$.

Note. The definition does not announce this, but the force of the definition is that the choice of N is possible *no matter how large M is chosen.* There may be no difficulty in finding an N if the M given is not very big.

Example 2.10 Let us prove that

$$\frac{n^2+1}{n+1} \to \infty$$

using the definition. If M is any positive number we need to find some point in the sequence after which all terms exceed M. Thus we need to consider the inequality

$$\frac{n^2+1}{n+1} \geq M.$$

After some arithmetic we see that this is equivalent to

$$n + \frac{1}{n+1} - \frac{n}{n+1} \geq M.$$

Since

$$\frac{n}{n+1} < 1$$

we see that, as long as $n \geq M+1$ this will be true. Thus take any integer $N \geq M+1$ and it will be true that

$$\frac{n^2+1}{n+1} \geq M.$$

for all $n \geq N$. (Any larger values of N would work too.) ◀

Exercises

2:5.1 Formulate the definition of a sequence diverging to $-\infty$.

2:5.2 Show, using the definition, that

$$\lim_{n\to\infty} n^2 = \infty.$$

2:5.3 Show, using the definition, that

$$\lim_{n\to\infty} \frac{n^3+1}{n^2+1} = \infty.$$

2:5.4 Prove that if $s_n \to \infty$ then $-s_n \to -\infty$.

2:5.5 Prove that if $s_n \to \infty$ then $(s_n)^2 \to \infty$ also.

2:5.6 Prove that if $x_n \to \infty$ then the sequence

$$\frac{x_n}{x_n+1}$$

is convergent. Is the converse true?

2:5.7 Suppose that a sequence $\{s_n\}$ of positive numbers satisfies $\lim_{n\to\infty} s_n = 0$. Show that $\lim_{n\to\infty} 1/s_n = \infty$. Is the converse true?

2:5.8 Suppose that a sequence $\{s_n\}$ of positive numbers satisfies the condition
$$s_{n+1} > \alpha s_n$$
for all n where $\alpha > 1$. Show that $s_n \to \infty$.

2:5.9 The sequence $s_n = (-1)^n$ does not diverge to ∞. For what values of M is it nonetheless true that there is an integer N so that
$$s_n > M$$
whenever $n \geq N$.

2:5.10 Show that the sequence
$$n^p + \alpha_1 n^{p-1} + \alpha_2 n^{p-2} + \ldots \alpha_p$$
diverges to ∞ where here p is a positive integer and α_1, $\alpha_2, \ldots, \alpha_p$ are real numbers (positive or negative).

2.6 Boundedness Properties of Limits

A sequence is said to be *bounded* if its range is a bounded set. Thus a sequence $\{s_n\}$ is bounded if there is a number M so that every term in the sequence satisfies
$$|s_n| \leq M.$$
For such a sequence, every term belongs to the interval $[-M, M]$.

It is fairly evident that a sequence that is not bounded could not converge. This is important enough to state and prove as a theorem.

Theorem 2.11 *Every convergent sequence is bounded.*

Proof. Suppose that $s_n \to L$. Then for every number $\varepsilon > 0$ there is an integer N so that
$$|s_n - L| < \varepsilon$$
whenever $n \geq N$. In particular we could take just one value of ε, say $\varepsilon = 1$, and find a number N so that
$$|s_n - L| < 1$$
whenever $n \geq N$. From this we see that
$$|s_n| = |s_n - L + L| \leq |s_n - L| + |L| < |L| + 1$$
for all $n \geq N$. This number $|L| + 1$ would be an upper bound for all the numbers $|s_n|$ except that we have no indication of the values for $|s_1|$, $|s_2|$, ... , $|s_{N-1}|$.

Thus if we write
$$M = \max\{|s_1|, |s_2|, \ldots, |s_{N-1}|, |L| + 1\}$$

we must have

$$|s_n| \leq M$$

for *every* value of n. This is our upper bound and the theorem is proved. ■

As a consequence of this theorem we can conclude that an unbounded sequence must diverge. Thus, even though it is a rather crude test, we can prove the divergence of a sequence if we are able somehow to show that it is unbounded. The next example illustrates this technique.

Example 2.12 We shall show that the sequence

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{n}$$

diverges. The easiest proof of this is to show that it is unbounded and hence, by Theorem 2.11, could not converge.

We watch only at the steps 1, 2, 4, 8, ... and make a rough lower estimate of s_1, s_2, s_4, s_8, ... in order to show that there can be no bound on the sequence. After a bit of arithmetic we see that

$$s_1 = 1$$

$$s_2 = 1 + \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right)$$

$$\begin{aligned} s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ &\geq 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) \end{aligned}$$

and, in general, that

$$s_{2^n} \geq 1 + n/2$$

for all $n = 0, 1, 2, \ldots$. Thus the sequence is not bounded and so must diverge. ◀

Example 2.13 As a variant of the sequence of the preceding example consider the sequence

$$t_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{n^p}$$

where p is any positive real number. The case $p = 1$ we have just found diverges.

For $p < 1$ the sequence is larger than it is for $p = 1$ and so the case is even stronger for divergence. For $p > 1$ the sequence is smaller and we cannot see immediately whether it is bounded or unbounded; in fact, with some effort we can show that such a sequence is bounded. What can we conclude? Nothing yet. An unbounded sequence diverges. A bounded sequence may converge or diverge. ◀

Exercises

2:6.1 Which statements are true?

(a) If $\{s_n\}$ is unbounded then either $\lim_{n\to\infty} s_n = \infty$ or else $\lim_{n\to\infty} s_n = -\infty$.

(b) If $\{s_n\}$ is unbounded then $\lim_{n\to\infty} |s_n| = \infty$

(c) If $\{s_n\}$ and $\{t_n\}$ are both bounded then so is $\{s_n + t_n\}$.

(d) If $\{s_n\}$ and $\{t_n\}$ are both unbounded then so is $\{s_n + t_n\}$.

(e) If $\{s_n\}$ and $\{t_n\}$ are both bounded then so is $\{s_n t_n\}$.

(f) If $\{s_n\}$ and $\{t_n\}$ are both unbounded then so is $\{s_n t_n\}$.

(g) If $\{s_n\}$ is bounded then so is $\{1/s_n\}$.

(h) If $\{s_n\}$ is unbounded then $\{1/s_n\}$ is bounded.

2:6.2 If $\{s_n\}$ is bounded prove that $\{s_n/n\}$ is convergent.

2:6.3 State the converse of Theorem 2.11. Is it true?

2:6.4 State the contrapositive of Theorem 2.11. Is it true?

2:6.5 Suppose that $\{s_n\}$ is a sequence of positive numbers converging to a positive limit. Show that there is a positive number c so that $s_n > c$ for all n.

2:6.6 As a computer experiment compute the values of the sequence

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

for large values of n. Is there any indication in the numbers that you see that this sequence fails to converge or must be unbounded?

2.7 Algebra of Limits

Sequences can be combined by the usual arithmetic operations (addition, subtraction, multiplication and division). Indeed most sequences we are likely to encounter can be seen to be composed of simpler sequences combined together in this way.

In Example 2.7 we suggested that the computations

$$\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \lim_{n\to\infty} \frac{1}{2+1/n^2} = \frac{1}{\lim_{n\to\infty}(2+1/n^2)} =$$
$$\frac{1}{2+\lim_{n\to\infty} 1/n^2} = \frac{1}{2}$$

could be justified. Note how this sequence has been obtained from simpler ones by ordinary processes of arithmetic. To justify such a method we need to investigate how the limit operation is influenced by algebraic operations.

Suppose that

$$s_n \to S \quad \text{and} \quad t_n \to T.$$

Then one would expect

$$Cs_n \to CS$$
$$s_n + t_n \to S + T$$
$$s_n - t_n \to S - T$$
$$s_n t_n \to ST$$

and

$$s_n/t_n \to S/T.$$

Each of these statements must be justified, however, solely on the basis of the definition of convergence, not on intuitive feelings that this should be the case. Thus we need to develop what could be called the "algebra of limits".

Theorem 2.14 (Multiples of Limits) *Suppose that $\{s_n\}$ is a convergent sequence and C a real number. Then*

$$\lim_{n\to\infty} Cs_n = C\left(\lim_{n\to\infty} s_n\right).$$

Proof. Let $S = \lim_{n\to\infty} s_n$. In order to prove that $\lim_{n\to\infty} Cs_n = CS$ we need to prove that no matter what positive number ε is given we can find an integer N so that

$$|Cs_n - CS| < \varepsilon$$

if $n \geq N$. Note that

$$|Cs_n - CS| = |C|\,|s_n - S|$$

by properties of absolute values. This gives us our clue.

Suppose first that $C \neq 0$ and let $\varepsilon > 0$. Choose N so that

$$|s_n - S| < \varepsilon/|C|$$

if $n \geq N$. Then if $n \geq N$ we must have

$$|Cs_n - CS| = |C|\,|s_n - S| < |C|\,(\varepsilon/|C|) = \varepsilon.$$

This is precisely the statement that

$$\lim_{n\to\infty} Cs_n = CS$$

and the theorem is proved in the case $C \neq 0$. The case $C = 0$ is obvious. (Now we should probably delete our first paragraph since it does not contribute to the proof; it only serves to motivate us in finding the correct proof.) ■

Theorem 2.15 (Sums and Differences of Limits) *Suppose that* $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences. Then*

$$\lim_{n\to\infty} (s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n$$

and

$$\lim_{n\to\infty} (s_n - t_n) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} t_n.$$

Proof. Let $S = \lim_{n\to\infty} s_n$ and $T = \lim_{n\to\infty} t_n$. In order to prove that $\lim_{n\to\infty}(s_n + t_n) = S + T$ we need to prove that no matter what positive number ε is given we can find an integer N so that

$$|(s_n + t_n) - (S + T)| < \varepsilon$$

if $n \geq N$. Note that

$$|(s_n + t_n) - (S + T)| \leq |s_n - S| + |t_n - T|$$

by the triangle inequality. Thus we can make this expression smaller than ε by making each of the two expressions on the right smaller than $\varepsilon/2$. This provides the method.

Suppose that $\varepsilon > 0$. Choose N_1 so that

$$|s_n - S| < \varepsilon/2$$

if $n \geq N_1$ and also choose N_2 so that

$$|t_n - T| < \varepsilon/2$$

if $n \geq N_2$. Then if n is greater than both N_1 and N_2 both of these two inequalities will be true. Set $N = \max\{N_1, N_2\}$ and note that if $n \geq N$ we must have

$$|(s_n + t_n) - (S + T)| \leq |s_n - S| + |t_n - T| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This is precisely the statement that

$$\lim_{n\to\infty} (s_n + t_n) = S + T$$

and the first statement of the theorem is proved. The second statement is similar and is left as an exercise. (Once again, for a more formal presentation, we would delete the first paragraph.) ■

Theorem 2.16 (Products of Limits) *Suppose that $\{s_n\}$ and $\{t_n\}$ are convergent sequences. Then*

$$\lim_{n\to\infty}(s_n t_n) = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty} t_n\right).$$

Proof. Let $S = \lim_{n\to\infty} s_n$ and $T = \lim_{n\to\infty} t_n$. In order to prove that $\lim_{n\to\infty}(s_n t_n) = ST$ we need to prove that no matter what positive number ε is given we can find an integer N so that

$$|s_n t_n - ST| < \varepsilon$$

if $n \geq N$. It takes some experimentation with different ways of writing this to find the most useful version. Here is an inequality that offers the best approach:

$$\begin{aligned}|s_n t_n - ST| &= |s_n(t_n - T) + s_n T - ST| \\ &\leq |s_n|\,|t_n - T| + |T|\,|s_n - S|.\end{aligned}$$

We can control the size of $|s_n - S|$ and $|t_n - T|$, T is constant, and $|s_n|$ cannot be too big. To control the size of $|s_n|$ we need to recall that convergent sequences are bounded (Theorem 2.11) and get a bound from there. With these preliminaries explained the rest of the proof should seem less mysterious. (Now this paragraph can be deleted for a more formal presentation.)

Suppose that $\varepsilon > 0$. Since $\{s_n\}$ converges it is bounded and hence, by Theorem 2.11, there is a positive number M so that $|s_n| \leq M$ for all n. Choose N_1 so that

$$|s_n - S| < \frac{\varepsilon}{2|T|+1}$$

if $n \geq N_1$. [We did not use $\varepsilon/(2T)$ since there is a possibility that $T = 0$.] Also choose N_2 so that

$$|t_n - T| < \frac{\varepsilon}{2M}$$

if $n \geq N_2$. Set $N = \max\{N_1, N_2\}$ and note that if $n \geq N$ we must have

$$\begin{aligned}|s_n t_n - ST| &\leq |s_n|\,|t_n - T| + |T|\,|s_n - S| \\ &\leq M\left(\frac{\varepsilon}{2M}\right) + |T|\left(\frac{\varepsilon}{2|T|+1}\right) < \varepsilon.\end{aligned}$$

This is precisely the statement that

$$\lim_{n\to\infty} s_n t_n = ST$$

and the theorem is proved. ∎

Theorem 2.17 (Quotients of Limits) *Suppose that $\{s_n\}$ and $\{t_n\}$ are convergent sequences. Suppose further that $t_n \neq 0$ for all n and that the limit*

$$\lim_{n\to\infty} t_n \neq 0.$$

Then

$$\lim_{n\to\infty} \left(\frac{s_n}{t_n}\right) = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n}.$$

Proof. Rather than prove the theorem at once as it stands let us prove just a special case of the theorem, namely that

$$\lim_{n\to\infty} \left(\frac{1}{t_n}\right) = \frac{1}{\lim_{n\to\infty} t_n}.$$

Let $T = \lim_{n\to\infty} t_n$. We need to show that no matter what positive number ε is given we can find an integer N so that

$$\left|\frac{1}{t_n} - \frac{1}{T}\right| < \varepsilon$$

if $n \geq N$. To work with this inequality requires us to consider

$$\left|\frac{1}{t_n} - \frac{1}{T}\right| = \frac{|t_n - T|}{|t_n|\,|T|}.$$

It is only the $|t_n|$ in the denominator that offers any trouble since if it is too small we cannot control the size of the fraction. This explains the first step in the proof that we now give, which otherwise might have seemed very strange.

Suppose that $\varepsilon > 0$. Choose N_1 so that

$$|t_n - T| < |T|/2$$

if $n \geq N_1$ and also choose N_2 so that

$$|t_n - T| < \varepsilon |T|^2/2$$

if $n \geq N_2$. From the first inequality we see that

$$|T| - |t_n| \leq |T - t_n| < |T|/2$$

and so

$$|t_n| \geq |T|/2$$

if $n \geq N_1$. Set $N = \max\{N_1, N_2\}$ and note that if $n \geq N$ we must have

$$\left|\frac{1}{t_n} - \frac{1}{T}\right| = \frac{|t_n - T|}{|t_n|\,|T|}$$

$$< \frac{\varepsilon|T|^2/2}{|T|^2/2} = \varepsilon.$$

This is precisely the statement that $\lim_{n\to\infty}(1/t_n) = 1/T$.

We now complete the proof of the theorem by applying the product theorem along with what we have just proved to obtain

$$\lim_{n\to\infty}\left(\frac{s_n}{t_n}\right) = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty}\frac{1}{t_n}\right) = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n}$$

as required. ■

Exercises

2:7.1 By imitating the proof given for the first part of Theorem 2.15 show that

$$\lim_{n\to\infty}(s_n - t_n) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} t_n.$$

2:7.2 Show that

$$\lim_{n\to\infty}(s_n)^2 = \left(\lim_{n\to\infty} s_n\right)^2$$

using the theorem on products and also directly from the definition of limit.

2:7.3 Explain which theorems are needed to justify the computation of the limit

$$\lim_{n\to\infty}\frac{n^2}{2n^2+1}$$

that introduced this section,

2:7.4 Which statements are true?

(a) If $\{s_n\}$ and $\{t_n\}$ are both divergent then so is $\{s_n + t_n\}$.

(b) If $\{s_n\}$ and $\{t_n\}$ are both divergent then so is $\{s_n t_n\}$.

(c) If $\{s_n\}$ and $\{s_n + t_n\}$ are both convergent then so is $\{t_n\}$.

(d) If $\{s_n\}$ and $\{s_n t_n\}$ are both convergent then so is $\{t_n\}$.

(e) Is $\{s_n\}$ is convergent so too is $\{1/s_n\}$.

(f) If $\{s_n\}$ is convergent so too is $\{(s_n)^2\}$.

(g) If $\{(s_n)^2\}$ is convergent so too is $\{s_n\}$.

2:7.5 Note that there are extra hypotheses in the quotient theorem (Theorem 2.17) that were not in the product theorem (Theorem 2.16). Explain why both of these hypotheses are needed.

2:7.6 Here is a flawed proof of Theorem 2.16. Find the flaw:

"Suppose that $\varepsilon > 0$. Choose N_1 so that

$$|s_n - S| < \frac{\varepsilon}{2|T|+1}$$

if $n \geq N_1$ and also choose N_2 so that

$$|t_n - T| < \frac{\varepsilon}{2|s_n| + 1}$$

if $n \geq N_2$. If $n \geq N = \max\{N_1, N_2\}$ then

$$|s_n t_n - ST| \leq |s_n|\,|t_n - T| + |T|\,|s_n - S|$$
$$\leq |s_n| \left(\frac{\varepsilon}{2|s_n| + 1}\right) + |T| \left(\frac{\varepsilon}{2|T| + 1}\right) < \varepsilon.$$

Well that works!"

2:7.7 Why are Theorems 2.15 and 2.16 no help in dealing with the limits

$$\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$$

and

$$\lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)?$$

What else can you do?

2:7.8 In calculus courses one learns that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at y if for every $\varepsilon > 0$ there is a $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon$ for all $|x - y| < \delta$. Show that if f is continuous at y and $s_n \to y$ then $f(s_n) \to f(y)$. Use this to prove that $\lim_{n\to\infty}(s_n)^2 = (\lim_{n\to\infty} s_n)^2$.

2.8 Order Properties of Limits

In the preceding section we discussed the algebraic structure of limits. It is a natural mathematical question to ask how the algebraic operations are preserved under limits. As it happens these natural mathematical questions usually are very important in applications. We have seen that the algebraic properties of limits can be used to great advantage in computations of limits.

There is another aspect of structure of the real number system that plays an equally important role as the algebraic structure and that is the order structure. Does the limit operation preserve that order structure the same way that it preserves the algebraic structure? For example, if

$$s_n \leq t_n$$

for all n can we conclude that

$$\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n?$$

In this section we solve this problem and several others related to the order structure. These results, too, will prove to be most useful in handling limits.

Theorem 2.18 *Suppose that* $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences and that*

$$s_n \leq t_n$$

for all n. *Then*

$$\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n.$$

Proof. Let $S = \lim_{n\to\infty} s_n$ and $T = \lim_{n\to\infty} t_n$ and suppose that $\varepsilon > 0$. Choose N_1 so that

$$|s_n - S| < \varepsilon/2$$

if $n \geq N_1$ and also choose N_2 so that

$$|t_n - T| < \varepsilon/2$$

if $n \geq N_2$. Set $N = \max\{N_1, N_2\}$ and note that if $n \geq N$ we must have

$$0 \leq t_n - s_n = T - S + (t_n - T) + (S - s_n) < T - S + \varepsilon/2 + \varepsilon/2.$$

This shows that

$$-\varepsilon < T - S.$$

This statement is true for *any* positive number ε. It would be false if $T - S$ is negative and hence $T - S$ is positive or zero, i.e., $T \geq S$ as required. ■

Note. There is a trap here that many students have fallen into. Since the condition $s_n \leq t_n$ implies

$$\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n$$

would it not follow "similarly" that the condition $s_n < t_n$ implies

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n?$$

Be very careful with this. It is false.

Corollary 2.19 *Suppose that* $\{s_n\}$ *is a convergent sequence and that*

$$\alpha \leq s_n \leq \beta$$

for all n. *Then*

$$\alpha \leq \lim_{n\to\infty} s_n \leq \beta.$$

Proof. Consider that the assumption here can be read as $\alpha_n \leq s_n \leq \beta_n$ where $\{\alpha_n\}$ and $\{\beta_n\}$ are constant sequences. Now apply the theorem. ■

Note. Again, don't forget the trap. The condition $\alpha < s_n < \beta$ for all n implies that

$$\alpha \leq \lim_{n\to\infty} s_n \leq \beta.$$

It would not imply that

$$\alpha < \lim_{n\to\infty} s_n < \beta.$$

The Squeeze Theorem The next theorem is another very useful variant on these themes. Here an unknown sequence is sandwiched between two convergent sequences, allowing us to conclude that that sequence converges. This theorem is often taught as "the squeeze theorem" which seems a convenient label.

Theorem 2.20 (Squeeze Theorem) *Suppose that $\{s_n\}$ and $\{t_n\}$ are convergent sequences, that*

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n$$

and that

$$s_n \leq x_n \leq t_n$$

for all n. Then $\{x_n\}$ is also convergent and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n.$$

Proof. Let L be the limit of the two sequences. Choose N_1 so that

$$|s_n - L| < \varepsilon$$

if $n \geq N_1$ and also choose N_2 so that

$$|t_n - L| < \varepsilon$$

if $n \geq N_2$. Set $N = \max\{N_1, N_2\}$. Note that

$$s_n - L \leq x_n - L \leq t_n - L$$

for all n and so

$$-\varepsilon < s_n - L \leq x_n - L \leq t_n - L < \varepsilon$$

if $n \geq N$. From this we see that

$$-\varepsilon < x_n - L < \varepsilon$$

or, to put it in a more familiar form,

$$|x_n - L| < \varepsilon$$

proving the statement of the theorem. ■

Example 2.21 Let θ be some real number and consider the computation of

$$\lim_{n\to\infty} \frac{\sin n\theta}{n}.$$

While this might seem hopeless at first sight since the values of $\sin n\theta$ are quite unpredictable, we recall that none of these values lies outside the interval $[-1,1]$. Hence

$$-\frac{1}{n} \leq \frac{\sin n\theta}{n} \leq \frac{1}{n}.$$

The two outer sequences converge to the same value 0 and so the inside sequence (the "squeezed" one) must converge to 0 as well. ◀

Absolute Values A further theorem on the theme of order structure is often needed. The absolute value, we recall, is defined directly in terms of the order structure. Is absolute value preserved by the limit operation?

Theorem 2.22 (Limits of Absolute Values) *Suppose that $\{s_n\}$ is a convergent sequence. Then the sequence $\{|s_n|\}$ is also a convergent sequence and*

$$\lim_{n\to\infty} |s_n| = \left| \lim_{n\to\infty} s_n \right|.$$

Proof. Let $S = \lim_{n\to\infty} s_n$ and suppose that $\varepsilon > 0$. Choose N so that

$$|s_n - S| < \varepsilon$$

if $n \geq N$. Observe that, because of the triangle inequality, this means that

$$||s_n| - |S|| \leq |s_n - S| < \varepsilon$$

for all $n \geq N$. By definition

$$\lim_{n\to\infty} |s_n| = |S|$$

as required. ■

Maxima and Minima Since maxima and minima can be expressed in terms of absolute values there is a corollary that is sometimes useful.

Corollary 2.23 (Max/Min of Limits) *Suppose that $\{s_n\}$ and $\{t_n\}$ are convergent sequences. Then the sequences*

$$\{\max\{s_n, t_n\}\}$$

and

$$\{\min\{s_n, t_n\}\}$$

are also convergent and

$$\lim_{n\to\infty} \max\{s_n, t_n\} = \max\{\lim_{n\to\infty} s_n, \lim_{n\to\infty} t_n\}$$

and

$$\lim_{n\to\infty} \min\{s_n, t_n\} = \min\{\lim_{n\to\infty} s_n, \lim_{n\to\infty} t_n\}.$$

Proof. The first of the these follows from the identity

$$\max\{s_n, t_n\} = \frac{|s_n + t_n|}{2} + \frac{|s_n - t_n|}{2}$$

and the theorem on limits of sums and the theorem on limits of absolute values. In the same way the second assertion follows from

$$\min\{s_n, t_n\} = \frac{|s_n + t_n|}{2} - \frac{|s_n - t_n|}{2}.$$

■

Exercises

2:8.1 Show that the condition

$$s_n < t_n$$

does not imply that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n.$$

(If the proof of Theorem 2.18 were modified in an attempt to prove this false statement where would the modifications fail?)

2:8.2 If $\{s_n\}$ is a sequence all of whose values lie inside an interval $[a, b]$ prove that $\{s_n/n\}$ is convergent.

2:8.3 Suppose that $s_n \leq t_n$ for all n and that $s_n \to \infty$. What can you conclude?

2:8.4 Suppose that

$$\lim_{n\to\infty} \frac{s_n}{n} > 0$$

Show that $s_n \to \infty$.

2:8.5 Suppose that $\{s_n\}$ and $\{t_n\}$ are sequences of positive numbers, that

$$\lim_{n\to\infty} \frac{s_n}{t_n} = \alpha$$

and that $s_n \to \infty$. What can you conclude?

2:8.6 Suppose that $\{s_n\}$ and $\{t_n\}$ are sequences of positive numbers, that
$$\lim_{n\to\infty} \frac{s_n}{t_n} = \infty$$
and that $t_n \to \infty$. What can you conclude?

2:8.7 Suppose that $\{s_n\}$ and $\{t_n\}$ are sequences of positive numbers, that
$$\lim_{n\to\infty} \frac{s_n}{t_n} = \infty$$
and that $\{s_n\}$ is bounded. What can you conclude?

2:8.8 Let $\{s_n\}$ be a sequence of positive numbers. Show that the condition
$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} < 1$$
implies that $s_n \to 0$.

2:8.9 Let $\{s_n\}$ be a sequence of positive numbers. Show that the condition
$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} > 1$$
implies that $s_n \to \infty$.

2.9 Monotone Convergence Criterion

In many applications of sequence theory we find that the sequences that arise are going in one direction: the terms steadily get larger or steadily get smaller. The analysis of such sequences is much easier than for general sequences.

Definition 2.24 (Increasing) We say that a sequence $\{s_n\}$ is *increasing* if
$$s_1 < s_2 < s_3 < \cdots < s_n < s_{n+1} < \ldots .$$

Definition 2.25 (Decreasing) We say that a sequence $\{s_n\}$ is *decreasing* if
$$s_1 > s_2 > s_3 > \cdots > s_n > s_{n+1} > \ldots .$$

Often we encounter sequences that "increase" except perhaps occasionally successive values are equal rather than strictly larger. The following language is usually used in this case.

Definition 2.26 (Nondecreasing) We say that a sequence $\{s_n\}$ is *nondecreasing* if
$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \ldots .$$

Definition 2.27 (Nonincreasing) We say that a sequence $\{s_n\}$ is *nonincreasing* if
$$s_1 \geq s_2 \geq s_3 \geq \cdots \geq s_n \geq s_{n+1} \geq \ldots .$$

Thus every increasing sequence is also nondecreasing but not conversely. A sequence that has any one of these four properties (increasing, decreasing, nondecreasing, or nonincreasing) is said to be *monotonic*. Monotonic sequences are often easier to deal with than sequences which can go both up and down.

Note. In some texts you will find that a nondecreasing sequence is said to be increasing and an increasing sequence is said to be *strictly* increasing. The way in which we intend these terms should be clear and intuitive. If your monthly salary occasionally rises but sometimes stays the same you would not likely say that it is increasing. You might, however, say "at least it never decreases", i.e., it is nondecreasing.

The convergence issue for a monotonic sequence is particularly straightforward. We can imagine that an increasing sequence could increase up to some limit, or we could imagine that it could increase indefinitely and diverge to $+\infty$. It is impossible to imagine a third possibility. We express this as a theorem which will become our primary theoretical tool in investigating convergence of sequences.

Theorem 2.28 (Monotone Convergence Theorem) *Let $\{s_n\}$ be a monotonic sequence. Then $\{s_n\}$ is convergent if and only if $\{s_n\}$ is bounded. More specifically*

1. *If $\{s_n\}$ is nondecreasing then either $\{s_n\}$ is bounded and converges to $\sup\{s_n\}$ or else $\{s_n\}$ is unbounded and $s_n \to \infty$.*

2. *If $\{s_n\}$ is nonincreasing then either $\{s_n\}$ is bounded and converges to $\inf\{s_n\}$ or else $\{s_n\}$ is unbounded and $s_n \to -\infty$.*

Proof. If the sequence is unbounded then it diverges. This is true for any sequence, not merely monotonic sequences.

Thus the proof is complete if we can show that for any bounded monotonic sequence $\{s_n\}$ the limit is $\sup\{s_n\}$ in case the sequence is nondecreasing or it is $\inf\{s_n\}$ in case the sequence is nonincreasing. Let us prove the first of these cases.

Let $\{s_n\}$ be assumed to be nondecreasing and bounded, and let $L = \sup\{s_n\}$. Then $s_n \leq L$ for all n and if $\beta < L$ there must be some term s_m say, with $s_m > \beta$. Let $\varepsilon > 0$. We know that there is an m so that

$$s_n \geq s_m > L - \varepsilon$$

for all $n \geq m$. But we already know that every term $s_n \leq L$. Putting these together we have that

$$L - \varepsilon < s_n \leq L < L + \varepsilon$$

or

$$|s_n - L| < \varepsilon$$

for all $n \geq m$. By definition then $s_n \to L$ as required. ■

How would we normally apply this theorem? Suppose a sequence $\{s_n\}$ were given that we recognize as increasing (or maybe just non-decreasing). Then to establish that $\{s_n\}$ converges we need only show that the sequence is bounded above, i.e., we need to find just one number M with

$$s_n \leq M$$

for all n. Any crude upper estimate would verify convergence.

Example 2.29 Let us show that the sequence $s_n = 1/\sqrt{n}$ converges. This sequence is evidently decreasing. Can we find a lower bound? Yes, all of the terms are positive so that 0 is a lower bound. Consequently the sequence must converge. If we wish to show that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

we need to do more. But to conclude convergence we needed only to make a crude estimate on how low the terms might go. ◀

Example 2.30 Let us examine the sequence

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}.$$

This sequence is evidently increasing. Can we find an upper bound? If we can then the series does converge. If we cannot then the series diverges. We have already (earlier) checked this sequence. It is unbounded and so $\lim_{n\to\infty} s_n = \infty$. ◀

Example 2.31 Let us examine the sequence

$$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}},\ldots$$

Handling such a sequence directly by the limit definition seems quite impossible. This sequence can be defined recursively by

$$x_1 = \sqrt{2} \quad x_n = \sqrt{2 + x_{n-1}}.$$

The computation of a few terms suggests that the sequence is increasing and so should be accessible by the methods of this section.

We prove this by induction. That $x_1 < x_2$ is just an easy computation (that the reader should do). Let us suppose that $x_{n-1} < x_n$ for some n and show that it must follow that $x_n < x_{n+1}$. But

$$x_n = \sqrt{2 + x_{n-1}} < \sqrt{2 + x_n} = x_{n+1}$$

where the middle step is the induction hypothesis (i.e., that $x_{n-1} < x_n$). It follows by induction that the sequence is increasing.

Now we show inductively that the sequence is bounded above. Any crude upper bound will suffice. It is clear that $x_1 < 10$. If $x_{n-1} < 10$ then

$$x_n = \sqrt{2 + x_{n-1}} < \sqrt{2 + 10} < 10$$

and so it follows, again by induction, that all terms of the sequence are smaller than 10. We conclude from the monotone convergence theorem that this sequence is convergent.

But to what? (Certainly it does not converges to 10 since that estimate was extremely crude.) That is not so easy to sort out it seems. But perhaps it is, since we know that the sequence converges to something, say L. In the equation

$$(x_n)^2 = 2 + x_{n-1},$$

obtained by squaring the recursion formula given to us, we can take limits as $n \to \infty$. Since $x_n \to L$ so too does $x_{n-1} \to L$ and $(x_n)^2 \to L^2$. Hence

$$L^2 = 2 + L.$$

The only possibilities for L in this quadratic equation are $L = -1$ and $L = 2$. We know the limit L exists and we know that it is either -1 or 2. We can clearly rule out -1 as none of the numbers in our sequence were negative. Hence $x_n \to 2$. ◀

Exercises

2:9.1 Define a sequence $\{s_n\}$ recursively by setting $s_1 = \alpha$ and

$$s_n = \frac{(s_{n-1})^2 + \beta}{2s_{n-1}}$$

where α, $\beta > 0$.

(a) Show that for $n = 1, 2, 3, \ldots$

$$\frac{(s_n - \sqrt{\beta})^2}{2s_n} = s_{n+1} - \beta.$$

(b) Show that $s_n > \sqrt{\beta}$ for all $n = 2, 3, 4, \ldots$ unless $\alpha = \sqrt{\beta}$. What happens if $\alpha = \sqrt{\beta}$?

(c) Show that $s_2 > s_3 > s_4 > \ldots s_n > \ldots$ except in the case $\alpha = \sqrt{\beta}$. .

(d) Does this sequence converge? To what?

(e) What is the relation of this sequence to the one introduced in Section 2.1 as Newton's method?

2:9.2 Define a sequence $\{t_n\}$ recursively by setting $t_1 = 1$ and

$$t_n = \sqrt{t_{n-1} + 1}.$$

Does this sequence converge? To what?

2:9.3 Consider the sequence

$$s_1 = 1 \quad \text{and} \quad s_n = \frac{2}{s_{n-1}^2}.$$

We argue that if $s_n \to L$ then

$$L = \frac{2}{L^2}$$

and so $L^3 = 2$ or $L = \sqrt[3]{2}$. Our conclusion is that $\lim_{n\to\infty} s_n = \sqrt[3]{2}$. Do you have any criticisms of this argument?

2:9.4 Does the sequence

$$\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$$

converge?

2:9.5 Does the sequence

$$\frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n) \cdot 1}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1) \cdot n^2}$$

converge?

2:9.6 Several nineteenth century mathematicians used, without proof, a principle in their proofs that has come to be known as the *nested interval property*:

> *Given a sequence of closed intervals*
>
> $$[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \ldots$$
>
> *arranged so that each interval is a subinterval of the one preceding it and so that the lengths of the intervals shrink to zero then there is exactly one point that belongs to every interval of the sequence.*

Prove this statement. Would it be true for a descending sequence of open intervals

$$(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3) \supset \ldots ?$$

2.10 Examples of Limits

The theory of sequence limits has now been developed far enough that we may investigate some interesting limits. Each of the limits in this section has some cultural interest. Most students would be expected to know and recognize these limits as they arise quite routinely. For us they are also an opportunity to show off our methods. Mostly we need to establish inequalities and use some of our theory. We have not needed to use an ε, N argument since we now have more subtle and powerful tools at hand.

Example 2.32 (Geometric Progressions) Let r be a real number. What is the limiting behavior of the sequence

$$1, r, r^2, r^3, r^4, \dots, r^n, \dots$$

forming a geometric progression? If $r > 1$ then it is not hard to show that $r^n \to \infty$. If $r \leq -1$ the sequence certainly diverges. If $r = 1$ this is just a constant sequence.

The interesting case is:

$$\lim_{n\to\infty} r^n = 0 \qquad \text{if } -1 < r < 1.$$

To prove this we shall use an easy inequality. Let $x > 0$ and n an integer. Then using the binomial theorem (or induction if you prefer) we can show that

$$(1+x)^n > nx.$$

Case (i): Let $0 < r < 1$. Then

$$r = \frac{1}{1+x}$$

(where $x = 1 - 1/r > 0$) and so

$$0 < r^n = \frac{1}{(1+x)^n} < \frac{1}{nx} \to 0$$

as $n \to \infty$. By the squeeze theorem we see that $r^n \to 0$ as required.

Case (ii): If $-1 < r < 0$ then $r = -t$ for $0 < t < 1$. Thus

$$-t^n \leq r^n \leq t^n.$$

By case (i) we know that $t^n \to 0$. By the squeeze theorem we see that $r^n \to 0$ again as required. ◀

Example 2.33 (Roots) An interesting and often useful limit is

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

To show this we once again derive an inequality from the binomial theorem: if $n \geq 2$ and $x > 0$ then

$$(1+x)^n > n(n-1)x^2/2.$$

For $n \geq 2$ write

$$\sqrt[n]{n} = 1 + x_n$$

(where $x_n = \sqrt[n]{n} - 1 > 0$) and so

$$n = (1 + x_n)^n > n(n-1)x_n^2/2$$

or

$$0 < x_n^2 < \frac{2}{n-1} \to 0$$

as $n \to \infty$. By the squeeze theorem we see that $x_n \to 0$ and it follows that $\sqrt[n]{n} \to 1$ as required.

As a special case of this example note that

$$\sqrt[n]{C} \to 1$$

as $n \to \infty$ for any positive constant C. This is true because if $C > 1$ then

$$1 < \sqrt[n]{C} < \sqrt[n]{n}$$

for large enough n. By the squeeze theorem this shows that $\sqrt[n]{C} \to 1$. If, however, $0 < C < 1$ then

$$\sqrt[n]{C} = \frac{1}{\sqrt[n]{1/C}} \to 1$$

by the first case since $1/C > 1$. ◀

Example 2.34 (Sums of Geometric Progressions) For all values of x in the interval $(-1, 1)$ the limit

$$\lim_{n\to\infty} \left(1 + x + x^2 + x^3 + \cdots + x^n\right) = \frac{1}{1-x}.$$

While, at first, a surprising result this is quite evident once one checks the identity

$$(1-x)\left(1 + x + x^2 + x^3 + \cdots + x^n\right) = 1 - x^{n+1}$$

which just requires a straightforward multiplication. Thus

$$\lim_{n\to\infty} \left(1 + x + x^2 + x^3 + \cdots + x^n\right) = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1-x} = \frac{1}{1-x}$$

where we have used the result we proved above, namely that

$$x^{n+1} \to 0 \quad \text{if } |x| < 1.$$

One special case of this is very useful to remember. Set $x = 1/2$. Then

$$\lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}\right) = 2.$$

◀

Example 2.35 (Decimal Expansions) What meaning is assigned to the infinite decimal expansion

$$x = 0.d_1d_2d_3d_4\ldots d_n\ldots$$

where the choices of integers $0 \le d_i \le 9$ can be made in any way? Repeating decimals can always be converted into fractions and so the infinite process can be avoided. But if the pattern does not repeat a different interpretation must be made.

The most obvious interpretation of the number x above is to declare that it is the limit of the sequence

$$\lim_{n\to\infty} 0.d_1d_2d_3d_4\ldots d_n.$$

But how do we know that the limit exists? Our theory provides an immediate answer: this sequence is nondecreasing and every term is a number smaller than 1. Thus by the monotone convergence theorem the sequence converges no matter what the choices of the decimal digits are. ◀

Example 2.36 (Expansion of e^x) Let $x > 0$ and consider the two closely related sequences

$$s_n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

and

$$t_n = \left(1 + \frac{x}{n}\right)^n.$$

The relation between the two sequences becomes more apparent once the binomial theorem is used to expand the latter.

In more advanced mathematics it is shown that both sequences converge to e^x. Let us be content to prove that

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n.$$

The sequence $\{s_n\}$ is clearly increasing since each new term is the preceding term with a positive number added to it. To show convergence then we need only show that the sequence is bounded. This takes some arithmetic, but not too much.

Choose N so large that

$$\frac{x}{N} < \frac{1}{2}.$$

Note then that

$$\frac{x^{N+1}}{(N+1)!} < \frac{1}{2}\left(\frac{x^N}{(N)!}\right)$$

that

$$\frac{x^{N+2}}{(N+2)!} < \frac{1}{4}\left(\frac{x^N}{(N)!}\right)$$

and that

$$\frac{x^{N+3}}{(N+3)!} < \frac{1}{8}\left(\frac{x^N}{(N)!}\right).$$

Thus

$$\begin{aligned} s_n &\leq \left[1 + x + \frac{x^2}{2!} + \ldots \frac{x^{N-1}}{(N-1)!}\right] + \frac{x^N}{(N)!}\left(1 + \frac{1}{2} + \frac{1}{4}\ldots\right) \\ &\leq \left[1 + x + \frac{x^2}{2!} + \ldots \frac{x^{N-1}}{(N-1)!}\right] + 2\frac{x^N}{(N)!}. \end{aligned}$$

Here we have used the limit for the sum of a geometric progression from Example 2.34 to make an upper estimate on how large this sum can get. Note that the N is fixed and so the number on the right hand side of this inequality is just a number, and it is larger than every number in the sequence $\{s_n\}$.

It follows now from the monotone convergence theorem that $\{s_n\}$ converges. To handle $\{t_n\}$ first apply the binomial theorem to obtain

$$t_n = 1 + x + \frac{1 - 1/n}{2!}x^2 + \frac{(1 - 1/n)(1 - 2/n)}{3!}x^3 + \cdots \leq s_n.$$

From this we see that $\{t_n\}$ is increasing and that it is smaller than the convergent sequence $\{s_n\}$. It follows, again from the monotone convergence theorem that $\{t_n\}$ converges. Moreover

$$\lim_{n\to\infty} t_n \leq \lim_{n\to\infty} s_n.$$

If we can obtain the opposite inequality we will have proved our assertion. Let m be a fixed number and let $n > m$. Then, from the expansion above, we note that

$$\begin{aligned} t_n &> 1 + x + \frac{1 - 1/n}{2!}x^2 + \frac{(1 - 1/n)(1 - 2/n)}{3!}x^3 \\ &+ \cdots + \frac{(1 - 1/n)(1 - 2/n)\ldots(1 - [m-1]/n)}{m!}x^m. \end{aligned}$$

We can hold m fixed and allow $n \to \infty$ in this inequality and obtain that

$$\lim_{n\to\infty} t_n \geq s_m$$

for each m. From this it now follows that

$$\lim_{n\to\infty} t_n \geq \lim_{n\to\infty} s_n$$

and we have completed our task. ◀

Exercises

2:10.1 Since we know that

$$1 + x + x^2 + x^3 + \cdots + x^n \to \frac{1}{1-x}$$

this suggests the formula

$$1 + 2 + 4 + 8 + 16 + \cdots = \frac{1}{1-2} = -1.$$

Do you have any criticisms? (By the way, do not be too harsh in your criticism; many great mathematicians, including Euler, would have accepted this formula.)

2:10.2 Let α and β be positive numbers. Discuss the convergence behavior of the sequence

$$\frac{\alpha^{\beta n}}{\beta^{\alpha n}}.$$

2:10.3 Define

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Show that $2 < e < 3$.

2:10.4 Show that

$$\lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^n = \sqrt{e}.$$

2:10.5 Check the simple identity

$$\left(1 + \frac{2}{n}\right) = \left(1 + \frac{1}{n+1}\right)\left(1 + \frac{1}{n}\right)$$

and use it to show that

$$\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n = e^2.$$

2.11 Subsequences

The sequence

$$1, -1, 2, -2, 3, -3, 4, -4, 5, -5, \ldots$$

appears to contain within itself the two sequences

$$1, 2, 3, 4, 5, \ldots$$

and

$$-1, -2, -3, -4, -5, \ldots.$$

In order to have a language to express this we introduce the term *subsequence*. We would say that the latter two sequences are subsequences of the first sequence. Often a sequence is best studied by looking at some of its subsequences. But what is a proper definition of this term? We need a formal mathematical way of expressing the vague idea that a subsequence is obtained by crossing out some of the terms of the original sequence.

Definition 2.37 (Subsequences) Let

$$s_1, s_2, s_3, s_4, \dots$$

be any sequence. Then by a *subsequence* of this sequence we mean any sequence

$$s_{n_1}, s_{n_2}, s_{n_3}, s_{n_4}, \dots$$

where n_1, n_2, n_3, $\dots$ is an increasing sequence of natural numbers.

Example 2.38 We can consider

$$1, 2, 3, 4, 5, \dots$$

to be a subsequence of sequence

$$1, -1, 2, -2, 3, -3, 4, -4, 5, -5, \dots$$

because it contains just the first, third, fifth, etc. terms of the original sequence. Here $n_1 = 1$, $n_2 = 3$, $n_3 = 5$, $\dots$. ◀

In many applications of sequences it is the subsequences that need to be studied. For example what can we say about the existence of monotonic subsequences, or bounded subsequences, or divergent subsequences, or convergent subsequences? The answers to these questions have important uses.

Existence of Monotonic Subsequences Our first question is easy to answer for any specific sequence, but harder to settle in general. Given a sequence can we always select a subsequence that is monotonic, either monotonic nondecreasing or monotonic nonincreasing?

Theorem 2.39 *Every sequence contains a monotonic subsequence.*

Proof. We construct first a nonincreasing subsequence if possible. We call the mth element x_m of the sequence $\{x_n\}$ a turn-back point if all later elements are less than or equal to it, in symbols if $x_m \geq x_n$ for all $n > m$. If there is an infinite subsequence of turn-back points $x_{m_1}, x_{m_2}, x_{m_3}, x_{m_4}, \dots$ then we have found our nonincreasing subsequence since

$$x_{m_1} \geq x_{m_2} \geq x_{m_3} \geq x_{m_4} \geq \dots .$$

This would not be possible if there are only finitely many turn-back points. Let us suppose that x_M is the last turn-back point so that any element x_n for $n > M$ is not a turn-back point. Since it is not there must be an element further on in the sequence greater than it, in symbols $x_m > x_n$ for some $m > n$. Thus we can choose $x_{m_1} > x_{M+1}$ with $m_1 > M+1$, then $x_{m_2} > x_{m+1}$ with $m_1 > m_2$, and then $x_{m_3} > x_{m_2}$ with $m_3 > m_2$, and so on to obtain an increasing subsequence

$$x_{M+1} < x_{m_1} < x_{m_2} < x_{m_3} < x_{m_4} < \dots$$

as required. ■

Existence of Convergent Subsequences Having answered this question about the existence of monotonic subsequences we can also now answer the question about the existence of convergent subsequences. This might, at first sight, seem just a curiosity but it will give us later on one of our most important tools in analysis.

Theorem 2.40 (Bolzano–Weierstrass) *Every bounded sequence contains a convergent subsequence.*

Proof. By Theorem 2.39 every sequence contains a monotonic subsequence. Here that subsequence would be both monotonic and bounded, and hence convergent. ■

Other (less important) questions of this type appear in the exercises.

Exercises

2:11.1 Show that, according to our definition, every sequence is a subsequence of itself. How would the definition have to be reworded to avoid this if, for some reason, this possibility were to have been avoided?

2:11.2 Show that every subsequence of a subsequence of a sequence $\{x_n\}$ is itself a subsequence of $\{x_n\}$.

2:11.3 If $\{s_{n_k}\}$ is a subsequence of $\{s_n\}$ and $\{t_{m_k}\}$ is a subsequence of $\{t_n\}$ then is it true that $\{s_{n_k} + t_{m_k}\}$ is a subsequence of $\{s_n + t_n\}$?

2:11.4 If $\{s_{n_k}\}$ is a subsequence of $\{s_n\}$ is $\{(s_{n_k})^2\}$ is a subsequence of $\{(s_n)^2\}$?

2:11.5 Describe those sequences that have only finitely many different subsequences.

2:11.6 Establish which statements are true?

(a) A sequence is convergent if and only if all of its subsequences are convergent.

(b) A sequence is bounded if and only if all of its subsequences are bounded.

(c) A sequence is monotonic if and only if all of its subsequences are monotonic.

(d) A sequence is divergent if and only if all of its subsequences are divergent.

2:11.7 Where possible find subsequences that are monotonic and subsequences that are convergent for the following sequences

(a) $\{(-1)^n n\}$.

(b) $\{\sin(n\pi/8)\}$.

(c) $\{n \sin(n\pi/8)\}$.

(d) $\left\{\frac{n+1}{n} \sin(n\pi/8)\right\}$.

(e) $\{1 + (-1)^n\}$.

(f) $\{r_n\}$ consists of all rational numbers in the interval $(0, 1)$ arranged in some order.

2:11.8 Describe all subsequences of the sequence

$$1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots.$$

Describe all convergent subsequences. Describe all monotonic subsequences.

2:11.9 If $\{s_{n_k}\}$ is a subsequence of $\{s_n\}$ show that $n_k \geq k$ for all $k = 1, 2, 3, \ldots$.

2:11.10 Give an example of a sequence that contains subsequences converging to every natural number (and no other numbers).

2:11.11 Give an example of a sequence that contains subsequences converging to every number in $[0, 1]$ (and no other numbers).

2:11.12 Show that there cannot exist a sequence that contains subsequences converging to every number in $(0, 1)$ and no other numbers.

2:11.13 Show that if $\{s_n\}$ has no convergent subsequences then $|s_n| \to \infty$ as $n \to \infty$.

2:11.14 If a sequence $\{x_n\}$ has the property that

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = L$$

show that the sequence $\{x_n\}$ converges to L.

2:11.15 If a sequence $\{x_n\}$ has the property that

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = \infty$$

show that the sequence $\{x_n\}$ diverges to ∞.

2:11.16 Let α and β be positive real numbers and define a sequence by setting $s_1 = \alpha$, $s_2 = \beta$ and $s_{n+2} = \frac{1}{2}(s_n + s_{n+1})$ for all $n = 1, 2, 3, \ldots$. Show that the subsequences $\{s_{2n}\}$ and $\{s_{2n-1}\}$ are monotonic and convergent. Does the sequence $\{s_n\}$ converge? To what?

2:11.17 Without appealing to any of the theory of this section prove that every unbounded sequence has a strictly monotonic subsequence (i.e., either increasing or decreasing).

2:11.18 Show that if a sequence $\{x_n\}$ converges to a finite limit or diverges to $\pm\infty$ then every subsequence has precisely the same behavior.

2:11.19 Suppose a sequence $\{x_n\}$ has the property that every subsequence has itself a further subsequence convergent to L. Show that $\{x_n\}$ converges to L.

2:11.20 Let $\{x_n\}$ be a bounded sequence and let $x = \sup\{x_n : n \in \mathbb{N}\}$. Suppose that, moreover, $x_n < x$ for all n. Prove that there is a subsequence convergent to x.

2:11.21 Let $\{x_n\}$ be a bounded sequence, let $y = \inf\{x_n : n \in \mathbb{N}\}$ and let $x = \sup\{x_n : n \in \mathbb{N}\}$. Suppose that, moreover, $y < x_n < x$ for all n. Prove that there is a pair of convergent subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ so that

$$\lim_{k\to\infty} |x_{n_k} - x_{m_k}| = x - y.$$

2:11.22 Does every divergent sequence contain a divergent monotonic sequence?

2:11.23 Does every divergent sequence contain a divergent bounded sequence?

2:11.24 Construct a proof of the Bolzano–Weierstrass theorem for bounded sequences using the nested interval property and not appealing to the existence of monotonic subsequences.

2:11.25 Construct a direct proof of the assertion that every convergent sequence has a convergent, monotonic subsequence (i.e., without appealing to Theorem 2.39).

2:11.26 Let $\{x_n\}$ be a bounded sequence that we do not know converges. Suppose that it has the property that every one of its convergent subsequences converges to the same number L. What can you conclude?

2:11.27 Let $\{x_n\}$ be a bounded sequence that diverges. Show that there is a pair of convergent subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ so that

$$\lim_{k\to\infty} |x_{n_k} - x_{m_k}| > 0.$$

2:11.28 Let $\{x_n\}$ be a sequence. A number z with the property that for all $\varepsilon > 0$ there are infinitely many terms of the sequence in the interval $(z-\varepsilon, z+\varepsilon)$ is said to be a *cluster point* of the sequence. Show that z is a cluster point of a sequence if and only if there is a subsequence $\{x_{n_k}\}$ converging to z.

2.12 Cauchy Convergence Criterion

What property of a sequence characterizes convergence? As a "characterization" we would like some necessary and sufficient condition for a sequence to converge. We could simply write the definition and consider that that is a characterization. Thus the following technical statement would, indeed, be a characterization of the convergence of a sequence $\{s_n\}$.

A sequence $\{s_n\}$ is convergent if and only if $\exists L$ so that $\forall \varepsilon > 0$ $\exists N$ with the property that

$$|s_n - L| < \varepsilon$$

whenever $n \geq N$.

In mathematics when we ask for a characterization of a property we can expect to find many answers, some more useful than others. The limitation of this particular characterization is that it requires us to find the number L which is the limit of the sequence in advance. Compare this with a characterization of convergence of a monotonic sequence $\{s_n\}$.

A monotonic sequence $\{s_n\}$ is convergent if and only if it is bounded.

This is a wonderful and most useful characterization. But it applies only to monotonic sequences.

A correct and useful characterization, applicable to all sequences, was found by Cauchy. This is the content of the next theorem. Note that it has the advantage that it describes a convergent sequence with no reference whatsoever to the actual value of the limit. Loosely it asserts that a sequence converges if and only if the terms of the sequence are eventually arbitrarily close together.

Theorem 2.41 (Cauchy Criterion) *A sequence $\{s_n\}$ is convergent if and only if for all $\varepsilon > 0$ there exists an integer N with the property that*

$$|s_n - s_m| < \varepsilon$$

whenever $n \geq N$ and $m \geq N$.

Proof. This property of the theorem is so important that it deserves some terminology. A sequence is said to be a *Cauchy sequence* if it satisfies this property. Thus the theorem states that a sequence is convergent if and only if it is a Cauchy sequence. The terminology is most significant in more advanced situations where being a Cauchy sequence is not necessarily equivalent with being convergent.

Our proof is a bit lengthy and will require an application of the Bolzano-Weierstrass Theorem.

The proof in one direction, however, is very easy. Suppose that $\{s_n\}$ is convergent to a number L. Let $\varepsilon > 0$. Then there must be an integer N so that

$$|s_k - L| < \frac{\varepsilon}{2}$$

whenever $k \geq N$. Thus if both m and n are larger than N,

$$|s_n - s_m| \leq |s_n - L| + |L - s_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which shows that $\{s_n\}$ is a Cauchy sequence.

Now let us prove the opposite (and more difficult) direction.

For the first step we show that every Cauchy sequence is bounded. Since the proof of this can be obtained by copying and modifying the proof of Theorem 2.11 we have left this as an exercise. (It is not interesting really that Cauchy sequences are bounded since, after the proof is completed we know that all Cauchy sequences are convergent and so must indeed be bounded.)

The second steps comes easily. We apply the Bolzano-Weierstrass theorem to the (bounded) sequence $\{s_n\}$ to obtain a convergent subsequence $\{s_{n_k}\}$.

The final step is a feature of Cauchy sequences. Once we know that $s_{n_k} \to L$ and that $\{s_n\}$ is Cauchy we can show that $s_n \to L$ also. Let $\varepsilon > 0$ and choose N so that

$$|s_n - s_m| < \varepsilon/2$$

for all m, $n \geq N$. Choose K so that

$$|s_{n_k} - L| < \varepsilon/2$$

for all $k \geq K$. Suppose that $n \geq N$. Set m equal to any value of n_k that is larger than N and so that $k \geq K$. For this value $s_m = s_{n_k}$

$$|s_n - L| \leq |s_n - s_{n_k}| + |s_{n_k} - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This is exactly the statement that the sequence $\{s_n\}$ converges to L and so the proof is complete. ■

Example 2.42 The Cauchy criterion is most useful in theoretical developments, rather than applied to concrete examples. Even so occasionally it is the fastest route to a proof of convergence. For example consider the sequence $\{x_n\}$ defined by setting $x_1 = 1$, $x_2 = 2$ and then recursively

$$x_n = \frac{x_{n-1} + x_{n-2}}{2}.$$

Each term after the second is the average of the preceding two terms. The distance between x_1 and x_2 is 1, that between x_2 and x_3 is $1/2$, between x_3 and x_4 is $1/4$, and so on. We see then that after the N stage all the distances are smaller than 2^{-N+1}, i.e., that for all $n \geq N$ and $m \geq N$

$$|x_n - x_m| \leq \frac{1}{2^{N-1}}.$$

This is exactly the Cauchy criterion and so this sequence converges. Note that the Cauchy criterion offers no information on what the sequence is converging to. The reader will have to come up with another method to find out. ◀

Exercises

2:12.1 Show directly that the sequence $s_n = 1/n$ is a Cauchy sequence.

2:12.2 Show directly that any multiple of a Cauchy sequence is again a Cauchy sequence.

2:12.3 Show directly that the sum of two Cauchy sequences is again a Cauchy sequence.

2:12.4 Show directly that any Cauchy sequence is bounded.

2:12.5 The following criterion is weaker than the Cauchy criterion. Show that it is not equivalent:

> For all $\varepsilon > 0$ there exists an integer N with the property that
> $$|s_{n+1} - s_n| < \varepsilon$$
> whenever $n \geq N$.

2:12.6 Is the following criterion weaker, stronger or equivalent to the Cauchy criterion?

> For all $\varepsilon > 0$ and all positive integers p there exists an integer N with the property that
> $$|s_{n+p} - s_n| < \varepsilon$$
> whenever $n \geq N$.

2:12.7 Show directly that if $\{s_n\}$ is a Cauchy sequence then so too is $\{|s_n|\}$ From this conclude that $\{|s_n|\}$ converges whenever $\{s_n\}$ converges.

2:12.8 Show that every subsequence of a Cauchy sequence is Cauchy. [Do not use the fact that every Cauchy sequence is convergent.]

2:12.9 Show that every bounded monotonic sequence is Cauchy. [Do not use the monotone convergence theorem.]

2:12.10 Show that the sequence in Example 2.42 converges to 5/3.

2.13 Upper and Lower Limits

If $\lim_{n\to\infty} x_n = L$ then, according to our definition, numbers α and β on either side of L, i.e, $\alpha < L < \beta$, have the property that

$$\alpha < x_n \text{ and } x_n < \beta$$

for all sufficiently large n. In many applications only *half* of this information is used.

Example 2.43 Here is an example showing how half a limit is as good as a whole limit. Let $\{x_n\}$ be a sequence of positive numbers with the property that

$$\lim_{n\to\infty} \sqrt[n]{x_n} = L < 1.$$

Then we can prove that $x_n \to 0$. To see this pick numbers α and β so that

$$\alpha < L < \beta < 1.$$

There must be an integer N so that

$$\alpha < \sqrt[n]{x_n} < \beta < 1$$

for all $n \geq N$. Forget half of this and focus on

$$\sqrt[n]{x_n} < \beta < 1.$$

Then we have

$$x_n < \beta^n$$

for all $n \geq N$ and it is clear now why $x_n \to 0$. ◀

This example suggests that the definition of limit might be weakened to handle situations where less is needed. This way we have a tool to discuss the limiting behavior of sequences that may not necessarily converge. Even if the sequence does converge this often offers a tool that can be used without first finding a proof of convergence.

We break the definition of sequence limit into two half-limits as follows.

Definition 2.44 (Lim Sup) A *limit superior* of a sequence $\{x_n\}$, denoted as $\limsup_{n\to\infty} x_n$, is defined to be the infimum of all numbers β with the following property:

$$\text{there is an integer } N \text{ so that } x_n < \beta \text{ for all } n \geq N.$$

Definition 2.45 (Lim Inf) A *limit inferior* of a sequence $\{x_n\}$, denoted as $\liminf_{n\to\infty} x_n$, is defined to be the supremum of all numbers α with the following property:

$$\text{there is an integer } N \text{ so that } \alpha < x_n \text{ for all } n \geq N.$$

Note. In interpreting this definition note that, by our usual rules on infs and sups, the values $-\infty$ and ∞ are allowed. If there are *no* numbers β with the property of the definition then the sequence is simply unbounded above. The infimum of the empty set is taken as ∞ and so

$$\limsup_{n\to\infty} x_n = \infty \Leftrightarrow \text{the sequence } \{x_n\} \text{ has no upper bound.}$$

On the other hand, if *every* number β has the property of the definition this means exactly that our sequence must be diverging to $-\infty$. The infimum of the set of *all* real numbers is taken as $-\infty$ and so

$$\limsup_{n\to\infty} x_n = -\infty \Leftrightarrow \text{the sequence } \{x_n\} \to -\infty.$$

The same holds in the other direction. A sequence that is unbounded below can be described by saying $\liminf_{n\to\infty} x_n = -\infty$. A sequence that diverges to ∞ can be described by saying $\liminf_{n\to\infty} x_n = \infty$.

We refer to these concepts as "upper limits" and "lower limits" or "extreme limits". They extend our theory describing the limiting behavior of sequences to allow precise descriptions of divergent sequences. Obviously we should establish very quickly that the upper limit is indeed greater than or equal to the lower limit since our language suggests this.

Theorem 2.46 *Let $\{x_n\}$ be a sequence of real numbers. Then*

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n.$$

Proof. If $\limsup_{n\to\infty} x_n = \infty$ or if $\liminf_{n\to\infty} x_n = -\infty$ we have nothing to prove. If not then take any number β larger than $\limsup_{n\to\infty} x_n$ and any number α smaller than $\liminf_{n\to\infty} x_n$. By definition then there is an integer N so that $x_n < \beta$ for all $n \geq N$ and an integer M so that $\alpha < x_n$ for all $n \geq M$. It must be true that $\alpha < \beta$. But β is *any* number larger than $\limsup_{n\to\infty} x_n$. Hence

$$\alpha \leq \limsup_{n\to\infty} x_n.$$

Similarly α is *any* number smaller than $\liminf_{n\to\infty} x_n$. Hence
$$\liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n$$
as required. ■

How shall we use the limit superior of a sequence $\{x_n\}$? If $\limsup_{n\to\infty} x_n = L$ then every number $\beta > L$ has the property that $x_n < \beta$ for all n large enough. This is because L is the infimum of such numbers β. On the other hand any number $b < L$ cannot have this property so $x_n \ge b$ for infinitely many indices n. Thus numbers slightly larger than L must be upper bounds for the sequence eventually. Numbers slightly less than L are not upper bounds eventually. To express this a little more precisely the number L is the limit superior of a sequence $\{x_n\}$ exactly when the following holds:

> For every $\varepsilon > 0$ there is an integer N so that $x_n < L + \varepsilon$ for all $n \ge N$ and $x_n > L - \varepsilon$ for infinitely many $n \ge N$.

The next theorem gives another characterization which is sometimes easier to apply. This version also better explains why we describe this notion as a "lim sup" and "lim inf".

Theorem 2.47 *Let $\{x_n\}$ be a sequence of real numbers. Then*
$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\}$$
and
$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\}$$

Proof. Let us prove just the statement for lim sups as the lim inf statement can be proved similarly.

Write
$$y_n = \sup\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\}.$$
Then $x_n \le y_n$ for all n and so, using the inequality promised in Exercise 2:13.5,
$$\limsup_{n\to\infty} x_n \le \limsup_{n\to\infty} y_n.$$
But $\{y_n\}$ is an increasing sequence and so
$$\limsup_{n\to\infty} y_n = \lim_{n\to\infty} y_n.$$
From this it follows that
$$\limsup_{n\to\infty} x_n \le \lim_{n\to\infty} \sup\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\}.$$

Let us now show the reverse inequality. If $\limsup_{n\to\infty} x_n = \infty$ then the sequence is unbounded above. Thus for all n

$$\sup\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\} = \infty$$

and so, in this case,

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\}$$

must certainly be true.

If $\limsup_{n\to\infty} x_n < \infty$ then take any number β larger than $\limsup_{n\to\infty} x_n$. By definition then there is an integer N so that $x_n < \beta$ for all $n \geq N$. It follows that

$$\lim_{n\to\infty} \sup\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\} \leq \beta.$$

But β is *any* number larger than $\limsup_{n\to\infty} x_n$. Hence

$$\lim_{n\to\infty} \sup\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\} \leq \limsup_{n\to\infty} x_n.$$

Having proved both inequalities the equality follows and the theorem is proved. ■

The connection between limits and extreme limits is very close. If a limit exists then the upper and lower limits must be the same.

Theorem 2.48 *Let $\{x_n\}$ be a sequence of real numbers. Then $\{x_n\}$ is convergent if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$ and these are finite. In this case*

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \lim_{n\to\infty} x_n.$$

Proof. Let $\varepsilon > 0$. If $\limsup_{n\to\infty} x_n = L$ then there is an integer N_1 so that $x_n < L + \varepsilon$ for all $n \geq N_1$. If it is also true that $\liminf_{n\to\infty} x_n = L$ then there is an integer N_2 so that $x_n > L - \varepsilon$ for all $n \geq N_2$. Putting these together we have

$$L - \varepsilon < x_n < L + \varepsilon$$

for all $n \geq N = \max\{N_1, N_2\}$. By definition then $\lim_{n\to\infty} x_n = L$.

Conversely if $\lim_{n\to\infty} x_n = L$ then for some N,

$$L - \varepsilon < x_n < L + \varepsilon$$

for all $n \geq N$. Thus

$$L - \varepsilon \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq L + \varepsilon.$$

Since ε is an arbitrary positive number we must have

$$L = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$$

as required. ■

In the exercises you will be asked to compute several lim sups and lim infs. This is just for familiarity with the concepts. Computations are not so important. What is important is the use of these ideas in theoretical developments. More critical is how these limit operations relate to arithmetic or order properties. The limit of a sum is the sum of the two limits. Is thus true for lim sups and lim infs? (See Exercise 2:13.9.) Do not skip these exercises.

Exercises

2:13.1 Complete Example 2.43 by showing that if $\{x_n\}$ is a sequence of positive numbers with the property that

$$\limsup_{n\to\infty} \sqrt[n]{x_n} < 1$$

then $x_n \to 0$. Show that if

$$\liminf_{n\to\infty} \sqrt[n]{x_n} > 1$$

then $x_n \to \infty$. What can you conclude if

$$\limsup_{n\to\infty} \sqrt[n]{x_n} > 1$$

or if

$$\liminf_{n\to\infty} \sqrt[n]{x_n} < 1?$$

2:13.2 Compute lim sups and lim infs for the following sequences

(a) $\{(-1)^n n\}$.
(b) $\{\sin(n\pi/8)\}$.
(c) $\{n\sin(n\pi/8)\}$.
(d) $\{[(n+1)\sin(n\pi/8)]/n\}$.
(e) $\{1+(-1)^n\}$.
(f) $\{r_n\}$ consists of all rational numbers in the interval $(0,1)$ arranged in some order.

2:13.3 Give examples of sequences of rational numbers $\{a_n\}$ with

(a) upper limit $\sqrt{2}$ and lower limit $-\sqrt{2}$,
(b) upper limit $+\infty$ and lower limit $\sqrt{2}$,
(c) upper limit π and lower limit e.

2:13.4 Show that

$$\limsup_{n\to\infty}(-x_n) = -(\liminf_{n\to\infty} x_n).$$

2:13.5 If two sequences $\{a_n\}$ and $\{b_n\}$ satisfy the inequality $a_n \le b_n$ for all sufficiently large n show that

$$\limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n.$$

$$\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n.$$

2:13.6 Show that $\lim_{n\to\infty} x_n = \infty$ if and only if

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \infty.$$

2:13.7 Show that if $\limsup_{n\to\infty} a_n = L$ for a finite real number L and $\varepsilon > 0$ then

$$a_n > L + \varepsilon$$

for only finitely many n and

$$a_n > L - \varepsilon$$

for infinitely many n.

2:13.8 Show that for any monotonic sequence $\{x_n\}$

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \lim_{n\to\infty} x_n$$

(including the possibility of infinite limits).

2:13.9 Show that for any sequences $\{a_n\}$ and $\{b_n\}$

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Give an example to show that the equality need not occur.

2:13.10 What is the correct version for the lim inf of Exercise 2:13.9?

2:13.11 Show that for any sequences $\{a_n\}$ and $\{b_n\}$ of positive numbers

$$\limsup_{n\to\infty}(a_n b_n) \le (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n).$$

Give an example to show that the equality need not occur.

2:13.12 What relation, if any, can you state for the lim sups and lim infs of a sequence $\{a_n\}$ and one of its subsequences $\{a_{n_k}\}$?

2:13.13 If a sequence $\{a_n\}$ has no convergent subsequences what can you state about the lim sups and lim infs of the sequence?

2:13.14 Let S denote the set of all real numbers t with the property that some subsequence of a given sequence $\{a_n\}$ converges to t. What is the relation between the set S and the lim sups and lim infs of the sequence $\{a_n\}$?

2:13.15 Prove the following assertion about the upper and lower limits for any sequence $\{a_n\}$ of positive real numbers:

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \le \liminf_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Give an example to show that each of these inequalities may be strict.

2:13.16 For any sequence $\{a_n\}$ write $s_n = (a_1 + a_2 + \dots a_n)/n$. Show that

$$\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} s_n \le \limsup_{n\to\infty} s_n \le \limsup_{n\to\infty} a_n.$$

Give an example to show that each of these inequalities may be strict.

2.14 Additional Problems for Chapter 2

2:14.1 Let α and β be positive numbers. Show that

$$\lim_{n\to\infty} \sqrt[n]{\alpha^n + \beta^n} = \max\{\alpha, \beta\}.$$

2:14.2 For any convergent sequence $\{a_n\}$ write $s_n = (a_1 + a_2 + \dots a_n)/n$, the sequence of averages. Show that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n.$$

Give an example to show that $\{s_n\}$ could converge even if $\{a_n\}$ diverges.

2:14.3 Let $a_1 = 1$ and define a sequence recursively by

$$a_{n+1} = \sqrt{a_1 + a_2 + \dots + a_n}.$$

Show that

$$\lim_{n\to\infty} \frac{a_n}{n} = 1/2.$$

2:14.4 Let $x_1 = \theta$ and define a sequence recursively by

$$x_{n+1} = \frac{x_n}{1 + x_n/2}.$$

For what values of θ is it true that $x_n \to 0$?

2:14.5 Let $\{a_n\}$ be a sequence of numbers in the interval $(0, 1)$ with the property that

$$a_n < \frac{a_{n-1} + a_{n+1}}{2}$$

for all $n = 2, 3, 4, \dots$. Show that this sequence is convergent.

2:14.6 For any convergent sequence $\{a_n\}$ write

$$s_n = \sqrt[n]{(a_1 a_2 \dots a_n)},$$

the sequence of geometric averages. Show that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n.$$

Give an example to show that $\{s_n\}$ could converge even if $\{a_n\}$ diverges.

2:14.7 If

$$\lim_{n\to\infty} \frac{s_n - \alpha}{s_n + \alpha} = 0$$

what can you conclude about the sequence $\{s_n\}$?

2:14.8 A function f is defined by

$$f(x) = \lim_{n\to\infty} \left(\frac{1-x^2}{1+x^2}\right)^n$$

at every value x for which this limit exists. What is the domain of the function?

2:14.9 A function f is defined by

$$f(x) = \lim_{n\to\infty} \frac{1}{x^n + x^{-n}}$$

at every value x for which this limit exists. What is the domain of the function?

2:14.10 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a positive function with a derivative f' that is everywhere continuous and negative. Apply Newton's method to obtain a sequence

$$x_1 = \theta \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Show that $x_n \to \infty$ for any starting value θ.

✂ **2:14.11** Let $f(x) = x^3 - 3x + 3$. Apply Newton's method to obtain a sequence

$$x_1 = \theta \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Show that for any positive integer p there is a starting value θ such that the sequence $\{x_n\}$ is periodic with period p.

2:14.12 A sequence $\{s_n\}$ is said to be *contractive* if there is a positive number $0 < r < 1$ so that

$$|s_{n+1} - s_n| \le r|s_n - s_{n-1}|$$

for all $n = 2, 3, 4, \ldots$.

(a) Show that the sequence defined by $s_1 = 1$ and $s_n = (4+s_{n-1})^{-1}$ for $n = 2, 3, \ldots$ is contractive.

(b) Show that every contractive sequence is Cauchy.

(c) Show that a sequence can satisfy the condition

$$|s_{n+1} - s_n| < |s_n - s_{n-1}|$$

for all $n = 2, 3, 4, \ldots$ and not be contractive, nor even convergent.

(d) Is every convergent sequence contractive?

✂ **2:14.13** The sequence defined recursively by

$$f_1 = 1, \; f_2 = 1 \quad f_{n+2} = f_n + f_{n+1}$$

is called the *Fibonacci sequence*. Let

$$r_n = f_{n+1}/f_n$$

be the sequence of ratios of successive terms of the Fibonacci sequence.

(a) Show that
$$r_1 < r_3 < r_5 \cdots < r_6 < r_4 < r_2.$$

(b) Show that $r_{2n} - r_{2n-1} \to 0$.

(c) This proves that the sequence $\{r_n\}$ converges. Can you find a way to determine that limit? (This is related to the roots of the equation $x^2 - x - 1 = 0$.)

2:14.14 A sequence of real numbers $\{x_n\}$ has the property that
$$(2 - x_n)x_{n+1} = 1.$$
Show that $\lim_{n\to\infty} x_n = 1$.

2:14.15 Let $\{a_n\}$ be an arbitrary sequence of positive real numbers. Show that
$$\limsup_{n\to\infty} \left(\frac{a_1 + a_{n+1}}{a_n}\right)^n \geq e.$$

2:14.16 Suppose that the sequence whose nth term is
$$s_n + 2s_{n+1}$$
is convergent. Show that $\{s_n\}$ is also convergent.

2:14.17 Show that the sequence
$$\sqrt{7}, \sqrt{7 - \sqrt{7}}, \sqrt{7 - \sqrt{7 + \sqrt{7}}}, \sqrt{7 - \sqrt{7 + \sqrt{7 - \sqrt{7}}}}, \ldots$$
converges and find its limit.

2:14.18 Let a_1 and a_2 be positive numbers and suppose that the sequence $\{a_n\}$ is defined recursively by
$$a_{n+2} = \sqrt{a_n} + \sqrt{a_{n+1}}.$$
Show that this sequence converges and find its limit.