Chapter 10

POWER SERIES

10.1 Introduction

One of the simplest and, arguably, the most important type of series of functions is the power series. This is a series of the form

$$\sum_0^\infty a_k x^k$$

or more generally

$$\sum_0^\infty a_k (x-c)^k.$$

It represents a notion of "infinitely long" polynomial

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \ldots.$$

The material we developed in Chapter 9 will allow us to show in this chapter that power series can be treated very much as if they were indeed polynomials in the sense that they can be integrated and differentiated term-by-term.

The main reason for developing this theory is that it allows a representation for functions as series. This enlarges considerably the class of functions that we can work with. Not all functions that arise in applications can be expressed as finite combinations of the elementary functions (i.e., as combinations of e^x, x^p, $\sin x$, $\cos x$, etc.). Thus if we remain at the level of an elementary calculus class we would be unable to solve many problems since we cannot express the functions needed for the solution in any way. For a large and important class of problems the functions that can be represented as power series (the so-called analytic functions) are precisely the functions needed.

10.2 Power Series: Convergence

We begin with the formal definition of power series.

Definition 10.1 Let $\{a_k\}$ be a sequence of real numbers and let $c \in \mathbb{R}$. A series of the form

$$\sum_{0}^{\infty} a_k(x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots$$

is called a *power series* centered at c. The numbers a_k are called the *coefficients* of the power series.

What can one say about the set of points on which the power series $\sum_0^\infty a_k(x-c)^k$ converges? It is immediately clear that the series converges at its center c. What possibilities are there? A collection of examples illustrates the methods and also essentially all of the possibilities.

Example 10.2 The simple example

$$\sum_{1}^{\infty} k^k x^k = x + 4x^2 + 27x^3 + \ldots$$

shows that a power series can diverge at every point other than its center. Observe that in this example $k^k x^k = (kx)^k$ does not approach 0 unless $x = 0$, so the series diverges for every $x \neq 0$ by the trivial test. Thus the set of convergence of this series is the set $\{0\}$. ◀

Example 10.3 The familiar geometric series

$$\sum_{k=0}^{\infty} x^k$$

should be considered the most elementary of all power series. We know that this series converges precisely on the interval $(-1, 1)$ and diverges everywhere else. ◀

Example 10.4 The series

$$\sum_{k=1}^{\infty} \frac{x^k}{k}$$

has as coefficients $a_k = 1/k$ and the root test[1] supplies

$$s = \limsup_k \sqrt[k]{|x|^k/k} = |x|.$$

(Verify this!) Thus the series converges on $(-1, 1)$ and diverges for $|x| > 1$. At the two endpoints of the interval $(-1, 1)$ a different test is required. We see that for $x = 1$ this is the familiar harmonic series and so diverges, while for $x = -1$ this is the familiar alternating harmonic series and so converges nonabsolutely. The interval of convergence is $[-1, 1)$. Observe that in this case, the series converges at only one of the two endpoints of the interval. ◀

Example 10.5 The series

$$\sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

converges on $[-1, 1]$ and diverges otherwise. Again the root test (or the ratio test) is helpful here. Simpler, though, is to notice that

$$\left|\frac{x^k}{k^2}\right| \le \frac{1}{k^2}$$

for all $|x| \le 1$ and so obtain convergence on $[-1, 1]$ by a comparison test with the convergent series $\sum_{k=0}^{\infty} 1/k^2$. If $|x| > 1$ the terms $|x^k/k^2| \to \infty$ and so, trivially, the series diverges. Note here that the series converges on the interval $[-1, 1]$ and is absolutely convergent there. ◀

Example 10.6 The root test applied to the series

$$\sum_1^k \frac{x^k}{k^k}$$

gives

$$\lim_{k\to\infty} \sqrt[k]{\frac{|x|^k}{k^k}} = |x| \lim_{k\to\infty} \frac{1}{k} = 0.$$

(The ratio test can also be used here.) It follows that the series converges for all $x \in \mathbb{R}$. Perhaps an easier method in this particular example is to use the comparison test and the fact that

$$\left|\frac{x}{k}\right|^k < \frac{1}{2^k} \text{ when } k \ge 2|x|.$$

[1] The form of the root test needed to discuss power series uses the limit superior. For that the study of Section 2.13 may be required.

Thus the series converges at any x by comparison with a geometric series. Thus the set of convergence of this series is $(-\infty, \infty)$, again as in the previous examples an interval. ◀

In general, as these examples seem to suggest, the set of points of convergence of a power series forms an interval and an application of the root test is an essential tool in determining that interval. Let us apply this test to the series

$$\sum_0^\infty a_k(x-c)^k.$$

Let $s = \limsup_k \sqrt[k]{|a_k|}$. Then

$$\limsup_k \sqrt[k]{|a_k||x-c|^k} = \limsup_k \sqrt[k]{|a_k|}|x-c| = s|x-c|.$$

By the root test the series converges absolutely if $s|x-c| < 1$ and diverges if $s|x-c| > 1$.

If $0 < s < \infty$, then the series converges on the interval

$$(c - 1/s, c + 1/s)$$

and diverges for x outside the interval

$$[c - 1/s, c + 1/s].$$

The root test is inconclusive about convergence at the endpoints $x = c \pm 1/s$ of these intervals. The interval of convergence is thus one of the four possibilities

$$(c - 1/s, c + 1/s) \text{ or } [c - 1/s, c + 1/s) \text{ or}$$

$$(c - 1/s, c + 1/s] \text{ or } [c - 1/s, c + 1/s].$$

If $s = 0$ then the series converges for all values of x. We could say that the interval of convergence is $(-\infty, \infty)$ in this case. If $s = \infty$ then the series converges for no values of x other than the trivial value $x = c$. We could say that the interval of convergence is the degenerate "interval" $\{c\}$.

Thus the set of convergence is an interval centered at c. This interval might be degenerate (consisting of only the center), might be all of the real line, and may contain none, one, or both of its endpoints.

Our next theorem summarizes the discussion of convergence to this point. We first give a formal definition.

Definition 10.7 Let $\sum_0^\infty a_k(x-c)^k$ be a power series. Then the

number
$$R = \frac{1}{\limsup_k \sqrt[k]{|a_k|}}$$
is called the *radius of convergence* of the series. (Here we interpret $R = \infty$ if $\limsup_k \sqrt[k]{|a_k|} = 0$ and $R = 0$ if $\limsup_k \sqrt[k]{|a_k|} = \infty$.)

Note. This book deals with *real* analysis, but a full theory of power series fits more naturally into the setting of *complex* analysis. In that setting, a power series converges in a "circle of convergence" centered at a complex number c in the complex plane and with radius
$$R = \frac{1}{\limsup_k \sqrt[k]{|a_k|}}.$$
This explains the origin of the term "radius of convergence".

Theorem 10.8 *Let $\sum_0^\infty a_k(x-c)^k$ be a power series with radius of convergence R.*

1. *If $R = 0$, then the series converges only at $x = c$.*
2. *If $R = \infty$, then the series converges absolutely for all x.*
3. *If $0 < R \leq \infty$, then the series converges absolutely for all x satisfying $|x-c| < R$ and diverges for all x satisfying $|x-c| > R$.*

Proof. We first consider the case $R = 0$. Here $\limsup_k \sqrt[k]{|a_k|} = \infty$ so, for $x \neq c$,
$$\limsup_k \sqrt[k]{|a_k||x-c|^k} = |x-c| \limsup_k \sqrt[k]{|a_k|} = \infty.$$
By the root test the series cannot converge for $x \neq c$. The other cases are similarly established by the root test as in the discussion following our examples. ∎

In general then a power series
$$\sum_{k=0}^\infty a_k x^k$$
with a finite radius of convergence R must have as its set of convergence one of the four intervals
$$(-R, R), \quad [-R, R], \quad (-R, R] \text{ or } \quad [-R, R).$$
As we have seen from the examples each of these four cases can occur. The only other possibilities are for series with radius of convergence

$R = 0$ in which case the set of convergence is trivially $\{0\}$ or $R = \infty$ in which case the set of convergence is the entire real line. Note too that if the series converges absolutely at $x = R$ or at $x = -R$ then it must converge absolutely on all of $[-R, R]$. It is possible, though, for the series to converge nonabsolutely at one endpoint but not at the other.

Exercises

10:2.1 Find the radius of convergence for each of the following series.

(a) $\sum_{0}^{\infty} (-1)^k x^{2k}$

(b) $\sum_{0}^{\infty} \frac{x^k}{k!}$

(c) $\sum_{0}^{\infty} k x^k$

(d) $\sum_{0}^{\infty} k! x^k$

10:2.2 Another expression for R is sometimes available. If

$$\lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right|$$

exists or equals ∞, then show that the following expression also gives the radius of convergence of a power series:

$$R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| .$$

10:2.3 For the examples

$$\sum_{k=0}^{\infty} x^k \quad , \quad \sum_{k=1}^{\infty} \frac{x^k}{k} \quad , \quad \sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

verify in each case that

$$R = \lim_{k} \left| \frac{a_k}{a_{k+1}} \right| = 1.$$

10:2.4 For the series

$$\sum_{1}^{\infty} k^k x^k \quad \text{and} \quad \sum_{1}^{\infty} \frac{x^k}{k^k}$$

check that the radius of convergence is $R = 0$ and ∞ respectively.

10:2.5 Give an example of a power series $\sum_0^\infty a_k x^k$ for which the radius of convergence R satisfies

$$R = \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_k|}}$$

but

$$\lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right|$$

does not exist.

10:2.6 Give an example of a power series $\sum_0^\infty a_k x^k$ for which the radius of convergence R satisfies

$$\liminf_k \left| \frac{a_{k+1}}{a_k} \right| < R < \limsup_k \left| \frac{a_{k+1}}{a_k} \right| .$$

10:2.7 Give an example of a power series $\sum_0^\infty a_k x^k$ with radius of convergence 1 that is nonabsolutely convergent at both endpoints 1 and -1 of the interval of convergence.

10:2.8 Give an example of a power series $\sum_0^\infty a_k x^k$ with interval of convergence exactly $[-\sqrt{2}, \sqrt{2})$.

10:2.9 If the power series $\sum_0^\infty a_k x^k$ has a radius of convergence R what must be the radius of convergence of the series $\sum_{k=0}^\infty k a_k x^k$ and $\sum_{k=1}^\infty k^{-1} a_k x^k$?

10:2.10 If the coefficients $\{a_k\}$ of a power series $\sum_0^\infty a_k x^k$ form a bounded sequence show that the radius of convergence is at least 1.

10:2.11 If the power series $\sum_0^\infty a_k x^k$ has a radius of convergence R_a and the power series $\sum_0^\infty b_k x^k$ has a radius of convergence R_b and $|a_k| \leq |b_k|$ for all k sufficiently large what relation must hold between R_a and R_b?

10:2.12 If the power series $\sum_0^\infty a_k x^k$ has a radius of convergence R what must be the radius of convergence of the series $\sum_{k=0}^\infty a_k x^{2k}$?

10:2.13 If the power series $\sum_0^\infty a_k x^k$ has a finite positive radius of convergence show that the radius of convergence of the series $\sum_{k=0}^\infty a_k x^{k^2}$ is 1.

10:2.14 Find the radius of convergence of the series

$$\sum_{k=0}^\infty \frac{(\alpha k)!}{(k!)^\beta} x^k$$

where α and β are positive and α is an integer.

10:2.15 Let $\{a_k\}$ be a sequence of real numbers and let $x_0 \in \mathbb{R}$. Suppose there exists $M > 0$ such that $|a_k x_0^k| \leq M$ for all $k \in \mathbb{N}$. Prove $\sum_0^\infty a_k x^k$ converges absolutely for all x satisfying the inequality $|x| < |x_0|$. What can you say about the radius of convergence of this series?

10.3 Uniform Convergence

So far we have reached a complete understanding of the nature of the set of convergence of any power series. In order to apply many of our theorems of Chapter 9 to questions concerning term-by-term integration or differentiation of power series, we need to check questions related to the *uniform* convergence of power series. Our next theorem does this and also summarizes the discussion of convergence to this point.

We repeat the convergence results of Theorem 10.8 but now add a discussion of uniform convergence.

Theorem 10.9 *Let $\sum_0^\infty a_k(x-c)^k$ be a power series with radius of convergence R.*

1. *If $R = 0$, then the series converges only at $x = c$.*
2. *If $R = \infty$, then the series converges absolutely and uniformly on any compact interval $[a, b]$.*
3. *If $0 < R < \infty$, then the series converges absolutely and uniformly on any interval $[a, b]$ contained entirely inside the interval $(c - R, c + R)$.*

Proof. To verify (2) and (3), let us choose $0 < \rho < R$ so that the interval $[a, b]$ is contained inside the interval $(c - \rho, c + \rho)$. Fix $\rho_0 \in (\rho, R)$. Then

$$\limsup_k \sqrt[k]{|a_k|} = \frac{1}{R} < \frac{1}{\rho_0}.$$

Thus there exists $N \in \mathbb{N}$ such that

$$\sqrt[k]{|a_k|} < \frac{1}{\rho_0} \quad \text{for all } k \geq N. \tag{1}$$

For $k \geq N$ and $|x - c| \leq \rho$ we calculate

$$|a_k(x-c)^k| \leq |a_k|\rho^k < \left(\frac{\rho}{\rho_0}\right)^k,$$

the last inequality following from (1).

Now since $\rho/\rho_0 < 1$, it follows that

$$\sum_0^\infty \left(\frac{\rho}{\rho_0}\right)^k < \infty.$$

It now follows from the Weierstrass M-test (Theorem 9.16) that the series converges absolutely and uniformly on the set $\{x : |x - c| < \rho\}$ and hence also on the subset $[a, b]$. ■

If the interval of convergence of a power series is $(-R, R)$ then certainly the assertion (3) of Theorem 10.9 is the best that can be made. (See Exercise 10:3.3.) The geometric series $\sum_{n=0}^{\infty} x^n$ furnishes the clearest example of this. This series converges on $(-1, 1)$ but does not converges uniformly on the entire interval of convergence $(-1, 1)$. It does, however, converge uniformly on any $[a, b] \subset (-1, 1)$.

To improve on this we can ask the following: if R is the radius of convergence of a power series and the interval of convergence is $[-R, R]$ or $(-R, R]$ or $[-R, R]$ can uniform convergence be extended to the endpoint(s)? If the convergence at an endpoint R (or $-R$) is absolute then an application of the Weierstrass M-test shows immediately that the convergence is absolute and uniform on $[-R, R]$. For non absolute convergence a more delicate test is needed and we need to appeal to material developed in Section 9.3.3. The theorem contains, for easy reference, a repeat of the third assertion in Theorem 10.9.

Theorem 10.10 *Suppose that the power series $\sum_0^{\infty} a_k(x-c)^k$ has a finite and positive radius of convergence R and an interval of convergence I.*

1. *If $I = [c-R, c+R]$ then the series converges uniformly (but not necessarily absolutely) on $[c-R, c+R]$.*
2. *If $I = (c-R, c+R]$ then the series converges uniformly (but not necessarily absolutely) on any interval $[a, c+R]$ for all $c-R < a < c+R$.*
3. *If $I = [c-R, c+R)$ then the series converges uniformly (but not necessarily absolutely) on any interval $[c-R, b]$ for all $c-R < b < c+R$.*
4. *If $I = (c-R, c+R)$ then the series converges uniformly and absolutely on any interval $[a, b]$ for $c-R < a < b < c+R$.*

Proof. For the purposes of the proof we can take $c = 0$. Let us examine the case $I = (c-R, c+R] = (-R, R]$ which is typical. Consider the intervals $[a, 0]$ for $-R < a < 0$ and $[0, R]$. The uniform convergence of the series on $[a, 0]$ is clear since this is contained entirely inside the interval of convergence.

Now we examine uniform convergence on $[0, R]$. We consider the series

$$\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} A_k t^k$$

where $A_k = a_k R^k$ and $t = (x/R)$. The series $\sum_{k=0}^{\infty} A_k t^k$ converges for $0 \leq t \leq 1$ by our assumptions. Note that $\sum_{k=0}^{\infty} A_k$ is convergent while the sequence $\{t^k\}$ converges monotonically on the interval $[0,1]$. By a variant of Theorem 9.19 (Exercise 9:3.26) this series converges uniformly for $t \in [0,1]$. This translates easily to the assertion that our original series converges uniformly for $x \in [0,R]$. Thus since the series converges uniformly on $[a,0]$ and on $[0,R]$ we have obtained the uniform convergence on $[a,R]$ as required. The other cases are similarly handled. ∎

Exercises

10:3.1 Characterize those power series $\sum_0^{\infty} a_k(x-c)^k$ that converge uniformly on $(-\infty,\infty)$.

10:3.2 Show that if $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely at a point $x_0 > 0$ then the convergence of the series is uniform on $[-x_0, x_0]$.

10:3.3 Show that if $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on an interval $(-r,r)$ then it must in fact converge uniformly on $[-r,r]$. Deduce that if the interval of convergence is exactly of the form $(-R,R)$, or $[-R,R)$ or $[-R,R)$ then the series cannot converge uniformly on the entire interval of convergence.

10.4 Functions Represented by Power Series

Suppose now that a power series $\sum_0^{\infty} a_k(x-c)^k$ has positive or infinite radius of convergence R. Then this series represents a function f on (at least) the interval $(c-R, c+R)$:

$$f(x) = \sum_0^{\infty} a_k(x-c)^k \quad \text{for } |x-c| < R. \tag{2}$$

If the series converges at one or both endpoints then this represents a function on $[c-R, c+R)$ or $(c-R, c+R]$ or $[c-R, c+R]$.

What can one say about the function f? In terms of the questions that have motivated us throughout Chapter 9 we can ask

1. Is the function f continuous on its domain of definition?
2. Can f be differentiated by termwise differentiation of its series?
3. Can f be integrated by termwise integration of its series?

We address each of these questions and find that generally the answer to each is yes.

10.4.1 Continuity of Power Series

A power series may represent a function on an interval. Is that function necessarily continuous?

Theorem 10.11 *A function f represented by a power series*

$$f(x) = \sum_0^\infty a_k(x-c)^k \quad \textit{for } |x-c| < R. \tag{3}$$

is continuous on its interval of convergence.

Proof. This follows from Theorem 10.10. For example if the interval of convergence is $(c-R, c+R]$ then we can show that f is continuous at each point of this interval. Since convergence is uniform on $[c, c+R]$ and since each of the functions $a_k(x-c)^k$ is continuous on $[c, c+R]$, the same is true of the function f (Corollary 9.23). For any point $x_0 \in (c-R, c)$ we can similarly prove that f is continuous at x_0 in the same way by noting that the series converges uniformly on an interval $[a, c]$ where a is chosen so that $c - R < a < x_0 < c$. ■

Example 10.12 The series

$$f(x) = \sum_{k=1}^\infty \frac{x^k}{k}$$

converges at every point of the interval $[-1, 1)$. Consequently this function is continuous at every point of that interval. We shall show in the next section that the identity

$$\log(1-x) = \sum_{k=1}^\infty \frac{x^k}{k}$$

holds for all $x \in (-1, 1)$ (by integrating the geometric series term by term). Since we are also assured of continuity at the endpoint $x = -1$ we can conclude that

$$\log 2 = \sum_{k=1}^\infty \frac{(-1)^k}{k}.$$

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10.4.2 Integration of Power Series

If a function is represented by a power series is it possible to integrate that function by integrating the power series term by term?

Theorem 10.13 *Let a function f be represented by a power series*

$$f(x) = \sum_0^\infty a_k(x-c)^k$$

with an interval of convergence I. Then for every point x in that interval f is integrable on $[c,x]$ (or $[x,c]$ if $x < c$) and

$$\int_c^x f(t)\, dt = \sum_0^\infty \frac{a_k}{k+1}(x-c)^{k+1}.$$

Proof. Let x be a point in the interval of convergence. The convergence of the series $\sum_0^\infty a_k(x-c)^k$ is uniform on $[c,x]$ (or on $[x,c]$ if $x < c$), so the series can be integrated term-by-term (Theorem 9.29). ■

Example 10.14 The geometric series

$$\frac{1}{1-x} = \sum_{k=0}^\infty x^k$$

has radius of convergence 1 and so can be integrated term by term provided we stay inside the interval $(-1,1)$. Thus

$$\log(1-x) = \int_0^x \frac{1}{1-t}\, dt = \sum_0^\infty \frac{1}{k+1} x^{k+1}$$

for all $-1 < x < 1$. We would not be able to conclude from this theorem that the integral can be extended to the endpoints of $(-1,1)$. The new series, however, also converges at $x = -1$ and so we can apply Theorem 10.11 to show that the identity just proved is actually valid on $[-1,1)$. ◀

10.4.3 Differentiation of Power Series

If a function is represented by a power series is it possible to differentiate that function by differentiating the power series term by term?

Note that for continuity and integration we were able to prove Theorems 10.11 and 10.13 immediately from general theorems on uniform convergence. To prove a theorem on term-by-term differentiation, we need to check uniform convergence of the series of *derivatives.* The following lemma gives us what we need.

Lemma 10.15 *Let $\sum_0^\infty a_k(x-c)^k$ have radius of convergence R.*

Then the series

$$\sum_1^\infty k a_k (x-c)^{k-1}$$

obtained via term-by-term differentiation also has the same radius of convergence R.

Proof. The radius of convergence of the series is given by

$$R = \frac{1}{\limsup_k \sqrt[k]{|a_k|}}.$$

The radius of convergence of the differentiated series is given by

$$R' = \frac{1}{\limsup_k \sqrt[k]{|k a_k|}}.$$

But since $\sqrt[k]{k} \to 1$ as $k \to \infty$ we see immediately that these two numbers are equal. (They may be both zero or both infinite.) ■

Theorem 10.16 *Let $\sum_0^\infty a_k(x-c)^k$ have radius of convergence $R > 0$, and let*

$$f(x) = \sum_0^\infty a_k (x-c)^k$$

whenever $|x-c| < R$. Then f is differentiable on $(c-R, c+R)$ and

$$f'(x) = \sum_1^\infty k a_k (x-c)^{k-1}$$

for each $x \in (c-R, c+R)$.

Proof. It follows from the preceding lemma that the series

$$\sum_1^\infty k a_k (x-c)^{k-1}$$

has radius of convergence R. Thus this series converges uniformly on any compact interval $[a,b]$ contained in $(c-R, c+R)$. Since each value of x in $(c-R, c+R)$ can be placed inside some such interval $[a,b]$ it now follows immediately from Corollary 9.35 that $f'(x) = \sum_1^\infty k a_k (x-c)^{k-1}$ whenever $|x-c| < R$. ■

We can apply the same argument to the differentiated series, and differentiate once more. From the expansion

$$f'(x) = \sum_1^\infty k a_k (x-c)^{k-1}$$

we obtain a formula for $f''(x)$:

$$f''(x) = \sum_{2}^{\infty} k(k-1)a_k(x-c)^{k-2}.$$

Let us express explicitly the formulas of $f(x)$, $f'(x)$ and $f''(x)$.

$$\begin{aligned} f(x) &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots \\ f'(x) &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \dots \\ f''(x) &= 2a_2 + 3 \cdot 2a_3(x-c) + \dots \end{aligned}$$

These expressions are valid in the interval $(c-R, c+R)$. For $x = c$ we obtain

$$\begin{aligned} f(c) &= a_0 \\ f'(c) &= a_1 \\ f''(c) &= 2a_2. \end{aligned}$$

If we continue in this way, we can obtain power series expansions for all the derivatives of f. This results in the following theorem. The proof (which requires mathematical induction) is left as Exercise 10:4.1.

Theorem 10.17 *Let $\sum_0^\infty a_k(x-c)^k$ have radius of convergence $R > 0$. Then the function*

$$f(x) = \sum_{0}^{\infty} a_k(x-c)^k$$

has derivatives of all orders and these derivatives can be calculated by repeated term-by-term differentiation. The coefficients a_k are related to the derivatives of f at $x = c$ by the formula

$$a_k = \frac{f^{(k)}(c)}{k!}.$$

Uniqueness of Power Series From Theorem 10.17 we deduce that any two power series representations of a function must be identical. Note that the centers have to be the same for this to be true.

Corollary 10.18 *Suppose two power series*

$$f(x) = \sum_{0}^{\infty} a_k(x-c)^k$$

and

$$g(x) = \sum_{0}^{\infty} b_k(x-c)^k$$

agree on some interval centered at c, that is $f(x) = g(x)$ for $|x-c| < \rho$ and some positive $< \rho$. Then $a_k = b_k$ for all $k = 0, 1, 2, \ldots$.

Proof. It follows immediately from Theorem 10.17 that

$$a_k = \frac{f^{(k)}(c)}{k!} = \frac{g^{(k)}(c)}{k!} = b_k$$

for all $k = 0, 1, 2, \ldots$. ■

Example 10.19 The series for the exponential function

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

reveals one of the key facts about the exponential function, namely that it is its own derivative. Note simply that

$$\frac{d}{dx} e^x = \frac{d}{dx} \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{d}{dx} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

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Example 10.20 The material in this section can also be used to obtain the power series expansion of the exponential function. Suppose that we know that the exponential function $f(x) = e^x$ does in fact have a power series expansion

$$f(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Then the coefficients must be given by the formulas we have obtained, namely

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

But for $f(x) = e^x$ it is clear that $f^{(k)}(0) = 1$ for all k and so the series must be indeed be given by $a_k = 1/k!$ as we well know. But how can we be assured that the exponential function does have a power series expansion? This argument shows only that if there is a series then that series is precisely $\sum_{k=0}^{\infty} \frac{x^k}{k!}$. There remains the possibility that there may not be a series after all. This is the subject of the next section. ◀

10.4.4 Power Series Representations

Corollary 10.18 shows that if one can obtain a power series representation for a function f by any means whatsoever, then that series must have its coefficients given by the equations $a_k = f^{(k)}(c)/k!$. In particular a power series representation for f about a given point must be unique.

Example 10.21 For example, the formula for the sum of a geometric series can be used to show that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^j x^{2j} + \ldots.$$

Thus this series represents the function $f(x) = \frac{1}{1+x^2}$ on the interval $(-1, 1)$. Note that the coefficients a_k are zero if k is odd and that $a_{2j} = (-1)^j$ for $k = 2j$ even. It now follows *automatically* that for even integers $k = 2j$

$$\frac{f^{(k)}(0)}{k!} = a_k = (-1)^j$$

while all the odd derivatives are zero. Thus

$$\frac{d^k}{dx^k}\left(\frac{1}{1+x^2}\right) = 0 \quad \text{at } x = 0$$

if k is odd and

$$\frac{d^k}{dx^k}\left(\frac{1}{1+x^2}\right) = (-1)^j (2j)! \quad \text{at } x = 0$$

if $k = 2j$ is even. One need not compute derivatives to obtain this result. ◀

Note. There is a curious fact here which should be puzzled upon. The formula

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^j x^{2j} + \ldots$$

is valid precisely for $-1 < x < 1$. But the function on the right hand side of this identity is defined for all values of x. We might have hoped for a representation valid for all x but we do not obtain one!

Sometimes the easiest way to obtain a power series expansion formula for a function is by using the formula $a_k = f^{(k)}(c)/k!$. For example, this is how we obtained the power series expansion for $f(x) = e^x$. We compute $f^{(k)}(x) = e^x$ for $k = 0, 1, 2 \ldots$, so $f^{(k)}(0) = 1$ for all k. Thus the series expansion for this function (if it has a series expansion) would have to be

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_0^\infty \frac{x^k}{k!}. \tag{4}$$

Note that the series converges for all $x \in \mathbb{R}$. In the next section we will show how to verify that the equality holds for all x.

If we had wanted a formula for $g(x) = e^{x^2}$ we might have used the same idea and determined all the derivatives $g^{(k)}(0)$. It would be simplest, however, to just substitute x^2 for x in the expansion (4), obtaining

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots = \sum_0^\infty \frac{x^{2k}}{k!}. \tag{5}$$

Also, from this expansion we can readily obtain an expansion for $2xe^{x^2}$ in either of two ways: We can multiply the expansion in (5) by $2x$ giving

$$2xe^{x^2} = 2x + 2x^3 + \frac{2x^5}{2!} + \frac{2x^7}{3!} + \cdots = \sum_0^\infty \frac{2x^{2k+1}}{k!}.$$

Alternatively we can use Theorem 10.16 and differentiate (5) term-by-term giving

$$2xe^{x^2} = \frac{d}{dx}e^{x^2} = 2x + \frac{4x^3}{2!} + \frac{6x^5}{3!} + \frac{8x^7}{4!} \cdots = \sum_0^\infty \frac{2x^{2k+1}}{k!}.$$

The reader may wish instead to try to obtain these expansions directly by using the formula $a_k = f^{(k)}(c)/k!$.

Exercises

10:4.1 Provide the details in the proof of Theorem 10.17.

10:4.2 Obtain expansions for

$$\frac{x}{1+x^2} \quad \text{and} \quad \frac{x}{(1+x^2)^2}.$$

10:4.3 Obtain expansions for

$$\frac{1}{1+x^3} \quad \text{and} \quad \frac{x^2}{1+x^3}.$$

10:4.4 Find a power series expansion about $x = 0$ for the function

$$f(x) = \int_0^1 \frac{1-e^{-sx}}{s}\,ds$$

10:4.5 The function

$$J_0(x) = \sum_{k=0}^\infty (-1)^k \frac{x^{2k}}{(k!)^2 2^{2k}}$$

is called a Bessel function of order zero of the first kind. Show that this is defined for all values of x. The function $J_1(x) = -J_0'(x)$ is called a Bessel function of order one of the first kind. Find a series expansion for $J_1(x)$.

10:4.6 Let

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

have a positive radius of convergence. If the function f is even (i.e., if $f(-x) = f(x)$ for all x) what can you deduce about the coefficients a_k? What can you deduce if the function is odd (i.e., if $f(-x) = -f(x)$ for all x)?

10:4.7 Let

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

have a positive radius of convergence. If zero is a critical point (i.e., if $a_1 = 0$) and if $a_2 > 0$ then the point $x = 0$ is a strict local minimum. Prove this and also formulate and prove a generalization of this that would allow $a_2 = a_3 = a_4 = \cdots = a_{N-1} = 0$ and $a_N \neq 0$.

10.5 The Taylor Series

We have seen that if a power series $\sum_0^\infty a_k(x-c)^k$ converges on an interval I then the series represents a function f that has derivatives of all orders. In particular the coefficients a_k are related to the derivatives of f at c:

$$a_k = \frac{f^{(k)}(c)}{k!}.$$

One then calls the series the *Taylor series* for f about the point $x = c$.

Let us turn the question around:

> What functions f have a Taylor series representation in their domain?

We see immediately that such a function must be infinitely differentiable in a neighborhood of c since for such a series to be valid we know that all of the derivatives $f^{(k)}(c)$ must exist. But is that enough?

If we start with a function that has derivatives of all orders on an interval I containing the point c, and write the series

$$\sum_0^\infty \frac{f^{(k)}(c)}{k!}(x-c)^k,$$

we might expect that this is exactly the representation we want. Indeed *if there is a valid representation* then this must be the one, since such representations are unique. But can we be sure the series converges to f on I? Or even that the series converges at all on I. The answer to both questions is "no".

Example 10.22 Consider, for example, the function

$$f(x) = 1/(1 + x^2).$$

This function is infinitely differentiable on all of the real line. Its Taylor series about $x = 0$ is, as we have seen in Example 10.21,

$$1 - x^2 + x^4 - x^6 + \cdots = \sum_0^\infty (-1)^k x^{2k}.$$

This series converges for $|x| < 1$ but diverges for $|x| \geq 1$. It does represent f on the interval $(-1, 1)$ but not on the full domain of f. Indeed there can be no series $\sum_{k=0}^\infty a_k x^k$ that represents f on $(-\infty, \infty)$ since that series would agree with this present series on $(-1, 1)$ and so could not be any different. We could, however, hope for series $\sum_{k=0}^\infty a_k (x - c)^k$ centered at different points c that might work. ◀

Worse situations are possible. For example, there are infinitely differentiable functions whose Taylor series have zero radius of convergence for every c; for these functions

$$\sum_0^\infty \frac{f^{(k)}(c)}{k!}(x - c)^k$$

diverges except at $x = c$ and this is true for all $c \in \mathbb{R}$.[2] For these functions the Taylor series cannot represent the function.

Another unpleasant situation occurs when a Taylor series converges to the wrong function. This possibility seems even more startling!

Example 10.23 Consider the function

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ e^{-1/x^2}, & \text{if } x \neq 0. \end{cases}$$

Exercise 10:5.4 provides an outline for showing that f is infinitely differentiable on the real line, and that $f^{(k)}(0) = 0$ for $k = 1, 2, 3, \ldots$. Thus the Taylor series for f about $x = 0$ takes the form $\sum_0^\infty 0x^k$ with all coefficients equal to zero. This series converges to the zero function on the real line, so it does not represent f except at the origin, even though the series converges for all x. ◀

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[2]See D. Morgenstern, Math. Nach. **12** (1954), p. 74. One finds here that in a certain sense "most" infinitely differentiable functions have this property!

10.5.1 Representing a Function by a Taylor Series

The preceding discussion shows that one should not automatically assume that a Taylor series for a function f represents f. It is true, however, that the developments in the earlier sections of this chapter help us see that many of the familiar Taylor series encountered in elementary calculus are valid.

Example 10.24 For example, starting with the geometric series

$$\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k,$$

we can apply Theorem 10.13 on integrating a power series term-by-term to obtain, for $|x| < 1$,

$$\begin{aligned} \ln(1+x) &= \int_1^x \frac{1}{1+t}\,dt = \sum_{k=0}^{\infty}\int_0^x (-1)^k t^k\,dt \\ &= \sum_0^{\infty}\frac{(-1)^k}{k+1}x^{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots . \end{aligned}$$

We can notice that the integrated series converges at $x = 1$ and so the convergence is uniform on $[0, 1]$. It follows that the representation is valid for $x \in (-1, 1]$ but for no other points. In this case we obtained a valid Taylor series expansion by integrating a series expansion that we already knew to be valid. ◀

To study the convergence of a Taylor series in general, let us return to fundamentals. Let f be infinitely differentiable in a neighborhood of c. For $n = 0, 1, 2, \dots$ let

$$P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k.$$

The polynomial P_n is called the *n-th Taylor polynomial* of f at c. The difference $R_n(x) = f(x) - P_n(x)$ is called the *n-th remainder function*. In order for the Taylor series about c to converge to f on an interval I, it is necessary and sufficient that $R_n \to 0$ pointwise on I.

Example 10.25 We know that the geometric series represents the function $f(x) = (1-x)^{-1}$ on the interval $(-1, 1)$. We could also prove this result by relying on the remainder term. For $x \neq 1$ and $n = 0, 1, 2, \dots$ we have

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \frac{x^{n+1}}{1-x}.$$

Here

$$P_n(x) = 1 + x + x + x^2 + \cdots + x^n$$

and

$$R_n(x) = \frac{x^{n+1}}{1-x}.$$

For $|x| < 1$, $R_n(x) \to 0$ as $n \to \infty$. But we have

$$f(x) = P_n(x) + R_n(x)$$

and so the Taylor series for $f(x) = 1/(1-x)$ represents f on the interval $(-1, 1)$. For $|x| \geq 1$, the remainder term does not tend to zero and (as before) we see that the representation is confined to the interval $(-1, 1)$. ◀

In a more general situation than this example we would not have an explicit formula for the remainder term. How then should we be able to show that the remainder term tends to zero? For functions that are infinitely differentiable in a neighborhood I of c, the various expressions we obtained in Section 7.12 for the remainder functions R_n can be used. In particular, Lagrange's form of the remainder allows us to write for $n = 0, 1, 2, 3, \ldots$

$$f(x) = P_n(x) + \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1},$$

where z is between x and c. With some information on the size of the derivatives $f^{(n+1)}(z)$ we can show that this remainder term tends to zero. The integral form of the remainder term, gives us

$$f(x) = P_n(x) + \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\,dt.$$

Again information on the size of the derivatives $f^{(n+1)}(t)$ might show that this remainder term tends to zero.

Example 10.26 Let us justify the familiar Taylor series for $\sin x$:

$$\sin x = \sum_0^\infty \frac{(-1)^k}{(2k+1)!}x^{2k+1}. \tag{6}$$

The remainder term is not expressible in any simple way but can be estimated by using the Lagrange's form of the remainder. The coefficients

$$\frac{(-1)^k}{(2k+1)!}$$

are easily verified by calculating successive derivatives of $f(x) = \sin x$

and using the formulas

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

To check convergence of the series, apply Lagrange's form for $R_n(x)$: For each $x \in \mathbb{R}$, there exists z such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}.$$

Now $|f^{(n+1)}(z)|$ equals either $|\cos z|$ or $|\sin z|$ (depending on n) so, in either case, $|f^{(n+1)}(z)| \leq 1$, and

$$|R_n(x)| \leq |x|^{n+1}/(n+1)!.$$

Since $|x|^{n+1}/(n+1)! \to 0$ as $n \to \infty$ for all $x \in \mathbb{R}$, we can see that the remainder term $|R_n(x)| \to 0$ as $n \to \infty$ for all $x \in \mathbb{R}$. Thus the series representation is completely justified for all real x.

Observe that our estimate for $|R_n(x)|$,

$$|R_n(x)| \leq |x|^{n+1}/(n+1)!,$$

gives also a sense of the *rate* of convergence of the series for fixed x. For example, for $|x| \leq 1$ we find

$$|R_n(x)| \leq 1/(n+1)!.$$

Thus, if we want to calculate $\sin x$ on $(-1, 1)$ to within .01, we need take only the first 5 terms of the series ($n = 4$) to achieve that degree of accuracy.

Had we used the integral form for $R_n(x)$ we would have obtained a similar estimate. We leave that calculation as Exercise 10:5.1.
◀

10.5.2 Analytic Functions

The class of functions that can be represented as power series is not large. As we have remarked the class of infinitely differentiable functions is much larger. The terminology that is commonly used for this very special class of functions is given by the definition.

Definition 10.27 A function f whose Taylor series converges to f in a neighborhood of c is said to be *analytic at c*.

The functions commonly encountered in elementary calculus are generally analytic except at certain "obviously non-analytic points": e.g., $|x|$ is not analytic at $x = 0$, and $1/(1-x)$ is not analytic at $x = 1$. These functions fail to have even a first derivative at the point in question. Similarly a function such as $f(x) = |x|^3$ cannot be analytic at $x = 0$ because, while $f'(0)$ and $f''(0)$ exist, $f^{(3)}(0)$ does not. It is not possible to write the complete Taylor series for such a function since some of the derivatives fail to exist.

Even if a function has infinitely many derivatives at a point it need not be analytic there. We would be able to write the complete Taylor expansion but, as we have already noted, the resulting series might not converge to f on any interval. In this connection, it is instructive to work Exercise 10:5.4.

In Example 10.26 we justified the Taylor expansion for $\sin x$. Part of the justification involved the fact that $\sin x$ and all of its derivatives are bounded on the real line. This suggests a general result.

Theorem 10.28 *Let f be infinitely differentiable in a neighborhood I of c. Suppose $x \in I$ and there exists $M > 0$ such that $|f^{(m)}(t)| \leq M$ for all $m \in I\!N$ and $t \in [c, x]$ (or $[x, c]$ if $x < c$). Then $\lim_{n\to\infty} R_n(x) = 0$. Thus, f is analytic at c.*

Proof. We prove the theorem for $x > c$. We leave the case $x < c$ as Exercise 10:5.5.

We use the integral form of the remainder (Theorem 7.43), obtaining

$$|R_n(x)| = \left| \frac{1}{n!} \int_c^x (x-t)^n f^{(n+1)}(t)\,dt \right|. \tag{7}$$

Using our hypothesis that $|f^{(m)}(t)| \leq M$ for all $t \in [c, x]$, we infer from (7) that

$$\begin{aligned} |R_n(x)| &\leq \frac{M}{n!} \int_c^x (x-t)^n\,dt \\ &= \frac{M}{n!} \left. \frac{(x-t)^{n+1}}{n+1} \right|_c^x \\ &= \frac{M}{(n+1)!}(x-c)^{n+1} \end{aligned}$$

For fixed x and c, $(x - c)$ is just a constant, so

$$\frac{M(x-c)^{n+1}}{(n+1)!} \to 0.$$

Thus $|R_n(x)| \to 0$ and f is analytic at c. ■

Example 10.29 The function $f(x) = e^x$ is analytic at $x = 0$. It is certainly infinitely differentiable but we need to prove more. This follows from the previous theorem. We choose, say, the interval

$[-1, 1]$ and note that $|f^{(n)}(x)| = |e^x| \leq e$ for all $x \in (-1, 1)$ and $n \in \mathbb{N}$. A similar observation applies to the analyticity of f at any point $c \in \mathbb{R}$. ◀

Exercise 10:5.6 provides another theorem similar to Theorem 10.28.

Exercises

10:5.1 Justify formula (6) for $\sin x$ using the integral form of the remainder $R_n(x)$.

10:5.2 Show that

$$f(x) = \sum_{k=0}^{n} \frac{f^k(0)}{k!} x^k + \frac{x^{n+1}}{n!} \int_0^1 f^{(n+1)}(sx)(1-s)^n \, ds$$

under appropriate assumptions on f.

10:5.3 Show that

$$\int_0^1 f^{(n+1)}(sb)(1-s)^n \, ds \leq \frac{n! f(b)}{b^{n+1}}$$

if f and all of its derivatives exist and are nonnegative on the interval $[0, b]$.

10:5.4 Let

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ e^{-1/x^2}, & \text{if } x \neq 0. \end{cases}$$

Prove that f is infinitely differentiable on the real line. Show that $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$. Explain why the Taylor series for f about $x = 0$ does not represent f in any neighborhood of zero. Is f analytic at $x = c$ for $c \neq 0$?

10:5.5 Prove Theorem 10.28 for $x < c$.

10:5.6 Prove Bernstein's Theorem: If f is infinitely differentiable on an interval I, and $f^{(n)}(x) \geq 0$ for all $n \in \mathbb{N}$ and $x \in I$, then f is analytic on I. Apply this result to $f(x) = e^x$.

10:5.7 Use the results of this section to verify that each of the functions below is analytic at $x = 0$, and write the Taylor series about $x = 0$.

(a) $\cos x^2$

(b) e^{-x^2}

10:5.8 Show that if f and g are analytic functions at each point of an interval (a, b) then so too is any linear combination $\alpha f + \beta g$.

10.6 Products of Power Series

Suppose that we have two power series representations

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

and

$$g(x) = \sum_{k=0}^{\infty} b_k (x - x_0)^k$$

valid in the intervals $(-R_f, R_f)$ and $(-R_g, R_g)$ respectively. How should we obtain a power series representation for the product $f(x)g(x)$? We might merely compute all the derivatives of this function and so construct its Taylor series. But is this the easiest or most convenient method? How do we know that such a representation would be valid?

The most direct approach to this problem is to apply here our study of products of series from Section 3.8. We know when such a product would be valid. Indeed, from that theory, we know immediately that

$$f(x)g(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$$

would hold in the interval $(-R, R)$ where $R = \min\{R_f, R_g\}$ and the coefficients are given by the formulas

$$c_k = \sum_{j=0}^{k} a_j b_{k-j}.$$

Example 10.30 The product of the series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

and the series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

gives the representation

$$\frac{f(x)}{1-x} = \sum_{k=0}^{\infty} (a_0 + a_1 + a_2 + \dots + a_k) x^k.$$

Where would this be valid? ◀

Example 10.31 A representation for the function $e^x \sin x$ might be most easily obtained by forming the product

$$\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots\right)\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots\right) = x + x^2 + \frac{1}{6}x^3 + \dots$$

and the series continued as far as is needed for the application at hand. ◀

10.6.1 Quotients of Power Series

Suppose that we have power series representations of two functions

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{and} \quad g(x) = \sum_{k=0}^{\infty} b_k x^k$$

both valid in some interval $(-r, r)$ at least. Can we find a representation of the quotient function $f(x)/g(x)$? Certainly we must demand that $g(0) \neq 0$ which amounts to asking for the leading coefficient in the series for g, the term b_0 not to be zero.

If there is a representation, say a series $\sum_{k=0}^{\infty} c_k x^k$ then, evidently, we require that

$$\frac{\sum_{k=0}^{\infty} a_k x^k}{\sum_{k=0}^{\infty} b_k x^k} = \sum_{k=0}^{\infty} c_k x^k.$$

This merely means that we want

$$\left(\sum_{k=0}^{\infty} b_k x^k\right)\left(\sum_{k=0}^{\infty} c_k x^k\right) = \sum_{k=0}^{\infty} a_k x^k.$$

The conditions for this are known to us since we have already studied how to form the product of two power series. For this to hold the coefficients $\{c_k\}$ which, at the moment, we do not know how to determine should satisfy

$$b_0 c_0 = a_0$$

$$b_0 c_1 + b_1 c_1 = a_1$$

$$b_0 c_2 + b_1 c_1 + b_2 c_2 = a_2$$

and, in general,

$$b_0 c_k + b_1 c_{k-1} + b_2 c_{k-2} + \cdots + b_k c_0 = a_k.$$

Since we know all the a_k's and b_k's we can readily solve these equations, one at a time starting from the first to obtain the coefficients for the quotient series. This algorithm (for that is what it is) for determining the c_k's is precisely "long division". Simply divide formally the expression (the denominator)

$$b_0 + b_1x + b_2x^2 + b_3x^3 + \dots$$

into the expression (the numerator)

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

and you will find yourself solving exactly these equations in our algorithm.

But what have we determined? We have shown that if there is a series representation for $f(x)/g(x)$ then this method will determine it. We do not have any assurances in advance that there is such a series though. We offer the next theorem, without proof, for those assurances. Alternatively in any computation we could construct the quotient series (all terms!) and determine that it has a positive radius of convergence. That, too, would justify the method although it is not likely the most practical approach.

Theorem 10.32 *Suppose that there are power series representations for two functions*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \textit{and} \quad g(x) = \sum_{k=0}^{\infty} b_k x^k$$

both valid in some interval $(-r, r)$ at least and that $b_0 \neq 0$. Then there is some positive $\delta > 0$ so that the function $f(x)/g(x)$ is analytic at zero and a quotient series can be found.

The proper setting for a proof of Theorem 10.32 is complex analysis, where one proves that a quotient of complex analytic functions is analytic if the denominator is not zero.

Exercises

10:6.1 Show that if f and g are analytic functions at each point of an interval (a, b) then so too is the product fg.

10:6.2 Under what conditions on the functions f and g on an interval (a, b) can you conclude that the quotient f/g is analytic?

10:6.3 Using long division find the first few terms of the power series expansion of

$$\frac{x+2}{x^2+x+1}.$$

centered at $x = 0$. What other method would have given you these same numbers?

10:6.4 Using long division and the power series expansions for $\sin x$ and $\cos x$ find the first few terms of the power series expansion of $\tan x$ centered at $x = 0$. What other method would have given you these same numbers?

10:6.5 Find a power series expansion centered at $x = 0$ for the function

$$\frac{\sin 2x}{\sin x}.$$

Did the fact that $\sin x = 0$ at $x = 0$ make you modify the method here?

10:6.6 Show that if

$$\frac{1}{\sum_{k=0}^{\infty} b_k x^k} = \sum_{k=0}^{\infty} c_k x^k$$

is valid then

$$c_k = \frac{(-1)^k}{b_0^{k+1}} \begin{vmatrix} b_1 & b_0 & 0 & 0 & \dots & 0 \\ b_2 & b_1 & b_0 & 0 & \dots & 0 \\ b_3 & b_2 & b_1 & b_0 & \dots & 0 \\ \dots & & & & \dots & \dots \\ b_k & b_{k-1} & b_{k-2} & b_{k-3} & \dots & b_1 \end{vmatrix}.$$

10.7 Composition of Power Series

Suppose that we wished to obtain a power series expansion for the function $e^{\sin x}$ using the two series expansions

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

and

$$\sin x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots.$$

Without pausing to decide if this makes any sense let us simply insert the series for $\sin x$ in the appropriate positions in the series for e^x. Then we might hope to justify that

$$e^{\sin x} = 1 + \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots\right)$$

$$+\frac{1}{2}\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots\right)^2 + \frac{1}{6}\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots\right)^3 + \dots$$

and expand grouping terms in the obvious way, getting (at least for a start)

$$e^{\sin x} = 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots.$$

Is this method valid?

To justify this method we state (without proof) a theorem giving some conditions when this could be verified. Note that the conditions are as one should expect for a composition of functions $f(g(x))$. The series for $g(x)$ is expanded about a point x_0. That is inserted into a series expanded about the value $g(x_0)$ thus obtaining a series for $f(g(x))$ expanded about the point x_0. The proof is not difficult if approached within a course in complex variables, but would be mysterious if attempted as a real variable theorem.

Theorem 10.33 *Suppose that there are power series representations for two functions*

$$g(x) = C + \sum_{k=1}^{\infty} a_k(x - x_0)^k \quad \textit{and} \quad f(x) = \sum_{k=0}^{\infty} b_k(x - C)^k$$

both valid in some nondegenerate intervals about their centers. Then there is a power series expansion for

$$f(g(x)) = \sum_{k=0}^{\infty} c_k(x - x_0)^k$$

with a positive radius of convergence whose coefficients can be obtained by inserting the series for $g(x) - C$ into the series for f, i.e., by expanding

$$\sum_{k=0}^{\infty} \left(\sum_{j=1}^{\infty} a_j(x - x_0)^j \right)^k$$

formally.

Exercises

10:7.1 Under what conditions on the functions f and g on an interval (a, b) can you conclude that the composition $f \circ g$ is analytic?

10:7.2 Find the first few terms in the power series expansion of $e^{\sin x}$ by a method different from that in this section.

10:7.3 Find the first few terms in the power series expansion of $e^{\tan x}$ using the method discussed in this section.

10.8 Trigonometric Series

In this section we present a short introduction to another way of representing functions, namely as trigonometric series or Fourier series.

There are deep connections between power series and Fourier series so this theory does belong in this chapter (see Exercise 10:8.1).

The origins of the subject go back to the middle of the eighteenth century. Certain problems in mathematical physics seemed to require that an arbitrary function f with a fixed period (taken here as 2π) be represented in the form of a trigonometric series

$$f(t) = \tfrac{1}{2}a_0 + \sum_{j=1}^{\infty} (a_j \cos jt + b_j \sin jt), \tag{8}$$

and such mathematicians as Daniel Bernoulli, d'Alembert, Lagrange, and Euler had debated whether such a thing should be possible. Bernoulli maintained that this would always be possible, while Euler and d'Alembert argued against it.

Joseph Fourier (1768–1830) saw the utility of these representations and, although he did nothing to verify his position other than to perform some specific calculations, claimed that the representation in (8) would be available for every function f and gave the formulas

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos jt\,dt \quad \text{and} \quad b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin jt\,dt$$

for the coefficients.

While his mathematical reasons were not very strong and much criticized at the time, his instincts were correct and series of this form with coefficients computed in this way are now known as Fourier series. The a_j and b_j are called the Fourier coefficients of f.

10.8.1 Uniform Convergence of Trigonometric Series

For a first taste of this theory we prove an interesting theorem that justifies some of Fourier's original intuitions. We show that if a trigonometric series converges *uniformly* to a function f then necessarily those coefficients given by Fourier are the correct ones.

Theorem 10.34 *Suppose that*

$$f(t) = \tfrac{1}{2}a_0 + \sum_{j=1}^{\infty} (a_j \cos jt + b_j \sin jt), \tag{9}$$

with uniform convergence on the interval $[-\pi, \pi]$. Then it follows that the function f is continuous and the coefficients are given by Fourier's formulas:

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos jt\,dt \quad \textit{and} \quad b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin jt\,dt$$

Proof. Fix $j \geq 1$, choose $n > j$ and write

$$S_n(t) = \tfrac{1}{2}a_0 + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt)$$

i.e., the partial sums of the series. A straightforward, if tiresome, calculation shows that

$$\int_{-\pi}^{\pi} S_n(t) \cos jt \, dt = \int_{-\pi}^{\pi} (\cos jt)^2 \, dt = \pi. \tag{10}$$

This is, remember, just a finite sum. The orthogonality formulas in Exercise 10:8.3 assist in this computation.

We are assuming that $S_n \to f$ uniformly and so it follows too, since $\cos jt$ is bounded that $S_n(t) \cos jt \to f(t) \cos jt$ uniformly for $t \in [-\pi, \pi]$. It follows, since all functions here are continuous, that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} S_n(t) \cos jt \, dt = \int_{-\pi}^{\pi} f(t) \cos jt \, dt.$$

In view of (10) this proves the formula for a_j and $j \geq 1$. The formulas for a_0 and b_j for $j \geq 1$ can be obtained by an identical method. ■

10.8.2 Fourier Series

Emboldened by the theorem we have just proved we make a dramatic move, the same move that Fourier made. We start with the function f (not the series) and construct a trigonometric series by using these coefficient formulas.

Note the twist in the logic. *If* there is a trigonometric series converging uniformly to a continuous function f then it would have to be given by the formulas of Theorem 10.34. Why not start with the series even if we have no knowledge that the series will converge uniformly, even if we do not know whether it will converge uniformly to the function we started with, indeed even if the series diverges?

Definition 10.35 Let f be a Riemann integrable function on the interval $[-\pi, \pi]$ and let

$$a_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos jt \, dt \quad \text{and} \quad b_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin jt \, dt.$$

Then the series

$$\tfrac{1}{2}a_0 + \sum_{j=1}^{\infty} (a_j \cos jt + b_j \sin jt), \tag{11}$$

is called the *Fourier series* of f.

There is a mild understanding here that the series should be somehow related to f and there is a hope that the series can be used as a "representation" of f. But uniform convergence is out of the question in general. Indeed even pointwise convergence is rather too much to hope for. To emphasize that this relation is not one of equality we usually write

$$f(t) \sim \tfrac{1}{2}a_0 + \sum_{j=1}^{\infty}(a_j \cos jt + b_j \sin jt).$$

Exercises

10:8.1 Let $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ be a complex power series with a radius of convergence larger than 1. By setting $z = e^{it}$ find a connection between complex power series and trigonometric series.

10:8.2 Explain why it is that for any Riemann integrable function f we can claim that the integrals defining the Fourier coefficients of f exist.

10:8.3 Check the so-called *orthogonality relations* by computing that for integers $k \neq j$, and all i

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(kt)\sin(jt)\,dt = 0, \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(kt)\sin(it)\,dt = 0,$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(kt)\cos(jt)\,dt = 0.$$

10:8.4 Check that for integers $i,\ k \neq 0$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (\sin kt)^2\,dt = 1$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (\cos it)^2\,dt = 1.$$

10.8.3 Convergence of Fourier Series

The theory of Fourier Series would have a much simpler, if less fascinating, development if the Fourier series of every continuous function converged uniformly to the original function. Not only is this false but the Fourier series of a continuous function can diverge at a large set of points. This leaves us with a serious difficulty. The Fourier series of a function is expected to represent the function but how?

If it does not converge to the function how can it be used as a representation?

There is a mistake in our reasoning. We know that if a series converges to a function in suitable ways then the function may be integrated and differentiated by termwise integration and differentiation of the series. But it may be true that a series may be manipulated in these ways *even if the series diverges* at some points. A representation need not be a pointwise or uniform representation to be useful.

In our next theorem we show that the Cesàro sums of the Fourier series of a suitable function do converge uniformly to the function even if the series itself is divergent. The reader should review the topic of Cesàro summability in Section 3.9.1. A young Hungarian mathematician Leopold Fejér (1880–1959) obtained this theorem in 1900.

Theorem 10.36 (Fejér) *Let f be a continuous function on $[-\pi, \pi]$ so that $f(-\pi) = f(\pi)$. Then the sequence of Cesàro means of the partial sums of the Fourier series for f converges uniformly to f on $[-\pi, \pi]$.*

Proof. Throughout the proof we may consider that f is defined on all of $\mathbb{R}$ and is 2π-periodic. We write

$$S_n(x) = \tfrac{1}{2}a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

for the partial sums of the Fourier series of f (this means the coefficients a_j, b_j are determined by using Fourier's formulas. Then we write

$$\sigma_n(x) = \frac{s_0(x) + s_1(x) + s_2(x) + \cdots + s_n(x)}{n+1}$$

for the sequence of averages (Cesàro means).

Our task is to prove that $\sigma_n \to f$ uniformly. Looking back we see that each $\sigma_n(x)$ is a finite sum of terms $s_k(x)$ and each $s_k(x)$ is a finite sum of terms involving a_j, b_j each of which is expressible as a an integral involving f and sin's and cos's. Thus after some considerable, but routine computations, we arrive at a formula

$$\sigma_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\left(f(x+t) + f(x-t)\right) K_n(t)\, dt$$

or the equivalent formula

$$\sigma_n(x) = \frac{1}{\pi}\int_{0}^{\pi} \left(f(x+t) + f(x-t)\right) K_n(t)\, dt. \qquad (12)$$

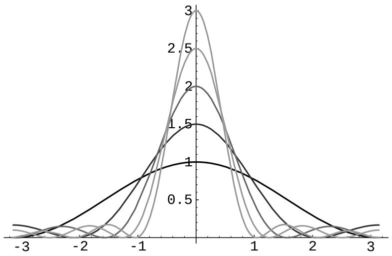

Figure 10.1: Fejér kernel $K_n(t)$ for $n = 1, 2, 3, 4$, and 5.

Here K_n is called the Fejér kernel and for each n,

$$K_n(t) = \frac{1}{2(n+1)} \left(\frac{\sin\left(\frac{1}{2}(n+1)t\right)}{\sin \frac{1}{2}t} \right)^2 .$$

The reader can just accept the computations for the purposes of our short introduction to the subject.

The Fejér kernel of order n enjoys the following properties, each of which is evident from its definition:

1. Each $K_n(t)$ is a nonnegative, continuous function.
2. For each n,
$$\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t)\,dt = \frac{2}{\pi}\int_0^{\pi} K_n(t)\,dt = 1.$$
3. For each n and $0 < |t| < \pi$,
$$0 \le K_n(t) \le \frac{\pi}{(n+1)t^2}.$$

Figure 10.1 illustrates the graph of this function for $n = 1, 2, 3, 4$, and 5.

Let $\varepsilon > 0$, and choose $\delta > 0$ so that

$$|f(x+t) + f(x-t) - 2f(x)| < \varepsilon$$

for every $0 \le t \le \delta$. This just uses the uniform continuity of f.

We note that

$$\frac{2}{\pi}\int_0^{\pi} f(x)K_n(t)\,dt = f(x)$$

by using the property 2 above. Thus we have

$$|\sigma_n(x) - f(x)| \le \frac{1}{\pi}\int_0^{\pi} |f(x+t) + f(x-t) - 2f(x)|\, K_n(t)\,dt$$

$$\leq I_1 + I_2,$$

where I_1 is the integral taken over $[0, \delta]$ and I_2 is the integral taken over $[\delta, \pi]$. Since K_n is nonnegative, we did not need to keep it inside the absolute value in the integral. The part I_1 will be small (for all n) because the expression in the absolute values is small for t in the interval $[0, \delta]$. The part I_2 will be small (for large n) because of the bound on the size of K_n for t away from zero in property (4) above. Here are the details: for I_1 we have

$$I_1 \leq \frac{\varepsilon}{\pi} \int_0^\delta K_n(t)\, dt \leq \varepsilon.$$

For I_2, let

$$\kappa_n = \sup\{K_n(t) : \delta \leq t \leq \pi\},$$

and note that property 3 supplies us with the fact that $\kappa_n \to 0$ as $n \to \infty$. Now we have

$$I_2 \leq \frac{\kappa_n \varepsilon}{\pi} \int_\delta^\pi \left(|f(x+t)| + |f(x-t)| + 2|f(x)|\right)\, dt$$

so that we can make I_2 as small as we please by choosing n large enough. It follows, since ε and x are arbitrary, that

$$\lim_{n\to\infty} \sigma_n(x) = f(x),$$

uniformly for $x \in [-\pi, \pi]$ as required. ■

Exercises

10:8.1 Let $s_n(x)$ be the sequence of partial sums of the Fourier series for a 2π-periodic integrable function f. Show that

$$s_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} \left(f(x+t) + f(x-t)\right) D_n(t)\, dt$$

and

$$s_n(x) = \frac{1}{\pi} \int_0^\pi \left(f(x+t) + f(x-t)\right) D_n(t)\, dt$$

where

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt$$

is called the Dirichlet kernel. Figure 10.2 illustrates the graph of this function for $n = 1, 3$, and 7. It should be contrasted with Figure 10.1.

10:8.2 Establish the following properties of the Dirichlet kernel:

(a) Each $D_n(t)$ is a continuous, 2π-periodic function.

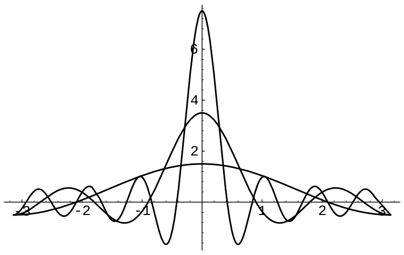

Figure 10.2: Dirichelet kernel $D_n(t)$ for $n = 1$, 3, and 7.

(b) Each $D_n(t)$ is an even function.

(c) For each n,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(t)\,dt = \frac{2}{\pi}\int_{0}^{\pi} D_n(t)\,dt = 1.$$

(d) For each n,

$$D_n(t) = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}.$$

(e) For each n, $D_n(0) = n + \frac{1}{2}$.

(f) For each n and all t, $|D_n(t)| \leq n + \frac{1}{2}$.

(g) For each n and $0 < |t| < \pi$,

$$|D_n(t)| \leq \frac{\pi}{2|t|}.$$

10:8.3 Let

$$K_n(t) = \frac{1}{n+1}\sum_{j=0}^{n} D_j(t)$$

where D_j are the Dirichlet kernels. Show that the formula for the averages σ_n given in the proof of Theorem 10.36 is correct.

10.8.4 Weierstrass Approximation Theorem

Fejér's theorem allows us to prove the famous Weierstrass approximation theorem. Note that a consequence of Fejér's theorem is that continuous, 2π-periodic functions can be uniformly approximated by trigonometric polynomials. Weierstrass' theorem asserts that continuous functions on a compact interval can be uniformly approximated by ordinary polynomials.

Theorem 10.37 (Weierstrass approximation) *Let f be a continuous function on an interval $[a,b]$, and let $\varepsilon > 0$. Then there is a polynomial*

$$g(x) = \alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1 x + \alpha_0$$

so that

$$|f(x) - g(x)| < \varepsilon$$

for all $x \in [a,b]$.

Proof. It is more convenient for this proof to assume that $[a,b] = [0,1]$. The general case can be obtained from this.

Let f be a continuous, function on $[0,1]$, let $\varepsilon > 0$, and write $F(t) = f(|\cos t|)$. Then F is a continuous, 2π-periodic function and can be approximated by a trigonometric polynomial within ε. This is because, in view of Theorem 10.36, for large enough n the Cesàro means $\sigma_n(f)$ are uniformly close to f.

Since F is even [i.e., $F(t) = F(-t)$] we can figure out what form that trigonometric polynomial may take. All the coefficients b_k involving $\sin kt$ in the Fourier series for F must be zero. Thus when we form the averages of the partial sums we obtain only sums of cosines. Consequently we can find c_0, c_1, c_2, $\ldots$ c_n so that

$$\left| F(t) - \sum_0^n c_j \cos jt \right| < \varepsilon \tag{13}$$

for all t. Each $\cos jt$ can be written using elementary trigonometric identities as $T_j(\cos t)$ for some jth order (ordinary) polynomial T_j, and so, by setting $x = \cos t$ for any $x \in [0,1]$, we have

$$\left| f(x) - \sum_0^n c_j T_j(x) \right| < \varepsilon,$$

which is exactly the polynomial approximation that we need. ∎

The polynomials T_j that appear in the proof are well known as the Tchebychev polynomials and are easily generated (see Exercise 10:8.2).

Exercises

10:8.1 Show that once Theorem 10.37 is proved for the interval $[0,1]$ it can be deduced for any interval $[a,b]$.

10:8.2 Define the Tchebychev polynomials by requiring T_j to be a polynomial so that

$$\cos jt = T_j(\cos t)$$

identically. Show that $T_0(x) = 1$, $T_1(x) = x$, and

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

Generate the first few of these polynomials.

10:8.3 Show that Theorem 10.37 can be interpreted as asserting that for any continuous function on an interval $[a, b]$ there is a sequence of polynomials p_n converging to f uniformly on $[a, b]$.

10:8.4 Does Exercise 10:8.3 also imply that there must be a power series expansion converging to f uniformly on $[a, b]$?

10:8.5 Let f be a continuous function on an interval $[a, b]$, and let $\varepsilon > 0$. Show that there must exist a polynomial p with rational coefficients so that

$$|f(x) - p(x)| < \varepsilon$$

for all $x \in [a, b]$.

10:8.6 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $\varepsilon > 0$. Must there exist a polynomial p so that

$$|f(x) - p(x)| < \varepsilon$$

for all $x \in \mathbb{R}$.

10:8.7 Let $f : (0, 1) \to \mathbb{R}$ be a continuous function and let $\varepsilon > 0$. Must there exist a polynomial p so that

$$|f(x) - p(x)| < \varepsilon$$

for all $x \in (0, 1)$.

10:8.8 Let $f : [0, 1] \to \mathbb{R}$ be a continuous function with the property that

$$\int_0^1 f(x)x^n \, dx = 0$$

for all $n = 0, 1, 2, 3, \ldots$. What can you conclude about the function f?

10:8.9 Let $f : [0, 1] \to \mathbb{R}$ be a continuous function with the property that $f(0) = 0$ and

$$\int_0^1 f(x) \sin \pi n x \, dx = 0$$

for all $n = 1, 2, 3, \ldots$. What can you conclude about the function f?