Instructors Preface

American colleges and universities have for many years offered courses with titles such as *Advanced Calculus* or *Introductory Real Analysis.* These courses are taken by a variety of students, serve a number of purposes, and are written at various levels of sophistication. The students range from ones who have just completed a course in elementary calculus to beginning graduate students in mathematics. The purposes are multifold:

(1) To present familiar concepts from calculus at a more rigorous level.

(2) To introduce concepts that are not studied in elementary calculus, but which are needed in more advanced undergraduate courses. This would include such topics as point set theory, uniform continuity of functions and uniform convergence of sequences of functions.

(3) To provide students with a level of mathematical sophistication that will prepare them for graduate work in mathematical analysis.

(4) To develop many of the topics that the authors feel "all students of mathematics should know."

There are now many texts that address some or all of these objectives. These books range from ones that do little more than address objective (1), to ones that try to address all four objectives. The books of the first extreme are generally aimed at one-term courses for students with minimal background. Books at the other extreme often contain substantially more material than can be covered in a one-year course.

The level of rigor varies considerably from one book to another. So does the style of presentation—some books endeavor to give a very efficient streamlined development, others try to be more "user-friendly". We have opted for the "user-friendly" approach. We feel this approach makes the concepts more meaningful to the student.

Our experience with students at various levels has shown that most students have difficulties when topics that are entirely new to them first appear. For some students that might occur almost immediately when rigorous proofs are required, for example, ones needing δ-ε arguments. For others, the difficulties begin with elementary point set theory, compactness arguments and the like.

To help students with the transition from elementary calculus to a more rigorous course we have included motivation for concepts most students have not seen before and provided more details in proofs when we introduce new methods. In addition, we have tried to give the students ample opportunity to see the use of the new tools in action.

For example, students often feel uneasy when they first encounter the various "compactness arguments" (Heine-Borel theorem, Bolzano-Weierstrass theorem, Cousin's lemma introduced in Section 4.5). To help the student see why such theorems are useful, we pose the problem of determining circumstances under which local boundedness of a function f on a set E implies global boundedness of f on E. We show by example that some conditions on E are needed, namely that E be closed and bounded, and then show how each of several theorems could be used to show that closed and boundedness of the set E suffices. Thus we introduce students to the theorems by showing how the theorems can be used in natural ways to solve a problem.

We have attempted to write this book in a manner sufficiently flexible to make it possible to use the book for courses of various lengths and a variety of levels of mathematical sophistication.

Much of the material in the book involves rigorous development of topics of a relatively elementary nature, topics that most students have studied at a nonrigorous level in a calculus course. A short course of moderate mathematical sophistication intended for students of minimal background can be based entirely on this material. (We indicate some possible choices for such courses below). Such a course might meet objective (1).

The remaining material falls roughly into two categories:

(E) Relatively elementary material that could be added to a longer course to provide enrichment and additional examples.

(A) Material of a more more mathematically sophisticated nature that would prepare a student for more advanced topics in real analysis. These topics might be needed later in this book, in a more advanced undergraduate course or in a beginning level graduate class.

We apply these markings to some entire chapters as well as to some sections within chapters and even to certain exercises. We do not view these markings as absolute. They can simply be interpreted in the following ways. Any unmarked material will not depend, in any substantial way, on earlier marked sections. In addition, if a section or exercised marked (A) will be used in a later section of this book, we indicate where it will be required.

The material marked (A) is in line with goals (2) and (3) above. We resist the temptation to address objective (4). There are simply too many additional topics that one might feel "every student should know", e.g., functions of several variables, vector analysis, Riemann-Stieltjes and Lebesgue integrals. To cover these topics in the manner we cover other material would render the book more like a reference book than a text that could reasonably be covered in a year. Students who have completed this book will be in a good position to study such topics at rigorous levels.

We include a chapter on metric spaces, however. We do this for two reasons: to offer a more general framework for viewing concepts treated in earlier chapters, and to illustrate how the abstract viewpoint can be applied to solving concrete problems.

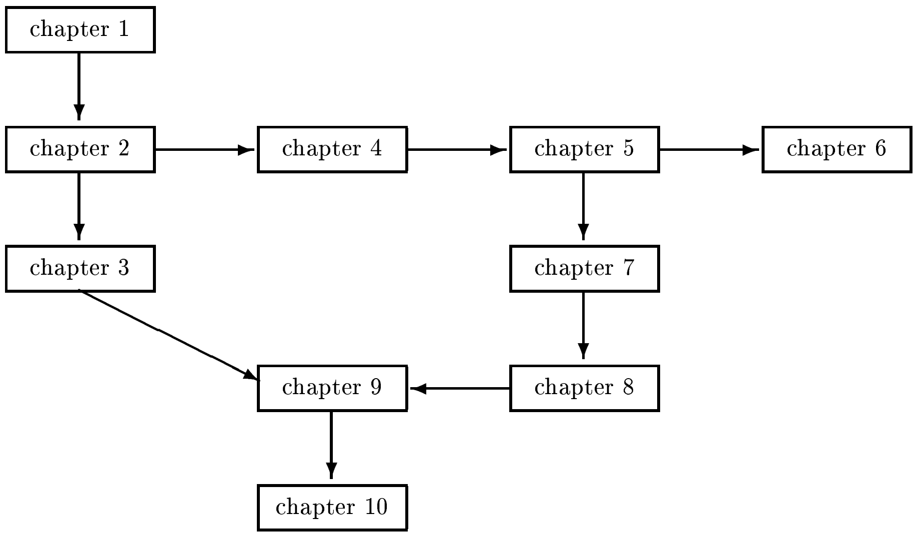
The exercises form an integral part of the book. Many of these exercises are routine in nature. Others are more demanding. A few provide examples that are not usually presented in books of this type, but that students have found challenging, interesting and instructive.

We should make one more point about the exposition. We do assume that students are familiar with nonrigorous calculus. In particular, we assume familiarity with the elementary functions and their elementary properties. We also assume some familiarity with computing derivatives and integrals. This allows us to illustrate various concepts using examples familiar to the students. For example, we begin Chapter 2 on sequences with a discussion of approximating $\sqrt{2}$ using Newton's method. This is merely a motivational discussion, so we are not bothered by the fact that that we don't treat the derivative formally until Chapter 7, and haven't yet proved that $\frac{d}{dx}(x^2 - 2) = 2x$. For students with minimal background we provide an appendix that informally covers such topics as notation, elementary set theory, functions, and proofs.

We have tried to write this book in a leisurely style. This allows us to provide motivational discussions and historical perspective in a number of places. Even though the book is relatively large (in terms of number of pages), we can comfortably cover virtually all of it in a full year course, including many of the interesting exercises.

Instructors teaching a short course have several options. One can base a course entirely on the unmarked material of Chapters 1, 2, 4, 5 and 7. As time permits, one can add parts of Chapters 3 and 8. Some of the material in Chapters 5 and 8 depend on Theorem 5.43 which states that a continuous function on a closed bounded interval $[a, b]$ is uniformly continuous. The proof of this theorem requires using at least one of the compactness arguments in Section 4.5. An instructor who skips that section may wish to accept Theorem 5.43 without proof.

Chapter Dependencies — (Unmarked Sections)



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Chapter 1

PROPERTIES OF THE REAL NUMBERS

1.1 Introduction

The goal of any analysis course is to do some analysis. There are some wonderfully important and interesting facts that can be established in a first analysis course.

Unfortunately all of the material we wish to cover rests on some foundations, foundations that may not have been properly set down in your earlier courses. Calculus courses traditionally avoid any foundational problems by simply not proving the statements that would need them. Here we cannot avoid this. We must start with the real number system.

Historically much of real analysis was undertaken without any clear understanding of the real numbers. To be sure the mathematicians of the time had a firm intuitive grasp of the nature of the real numbers and often found precisely the right tool to use in their proofs, but in many cases the tools could not be justified by any line of reasoning.

By the 1870's mathematicians such as Cantor and Dedekind had found ways to describe the real numbers in a way that seemed rigorous. We could follow their example and find a presentation of the real numbers that starts at the very beginning and leads up slowly (very slowly) to the exact tools that we need to study analysis. This subject is, perhaps, best left to courses in logic where other important foundation issues can be discussed.

The approach we shall take (and most textbooks take) is simply to list all the tools that are needed in such a study and take them for granted. You may consider that the real number system is exactly as you have always imagined it. You can sketch pictures of the real line and measure distances and consider the order just as before. Nothing is changed from high school algebra or calculus. But when we come to prove assertions about real numbers or real functions or real sets we must use exactly the tools here and not rely on our intuition.

1.2 The Real Number System

To do real analysis we should know exactly what the real numbers are. Here is a very loose exposition, suitable for calculus students but (as we will see) not suitable for us.

The Natural Numbers $\mathbb{N}$ We start with the natural numbers. These are the counting numbers

$$1, 2, 3, 4, \ldots .$$

The symbol $\mathbb{N}$ is used to indicate this collection. Thus $n \in \mathbb{N}$ means that n is a natural number, one of these numbers $1, 2, 3, 4, \ldots$.

There are two operations on the natural numbers, addition and multiplication:

$$m + n \text{ and } m \cdot n.$$

There is also an order relation

$$m < n.$$

Large amounts of time in elementary school are devoted to an understanding of these operations and the order relation.

(Subtraction and division can also be defined, but not for all pairs in $\mathbb{N}$. While $7 - 5$ and $10/5$ are assigned a meaning (we say $x = 7 - 5$ if $x + 5 = 7$ and we say $x = 10/5$ if $5 \cdot x = 10$) there is no meaning that can be attached to $5 - 7$ and $5/10$ in this number system.)

The Integers $\mathbb{Z}$ For various reasons, usually well motivated in the lower grades, the natural numbers prove to be rather limited in representing problems that arise in applications of mathematics to the real world. Thus they are enlarged by adjoining the negative integers and zero. Thus the collection

$$\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$$

is denoted $\mathbb{Z}$ and called the integers. (The symbol $\mathbb{N}$ seems obvious enough [N for "natural"] but the symbol $\mathbb{Z}$ for the integers originates in the German word for whole number.)

Once again there are two operations on $\mathbb{Z}$, addition and multiplication:

$$m + n \quad \text{and} \quad m \cdot n.$$

Again there is an order relation

$$m < n.$$

Fortunately the rules of arithmetic and order learned for the simpler system $\mathbb{N}$ continue to hold for $\mathbb{Z}$ and the young students extend their abilities perhaps painlessly.

(Subtraction can now be defined in this larger number system, but division still may not be defined [(e.g., $-9/3$ is defined but $3/(-9)$ is not]).

The Rational Numbers $\mathbb{Q}$ At some point the problem of the failure of division in the sets $\mathbb{N}$ and $\mathbb{Z}$ becomes acute and the student must progress to an understanding of fractions. This larger number system is denoted $\mathbb{Q}$ where the Q here is meant to suggest quotients, which is after all what fractions are.

The collection of all "numbers" of the form

$$\frac{m}{n}$$

where m and n are integers and n is not zero, is called the set of rational numbers and is denoted $\mathbb{Q}$.

A higher level of sophistication is demanded for the young students. Now they need to understand that equality has a new meaning. In $\mathbb{N}$ or $\mathbb{Z}$ a statement $m = n$ meant merely that m and n were the same object. Now

$$\frac{m}{n} = \frac{a}{b}$$

for m, n, a, $b \in \mathbb{Z}$ (and $n \neq 0$, $b \neq 0$) means that

$$m \cdot b = a \cdot n.$$

Addition and multiplication present major challenges too. It may be obscured by teachers who dwell too long on physical models and colored sticks, but ultimately the students must learn that

$$\frac{m}{n} + \frac{a}{b} = \frac{mb + na}{nb}$$

and

$$\frac{m}{n} \cdot \frac{a}{b} = \frac{ma}{nb}.$$

Subtraction and division are similarly defined. Fortunately once again the rules of arithmetic are unchanged. The associative rule, distributive rule, etc. all remain true even in this number system. Even though the rules are the same the young students may still suffer under their early lessons in the dreaded world of fractions.

Again too an order relation

$$\frac{m}{n} < \frac{a}{b}$$

is available. It can be defined by requiring $mb < na$. Again, too, the same rules for inequalities learned for integers and natural numbers are valid for rationals.

The Real Numbers $\mathbb{R}$ Up to this point in developing the real numbers we have encountered only arithmetic operations. The progression from $\mathbb{N}$ to $\mathbb{Z}$ to $\mathbb{Q}$ is simply algebraic. All this algebra might have been a burden to the weaker students in the lower grades, but conceptually the steps are easy to grasp with a bit of familiarity.

The next step, needed for all calculus students, is to develop the still larger system of real numbers, denoted as $\mathbb{R}$. We often refer to the real number system as *the real line* and think about it as a geometrical object, even though nothing in our definitions would seem at first sight to allow this.

Most calculus students would be hard pressed to say exactly what these numbers are. They recognize that $\mathbb{R}$ includes all of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ and also many new numbers such as $\sqrt{2}$, e, and π. But asked what a real number is many would return a blank stare. Even just asked what $\sqrt{2}$, e, or π are often produces puzzlement. Well $\sqrt{2}$ is a number whose square is 2. But is there a number whose square is 2? A calculator might oblige with 1.4142136 but $(1.4142136)^2$ is not 2. So what exactly "is" this number $\sqrt{2}$? If we are unable to write down a number whose square is 2 why can we claim that there is a number whose square is 2? And π and e are worse.

Some calculus texts handle this for the students by proclaiming that real numbers are obtained by infinite decimal expansions. Thus while rational numbers have infinite decimal expansions that terminate (e.g., $1/4 = 0.25$) or repeat (e.g., $1/3 = 0.333333\dots$) the collection of real numbers would include *all* infinite decimal expansions whether repeating, terminating or not. In that case the claim would be that there is some infinite decimal expansion $1.414213\dots$ whose square really is 2 and that infinite decimal expansion is the number we mean by the symbol $\sqrt{2}$.

This approach is adequate for applications of the calculus and is a useful way to avoid doing any hard mathematics in introductory calculus courses. But the reader should recall that, at certain stages in the calculus textbook that you used, would have appeared a phrase such as "the proof of this next theorem is beyond the level of this text". It was beyond the level of the text only because the real numbers had not been properly treated and so there was no way that a proof could have been attempted.

We need to construct such proofs and so we need to abandon this loose, descriptive way of thinking about the real numbers. Instead we will define the real numbers to be a complete, ordered field. In the next sections each of these terms is defined.

1.3 Algebraic Structure

We describe the real numbers by assuming that they have a collection of properties. We do not construct the real numbers, we just announce what properties they are to have. Since the properties that we develop are familiar and acceptable and do in fact describe the real numbers that we are accustomed to using, this approach should not cause any distress. We are just stating rather clearly what it is about the real numbers that we need to use.

We begin with the algebraic structure.

In elementary algebra courses one learns many formulas that are valid for real numbers. For example the formula

$$(x+y)+z = x+(y+z)$$

called the associative rule is learned. So also is the useful factoring rule

$$x^2 - y^2 = (x-y)(x+y).$$

It is possible to reduce the many rules down to one small set of rules that can be used to prove all the other rules.

These rules can be used for other kinds of algebra, algebras where the objects are not real numbers but some other kind of mathematical constructions. This particular structure occurs so frequently, in fact, and in so many different applications that it has its own name. Any set of objects which has these same features is called a *field.* Thus we can say that the first important structure of the real number system is the field structure.

The following nine properties are called the *field axioms.* When we are doing algebra in the real number system it is the field axioms that we are really using.

Assume that the set of real numbers $\mathbb{R}$ has two operations, called addition "+" and multiplication "·" and that these operations satisfy the field axioms. The operation $a \cdot b$ (multiplication) is most often written without the dot as ab.

A1 For any a, $b \in \mathbb{R}$ there is a number $a+b \in \mathbb{R}$ and $a+b = b+a$.

A2 For any a, b, $c \in \mathbb{R}$ the identity

$$(a+b)+c = a+(b+c)$$

is true.

A3 There is a unique number $0 \in \mathbb{R}$ so that

$$a+0 = 0+a = a$$

for all $a \in \mathbb{R}$.

A4 For any number $a \in \mathbb{R}$ there is a corresponding number denoted $-a$ with the property that

$$a+(-a) = 0.$$

M1 For any a, $b \in \mathbb{R}$ there is a number $ab \in \mathbb{R}$ and $ab = ba$.

M2 For any a, b, $c \in \mathbb{R}$ the identity

$$(ab)c = a(bc)$$

is true.

M3 There is a unique number $1 \in \mathbb{R}$ so that

$$a1 = 1a = a$$

for all $a \in \mathbb{R}$.

M4 For any number $a \in \mathbb{R}$, $a \neq 0$, there is a corresponding number denoted a^{-1} with the property that

$$aa^{-1} = 1.$$

AM1 For any a, b, $c \in \mathbb{R}$ the identity

$$(a+b)c = ab+bc$$

is true.

Note that we have labeled the axioms with letters indicating which operations are affected, thus A for addition and M for multiplication. The distributive rule AM1 connects addition and multiplication.

How are we to use these axioms? The answer likely is that, in an analysis course, you would not. You might try some of the exercises to understand what a field is and why the real numbers form a field. In an algebra course it would be most interesting to consider many other examples of fields and some of their applications. For an analysis course, understand that we are trying to specify exactly what we mean by the real number system and these axioms are just the beginning of that process.. The first step in that is to declare that the real numbers form a field under the two operations of addition and multiplication.

Exercises

1:3.1 The field axioms include rules known often as "associative rules", "commutative rules" and "distributive rules". Which are which and why do they have these names?

1:3.2 To be precise we would have to say what is meant by the operations of addition and multiplication. Let S be a set and let $S \times S$ be the set of all ordered pairs (s_1, s_2) for $s_1,\ s_2 \in S$. A *binary operation* on S is a function $B : S \times S \to S$. Thus the operation takes the pair (s_1, s_2) and outputs the element $B(s_1, s_2)$. For example addition is a binary operation. We could write $(s_1, s_2) \to A(s_1, s_2)$ rather than the more familiar $(s_1, s_2) \to s_1 + s_2$.

(a) Rewrite axioms A1-A4 using this notation $A(s_1, s_2)$ instead of the sum notation.

(b) Define a binary operation on $\mathbb{R}$ different than addition, subtraction, multiplication or division and determine some of its properties.

(c) For a binary operation B define what you might mean by the commutative, associative and distributive rules.

(d) Does the binary operation of subtraction satisfy any one of the commutative, associative or distributive rules?

1:3.3 If in the field axioms for $\mathbb{R}$ we replace $\mathbb{R}$ by any other set with two operations $+$ and $\cdot$ that satisfy these nine properties then we say that that structure is a field. For example $\mathbb{Q}$ is a field. The rules are valid since $\mathbb{Q} \subset \mathbb{R}$. The only thing that needs to be checked is that $a + b$ and $a \cdot b$ are in $\mathbb{Q}$ if both a and b are. For this reason $\mathbb{Q}$ is called a *subfield* of $\mathbb{R}$. Find another subfield.

1:3.4 Let S be a set consisting of two elements labeled as A and B. Define $A + A = A$, $B + B = A$, $A + B = B + A = B$, $A \cdot A = A$, $A \cdot B = B \cdot A = A$ and $B \cdot B = B$. Show that all nine of the axioms of a field hold for this structure.

1:3.5 Using just the field axioms show that

$$(x + 1)^2 = x^2 + 2x + 1$$

for all $x \in \mathbb{R}$. Would this identity be true in any field?

1:3.6 Define operations of addition and multiplication on the finite set $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ as follows:

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Show that $\mathbb{Z}_5$ satisfies all the field axioms.

1:3.7 Define operations of addition and multiplication on the finite set $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ as follows:

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\times$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Which of the field axioms does $\mathbb{Z}_5$ fail to satisfy?

1:3.8 The complex numbers $\mathbb{C}$ are defined as equal to the set $\mathbb{R}^2$ of all ordered pairs of real numbers subject to these operations:

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

and

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1).$$

(a) Show that $\mathbb{C}$ is a field.

(b) What are the additive and multiplicative identity elements?

(c) What are the additive and multiplicative inverses of an element (a, b)?

(d) Solve $(a, b)^2 = (1, 0)$ in $\mathbb{C}$.

(e) We identify $\mathbb{R}$ with a subset of $\mathbb{C}$ by identifying the elements $x \in \mathbb{R}$ with the element $(x, 0)$ in $\mathbb{C}$. Explain how this can be interpreted as saying that "$\mathbb{R}$ is a subfield of $\mathbb{C}$".)

(f) Show that there is an element $i \in \mathbb{C}$ with $i^2 = -1$ so that every element $z \in \mathbb{C}$ can be written as $z = x + iy$ for $x, y \in \mathbb{R}$.

(g) Explain why the equation $x^2 + x + 1 = 0$ has no solution in $\mathbb{R}$ but two solutions in $\mathbb{C}$.

1.4 Order Structure

The real number system also enjoys an order structure. Part of our usual picture of the reals is the sense that some numbers are "bigger" than others or more to the "right" than others. We express this by using inequalities $x < y$ or $x \leq y$. The order structure is closely related to the field structure. For example when we use inequalities in elementary courses we frequently use the fact that if $x < y$ and $0 < z$ then $xz < yz$, i.e., that inequalities can be multiplied through by positive numbers.

This structure, too, can be axiomatized and reduced to a small set of rules. Once again these same rules can be found in other applications of mathematics. When these rules are added to the field axioms the result is called an *ordered field.*

The real numbers system is an ordered field, satisfying the further four axioms. Here $a < b$ is now a statement which is either true or false. (Before $a + b$ and $a \cdot b$ were not statements, but elements of $\mathbb{R}$.)

O1 For any $a, b \in \mathbb{R}$ exactly one of the statements $a = b$, $a < b$ or $b < a$ is true.

O2 For any $a, b, c \in \mathbb{R}$ if $a < b$ is true and $b < c$ is true then $a < c$ is true.

O3 For any $a, b \in \mathbb{R}$ if $a < b$ is true then $a + c < b + c$ is also true for any $c \in \mathbb{R}$.

O4 For any $a, b \in \mathbb{R}$ if $a < b$ is true then $a \cdot c < b \cdot c$ is also true for any $c \in \mathbb{R}$ for which $c > 0$.

How are we to use these axioms? The answer once again likely is that, in an analysis course, you would not. You can rely on your earlier practice with inequalities in more elementary courses. Try some exercises in order to appreciate what an ordered field is and why the real numbers form an ordered field.

Exercises

1:4.1 Using just the axioms here prove that
$$ad + bc < ac + bd$$
if $a < b$ and $c < d$.

1:4.2 Show for every $n \in \mathbb{N}$ that $n^2 \geq n$.

1:4.3 Using just the axioms here prove the *arithmetic-geometric mean inequality*:
$$\sqrt{ab} \leq \frac{a+b}{2}$$
for any a, $b \in \mathbb{R}$ with $a > 0$ and $b > 0$. (Assume, for the moment, the existence of square roots.)

✂ **1:4.4** Can an order be defined on the field $\mathbb{C}$ of Exercise 1:3.8 in such a way so to make it an ordered field?

1.5 Bounds

Let E be some set of real numbers. There may or may not be a number M that is bigger than every number in the set E. If there is we say that M is an upper bound for the set. If there is no upper bound then the set is said to be *unbounded above* or to have no upper bound. This is a simple enough idea but it is critical to an understanding of the real numbers and so we shall look more closely at it and give some precise definitions.

Definition 1.1 (Upper Bounds) Let E be a set of real numbers. A number M is said to be an *upper bound* for E if $x \leq M$ for all $x \in E$.

Definition 1.2 (Lower Bounds) Let E be a set of real numbers. A number m is said to be a *lower bound* for E if $m \leq x$ for all $x \in E$.

It is often very important to note whether a set has bounds or not. A set that has an upper bound and a lower bound is called *bounded*.

A set can have many upper bounds. Indeed every number is an upper bound for the empty set $\emptyset$. A set may have no upper bounds. We can use the phrase "E is unbounded above" if there are no upper bounds. For some sets the most natural upper bound (from among the infinitely many to choose) is just the largest member of the set. This is called the maximum. Similarly the most natural lower bound for some sets is the smallest member of the set, the minimum.

Definition 1.3 (Maximum) Let E be a set of real numbers. If there is a number M that belongs to E and is larger than every other member of E then M is called the maximum of the set E and we write $M = \max E$.

Definition 1.4 (Minimum) Let E be a set of real numbers. If there is a number m that belongs to E and is smaller than every other member of E then m is called the minimum of the set E and we write $m = \min E$.

Example 1.5 The interval

$$[0,1] = \{x : 0 \leq x \leq 1\}$$

has a maximum and a minimum. The maximum is 1 and 1 is also an upper bound for the set. (Indeed if a set has a maximum then that number must certainly be an upper bound for the set.) Any number larger than 1 is also an upper bound. The number 0 is the minimum and also a lower bound.

The interval

$$(0,1) = \{x : 0 < x < 1\}$$

has no maximum and no minimum. At first glance some novices insist that the maximum should be 1 and the minimum 0 as before. But look at the definition. The maximum must be both an upper bound and also a member of the set. Here 1 and 0 are upper and lower bounds respectively but do not belong to the set. ◀

Example 1.6 The set $\mathbb{N}$ of natural numbers has a minimum but no maximum and no upper bounds at all. We would say that it is bounded below but not bounded above. ◀

1.6 Sups and Infs

Let us return to the subject of maxima and minima again. If E has a maximum, say M, then that maximum could be described by the statement

> M is the least of all the upper bounds of E,

that is to say, M is the minimum of all the upper bounds. The most frequent language used here is "M is the least upper bound". It is possible for a set to have no maximum and yet be bounded above: in any example that comes to mind you will see that the set appears to have a least upper bound. For example, the open interval $(0,1)$ has no maximum, but many upper bounds. The least of all the upper bounds is the number 1 that cannot be described as a maximum because it fails to be in the set.

Definition 1.7 (Least Upper Bound/Supremum) Let E be a nonempty set of real numbers that is bounded above. If M is the least of all the upper bounds then M is said to be *the least upper bound of* E or *the supremum* of E and we write $M = \sup E$.

Definition 1.8 (Greatest Lower Bound/Infimum) Let E be a nonempty set of real numbers that is bounded below. If m is the greatest of all the lower bounds of E then m is said to be *the greatest lower bound of* E or *the infimum* of E and we write $M = \inf E$.

To complete the definition of $\inf E$ and $\sup E$ it is most convenient to be able write this expression even for $E = \emptyset$ or for unbounded sets. Thus we write

1. $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.
2. If E is unbounded above then $\sup E = \infty$.
3. If E is unbounded below then $\inf E = -\infty$.

The Axiom of Completeness Any example of a nonempty set that you are able to visualize that has an upper bound, will also have a least upper bound. Pages of examples might convince you that all nonempty sets bounded above must have a least upper bound. Indeed your intuition will forbid you to accept the idea that this could not always be the case. To prove such an assertion is not possible using only the axioms for an ordered field. Thus we shall assume one further axiom, known as the axiom of completeness.

> **Completeness Axiom** Every nonempty set E of real numbers that is bounded above has a least upper bound, i.e., $\sup E$ exists and is a real number.

This now is the totality of all the axioms we need to assume. We have assumed that $\mathbb{R}$ is a field with two operations of addition and multiplication, that $\mathbb{R}$ is an ordered field with an inequality relation "$<$", and finally that $\mathbb{R}$ is a complete ordered field. This is enough to characterize the real numbers and the phrase "complete ordered field" refers to the system of real numbers and to no other system. (We shall not prove this statement; see Exercise 1:6.24 for a discussion.)

Exercises

1:6.1 Show that a set of real numbers E is bounded if and only if there is a positive number r so that $|x| < r$ for all $x \in E$.

1:6.2 Write down $\sup E$ and $\inf E$ and (where possible) $\max E$ and $\min E$ for the following examples of sets:

(a) $E = \mathbb{N}$.

(b) $E = \mathbb{Z}$.

(c) $E = \mathbb{Q}$.

(d) $E = \mathbb{R}$.

(e) $E = \{-3, 2, 5, 7\}$.

(f) $E = \{x : x^2 < 2\}$.

(g) $E = \{x : x^2 - x - 1 < 0\}$.

(h) $E = \{1/n : n \in \mathbb{N}\}$.

(i) $E = \{\sqrt[n]{n} : n \in \mathbb{N}\}$.

1:6.3 Under what conditions does $\sup E = \max E$?

1:6.4 Show for every nonempty, finite set E, that $\sup E = \max E$.

1:6.5 For every $x \in \mathbb{R}$ define

$$[x] = \max\{n \in \mathbb{Z} : n \leq x\}$$

called the *greatest integer function.* Show that this is well defined and sketch the graph of the function.

1:6.6 Let A be a set of real numbers and let $B = \{-x : x \in A\}$. Find a relation between $\max A$ and $\min B$ and between $\min A$ and $\max B$.

1:6.7 Let A be a set of real numbers and let $B = \{-x : x \in A\}$. Find a relation between $\sup A$ and $\inf B$ and between $\inf A$ and $\sup B$.

1:6.8 Let A be a set of real numbers and let $B = \{x + r : x \in A\}$ for some number r. Find a relation between $\sup A$ and $\sup B$.

1:6.9 Let A be a set of real numbers and let $B = \{xr : x \in A\}$ for some positive number r. Find a relation between $\sup A$ and $\sup B$. (What happens if r is negative?)

1:6.10 Let A and B be sets of real numbers such that $A \subset B$. Find a relation among $\inf A$, $\inf B$, $\sup A$ and $\sup B$.

1:6.11 Let A and B be sets of real numbers and write $C = A \cup B$. Find a relation among $\sup A$, $\sup B$ and $\sup C$.

1:6.12 Let A and B be sets of real numbers and write $C = A \cap B$. Find a relation among $\sup A$, $\sup B$ and $\sup C$.

1:6.13 Let A and B be sets of real numbers and write
$$C = \{x + y : x \in A,\ y \in B\}.$$
Find a relation among $\sup A$, $\sup B$ and $\sup C$.

1:6.14 Let A and B be sets of real numbers and write
$$C = \{x + y : x \in A,\ y \in B\}.$$
Find a relation among $\inf A$, $\inf B$ and $\inf C$.

1:6.15 Let A be a set of real numbers and write $A^2 = \{x^2 : x \in A\}$. Is there any relations you can find between the infs and sups of the two sets?

1:6.16 Let E be a set of real numbers. Show that x is not an upper bound of E if and only if there exists a number $e \in E$ such that $e > x$.

1:6.17 Let A be a set of real numbers. Show that a real number x is the supremum of A if and only if $a \leq x$ for all $a \in A$ and for every positive number ε there is an element $a' \in A$ such that $x - \varepsilon < a'$.

1:6.18 Formulate a condition analogous to the preceding exercise for an infimum.

1:6.19 Using the completeness axiom show that every nonempty set E of real numbers that is bounded below has a greatest lower bound, i.e., $\inf E$ exists and is a real number.

1:6.20 A function is said to be *bounded* if its range is a bounded set. Give examples of functions $f : \mathbb{R} \to \mathbb{R}$ that are bounded and examples of such functions that are unbounded. Give an example of one that has the property that
$$\sup\{f(x) : x \in \mathbb{R}\}$$
is finite but $\max\{f(x) : x \in \mathbb{R}\}$ does not exist.

1:6.21 The rational numbers $\mathbb{Q}$ satisfy the axioms for a an ordered field. Show that the completeness axiom would not be satisfied. That is show that this statement is false: Every nonempty set E of rational numbers that is bounded above has a least upper bound, i.e., $\sup E$ exists and is a rational number.

1:6.22 Let F be the set of all numbers of the form $x + \sqrt{2}y$ where x and y are rational numbers. Show that F has all the properties of an ordered field but does not have the completeness property.

1:6.23 Let A and B be nonempty sets of real numbers and let
$$\delta(A, B) = \inf\{|a - b| : a \in A,\ b \in B\}.$$
$\delta(A, B)$ is often called the "distance" between the sets A and B.

(a) Let $A = \mathbb{N}$ and $B = \mathbb{R} \setminus \mathbb{N}$. Compute $\delta(A, B)$

(b) If A and B are finite sets what does $\delta(A, B)$ represent?

(c) Let $A = \{x\}$ and $B = [0, 1]$. What does the statement $\delta(A, B) = 0$ mean for the point x?

(d) Let $A = \{x\}$ and $B = (0, 1)$. What does the statement $\delta(A, B) = 0$ mean for the point x?

1:6.24 The statement that every complete ordered field "is" the real number system means the following. Suppose that F is a nonempty set with operations of addition "+" and multiplication "·" and an order relation "<" that satisfies all the axioms of an ordered field and also the axiom of completeness. Then there is a one-one onto function $f : \mathbb{R} \to F$ that has the following properties: ✂

(a) $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

(b) $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in \mathbb{R}$.

(c) $f(y) < f(x)$ if and only if $x < y$ for $x, y \in \mathbb{R}$.

Thus, in a certain sense, F and $\mathbb{R}$ are essentially the same object. Attempt a proof of this statement. [Note that $x + y$ for $x, y \in \mathbb{R}$ refers to the addition in the reals whereas $f(x) + f(y)$ refers to the addition in the set F.]

1.7 The Archimedean Property

There is an important relationship holding between the set of natural numbers $\mathbb{N}$ and the larger set of real numbers $\mathbb{R}$. Because we have a well formed mental image of what the set of reals "looks like" this property is entirely intuitive and natural. It hardly seems that it would require a proof. It says that the set of natural numbers $\mathbb{N}$ has no upper bound; i.e., that there is no real number x so that $n \le x$ for all $n = 1, 2, 3, \ldots$.

At first sight this seems to be a purely algebraic and order property of the reals. In fact it cannot be proved without invoking the completeness property of Section 1.6.

Theorem 1.9 (Archimedean Property of $\mathbb{R}$) *The set of natural numbers $\mathbb{N}$ has no upper bound.*

Proof. The proof is obtained by contradiction. If the set $\mathbb{N}$ does have an upper bound, then it must have a least upper bound. Let $x = \sup \mathbb{N}$, supposing that such does exist as a finite real number. Then $n \le x$ for all integers n but $n \le x - 1$ cannot be true for all integers n. Choose some integer m with $m > x - 1$. Then $m + 1$ is also an integer and $m + 1 > x$. But that cannot be so since we defined x as the supremum. From this contradiction the theorem follows. ■

The archimedean theorem has some consequences that have a great impact on how we must think of the real numbers.

1. No matter how large a real number x is given, there is always an integer n larger.

2. Given any positive number y, no matter how large, and any positive number x, no matter how small, one can add x to itself sufficiently many times so that the result exceeds y, i.e., $nx > y$ for some integer $n \in \mathbb{N}$.

3. Given any positive number x, no matter how small, one can always find a fraction $1/n$ with n a positive integer that is smaller, i.e., so that $1/n < x$.

Each of these is a consequence of the archimedean theorem and the archimedean theorem in turn can be derived from any one of these.

Exercises

1:7.1 Using the archimedean theorem, prove each of the three statements that follow the proof of the archimedean theorem.

1:7.2 Suppose that it is true that for each $x > 0$ there is an $n \in \mathbb{N}$ so that $1/n < x$. Prove the the archimedean theorem using this assumption.

1:7.3 Without using the archimedean theorem show directly that for each $x > 0$ there is an $n \in \mathbb{N}$ so that $1/n < x$.

1:7.4 Let X be any real number. Show that there is an integer $m \in \mathbb{Z}$ so that

$$m \leq X < m + 1.$$

Show that m is unique.

1:7.5 The mathematician Liebnitz based his calculus on the assumption that there were "infinitesimals", positive real numbers that are extremely small—smaller than all positive rational numbers certainly. Some calculus students also believe, apparently, in the existence of such numbers since they can imagine a number that is "just next to zero". Is there a positive real number smaller than all positive rational numbers?

1:7.6 The Archimedean property asserts that if $x > 0$ then there is an integer N so that $1/N < x$. The proof requires the completeness axiom. Give a proof that does not use the completeness axiom that works for x rational. Find a proof that is valid for $x = \sqrt{y}$ where y is rational.

✂ **1:7.7** In Section 1.2 we made much of the fact that there is a number whose square is 2 and so $\sqrt{2}$ does exist as a real number. Show that

$$\alpha = \sup\{x \in \mathbb{R} : x^2 < 2\}$$

exists as a real number and that $\alpha^2 = 2$.

1.8 Inductive Property of $\mathbb{N}$

Since the natural numbers are included in the set of real numbers there are further important properties of $\mathbb{N}$ that can be deduced from the axioms. The most important of these is the principle of induction. This is the basis for the technique of proof known as induction which is often used in this text. For an elementary account and some practice the reader should look at Section A.8 in the appendix.

We first prove a statement that is equivalent.

Theorem 1.10 (Well-Ordering Property) *Every nonempty subset of $I\!N$ has a smallest element.*

Proof. Let $S \subset \mathbb{N}$ and $S \neq \emptyset$. Then $\alpha = \inf S$ must exist and be a real number since S is bounded below. If $\alpha \in S$ then we are done since we have found a minimal element.

Suppose not. Then, while α is the greatest lower bound of S, α is not a minimum. There must be an element of S which is smaller than $\alpha + 1$ since α is the greatest lower bound of S. That element cannot be α since we have assumed that $\alpha \notin S$. Thus we have found $x \in S$ with $\alpha < x < \alpha + 1$. Now x is not a lower bound of S, since it is greater than the greatest lower bound of S so there must be yet another element y of S such that

$$\alpha < y < x < \alpha + 1.$$

But now we have reached an impossibility, for x and y are in S and both integers, but $0 < x - y < 1$ which cannot happen for integers. From this contradiction the proof now follows. ■

Now we can state and prove the principle of induction.

Theorem 1.11 (Principle of Induction) *Let $S \subset I\!N$ so that $1 \in S$ and for every integer n if $n \in S$ then so also is $n+1$. Then $S = I\!N$.*

Proof. Let $E = \mathbb{N} \setminus S$. We claim that $E = \emptyset$ and then it follows that $S = \mathbb{N}$ proving the theorem. Suppose not, i.e., suppose $E \neq \emptyset$. By Theorem 1.10 there is a first element α of E. Can $\alpha = 1$? No, because $1 \in S$ by hypothesis. Thus $\alpha - 1$ is also an integer and, since it cannot be in E it must be in S. By hypothesis it follows that $\alpha = (\alpha - 1) + 1$ must be in S. But it is in E. This is impossible and so we have obtained a contradiction, proving our theorem. ■

Exercises

1:8.1 Show that any bounded nonempty set of natural numbers has a maximal element.

1:8.2 Show that any bounded nonempty subset of $\mathbb{Z}$ has a maximum and a minimum.

1:8.3 For further exercises on proving statements using induction as a method see Section A.8.

✂ **1:8.4** We have assumed in the text that the set $\mathbb{N}$ is obviously contained in $\mathbb{R}$. After all 1 is a real number (it's in the axioms), 2 is just $1 + 1$ and so real, 3 is $2 + 1$ etc.. In that way we have been able to prove the material of this section. But there is a logical flaw here. We would need induction really to define $\mathbb{N}$ in this way (and not just say "etc."). Here is a set of exercises that would remedy that for students with some background in set manipulations.

(a) Define a set $S \subset \mathbb{R}$ to be *inductive* if $1 \in S$ and $x \in S$ implies that $x + 1 \in S$. Show that $\mathbb{R}$ is inductive.

(b) Show that there is a smallest inductive set, i.e., show that the intersection of the family of all inductive sets is itself inductive.

(c) Define $\mathbb{N}$ to be that smallest inductive set.

(d) Prove Theorem 1.11 now. (That is show that any set S with the property stated there is inductive and conclude that $S = \mathbb{N}$.)

(e) Prove Theorem 1.10 now. (That is with this definition of $\mathbb{N}$ prove the well ordering property.)

1.9 The Rational Numbers are Dense

There is an important relationship holding between the set of rational numbers $\mathbb{Q}$ and the larger set of real numbers $\mathbb{R}$. The rational numbers are dense. They make an appearance in every interval; there are no gaps, no intervals that miss having rational numbers.

For practical purposes this has great consequences. We need never actually compute with arbitrary real numbers, since close by are rational numbers that can be used. Thus, while π is irrational, in routine computations with a practical view any nearby fraction might do. At various times people have used 3, 22/7, and 3.14159, for example.

For theoretical reasons this fact is of great importance too. It allows many arguments to replace a consideration of the set of real numbers with the smaller set of rationals. Since every real is as close as we please to a rational and since the rationals can be carefully described and easily worked with, many simplifications are allowed.

Definition 1.12 (Dense Sets) A set E of real numbers is said to be *dense* (or *dense in* $\mathbb{R}$) if every interval (a, b) contains a point of E.

Theorem 1.13 *The set* $\mathbb{Q}$ *of rational numbers is dense.*

Proof. Let $x < y$ and consider the interval (x, y). We must find a rational number inside this interval.

By the archimedean theorem, Theorem 1.9, there is a positive integer

$$n > \frac{1}{y-x}.$$

This means that $ny > nx + 1$.

Let m be chosen as the integer just less than $nx+1$; more precisely (using Exercise 1:7.4), find an integer $m \in \mathbb{Z}$ so that

$$m \leq nx + 1 < m + 1.$$

Now some arithmetic on these inequalities shows that

$$m - 1 \leq nx < ny$$

and then

$$x < \frac{m}{n} \leq x + \frac{1}{n} < y$$

thus exhibiting a rational number m/n in the interval (x, y). ■

Exercises

1:9.1 Show that the definition of "dense" could be given as

> A set E of real numbers is said to be *dense* if every interval (a, b) contains infinitely many points of E.

1:9.2 Find a rational number between $\sqrt{10}$ and π.

1:9.3 If a set E is dense what can you conclude about a set $A \supset E$?

1:9.4 If a set E is dense what can you conclude about the set $\mathbb{R} \setminus E$?

1:9.5 If two sets E_1 and E_2 are dense what can you conclude about the set $E_1 \cap E_2$?

1:9.6 Show that the dyadic rationals, i.e., rational numbers of the form $m/2^n$ for $m \in \mathbb{Z}$, $n \in \mathbb{N}$ are dense.

1:9.7 Are the numbers of the form $\pm m/2^{100}$ for $m \in \mathbb{N}$ dense? What is the length of the smallest interval that contains no such number?

1:9.8 Show that the numbers of the form $\pm m\sqrt{2}/n$ for m, $n \in \mathbb{N}$ are dense.

✂ **1:9.9** Use this definition of "dense in a set" to answer the following questions:

> A set E of real numbers is said to be *dense in a set* A if every interval (a, b) that contains a point of A also contains a point of E.

(a) Show that dense in the set of all reals is the same as dense.

(b) Give an example of a set E dense in $\mathbb{N}$ but with $E \cap \mathbb{N} = \emptyset$.

(c) Show that the irrationals are dense in the rationals. (A real number is irrational if it is not rational, i.e. if it belongs to $\mathbb{R}$ but not to $\mathbb{Q}$.)

(d) Show that the rationals are dense in the irrationals.

(e) What property does a set E have that is equivalent to the assertion that $\mathbb{R} \setminus E$ is dense in E?

1.10 The Metric Structure of $\mathbb{R}$

In addition to the algebraic and order structure of the real numbers we need, too, to make measurements. We need to describe distances between points. These are the metric properties of the reals, to borrow a term from the Greek for measure (*metron*).

As usual the distance between a point x and another point y is either $x - y$ or $y - x$ depending on which is positive. Thus the distance between 3 and -4 is 7. The distance between π and $\sqrt{10}$ is $\sqrt{10} - \pi$. To describe this in general requires the absolute value function which simply makes a choice between positive and negative.

Definition 1.14 (Absolute Value) For any real number x write

$$|x| = x \quad \text{if } x \geq 0$$

and

$$|x| = -x \quad \text{if } x < 0 \; .$$

(Beginners tend to think of the absolute value function as "stripping off the negative sign" but the example $|\pi - \sqrt{10}| = \sqrt{10} - \pi$ shows that this is a limited viewpoint.)

Properties of the Absolute Value Since the absolute value is defined directly in terms of inequalities (i.e., the choice $x \geq 0$ or $x < 0$) there are a number of properties that can be proved directly from properties of inequalities. These properties are used routinely and the student will need to have a complete mastery of them.

Theorem 1.15 *The absolute value function has the following properties:*

1. *For any $x \in \mathbb{R}$, $-|x| \leq x \leq |x|$.*
2. *For any x, $y \in \mathbb{R}$, $|xy| = |x|\,|y|$.*
3. *For any x, $y \in \mathbb{R}$, $|x+y| \leq |x| + |y|$.*
4. *For any x, $y \in \mathbb{R}$, $|x| - |y| \leq |x-y|$ and $|y| - |x| \leq |x-y|$.*

Distances on the Real Line Using the absolute value function we can define the distance function or metric.

Definition 1.16 (Distance) The distance between two real numbers x and y is

$$d(x,y) = |x-y|.$$

We hardly ever use the notation $d(x,y)$ in elementary analysis, preferring to write $|x-y|$ even while we are thinking of this as the distance between the two points. Thus if a sequence of points x_1, x_2, x_3, ... is growing ever closer to a point c we should perhaps describe $d(x_n, c)$ as getting smaller and smaller thus emphasizing that the distances are shrinking; more often we would simply write $|x_n - c|$ and expect the reader to interpret this as a distance.

Properties of the Distance Function The main properties of the distance function are just interpretations of the absolute value function. Expressed in the language of a distance function they are geometrically very intuitive:

1. $d(x,y) \geq 0$ [all distances are positive or zero].
2. $d(x,y) = 0$ if and only if $x = y$ [different points are at positive distance apart].
3. $d(x,y) = d(y,x)$ [distance is symmetric, i.e., the distance from x to y is the same as from y to x].

4. $d(x, y) \leq d(x, z) + d(z, y)$ [the triangle inequality, i.e, it is no longer to go directly from x to y than to go from x to z and then to y.]

Later on in Chapter **??** we will study general structures called metric spaces where exactly such a notion of distance satisfying these four properties is used. For now we prefer to rewrite these properties in the language of the absolute value where they lose some of their intuitive appeal. But it is in this form that we are likely to use them.

1. $|a| \geq 0$.
2. $|a| = 0$ if and only if $a = 0$.
3. $|a| = |-a|$.
4. $|a + b| \leq |a| + |b|$. [the triangle inequality].

Exercises

1:10.1 Show that $|x| = \max\{x, -x\}$.

1:10.2 Show that $\max\{x, y\} = |x + y|/2 + |x - y|/2$. What expression would give $\min\{x, y\}$?

1:10.3 Show that the inequalities

$$|x - a| < \varepsilon$$

and

$$a - \varepsilon < x < a + \varepsilon$$

are equivalent.

1:10.4 Show that if $\alpha < x < \beta$ and $\alpha < y < \beta$ then $|x - y| < \beta - \alpha$ and interpret this geometrically as a statement about the interval (α, β).

1:10.5 Show that

$$||x| - |y|| \leq |x - y|$$

assuming the triangle inequality, i.e., that $|a + b| \leq |a| + |b|$. This inequality is also called the triangle inequality.

1:10.6 Under what conditions is it true that

$$|x + y| = |x| + |y|?$$

1:10.7 Under what conditions is it true that

$$|x - y| + |y - z| = |x - z|?$$

1:10.8 Show that

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

for any numbers $x_1, x_2, \ldots, x_n$.

1:10.9 Let E be a a set of real numbers and let $A = \{|x| : x \in E\}$. What relations can you find between the infs and sups of the two sets?

1:10.10 Find the inf and sup of the set $\{x : |2x + \pi| < \sqrt{2}\}$.