Zeros of truncated taylor series approximations to the symmetric Zeta function

Greg Fee

Riemann's symmetric Zeta function

Denote Riemann's symmetric Zeta function by $f(s)$, defined by

$$f(s) = \frac{1}{2}\, s\,(s-1)\,\text{Pi}^{-\frac{1}{2}s}\,\text{GAMMA}\left(\frac{1}{2}\,s\right)\text{Zeta}(s) \qquad (1.1)$$

where $\Gamma(s) = \int_0^\infty t^{s-1}\exp(-t)\,dt$, is the Gamma function

and $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, is the Riemann Zeta function.

$\zeta(s)$ only converges for $\Re e(s) > 1$.

Use the Euler-Maclaurin summation formula to analytically continue the above sum.

$f(s)$ is symmetric about $s = \frac{1}{2}$, that is $f\left(\frac{1}{2} + s\right) = f\left(\frac{1}{2} - s\right)$.

$f(s)$ is real valued on the real line so it is conjugate symmetric in the complex plane.

The trivial zeros of the Zeta function at negative even integers, and the pole at 1 are removed by using the symmetric Zeta function.

Taylor series attempts

First try to compute the exact truncated taylor series of $f(s)$ about $s = \frac{1}{2}$.

This was too slow.

Next try to compute the taylor series using floating point coefficients, but this was also too slow.

Next try to use polynomial interpolation at the even integers, but this did not converge.

Taylor series

Use the inverse **FFT** (**F**ast **F**ourier **T**ransform) to compute the taylor series.

Evaluate $f(s)$ at 4096 equally spaced points around a circle centered at $s = 0.5$ with $radius = 0.25$.

Use 6144 significant decimal digit arithmetic.

From the conjugate and even symmetry, only need to compute 1025 values.

Use 16 computers computing 64 values each. All computers finish in one overnight computation using 15 hours each.

Root finding

Use Madsen's method to find the roots of a polynomial.

The roots along the line real part = 1/2, correspond to roots of the Zeta function.

All the other roots are extraneous roots.

Curve fitting

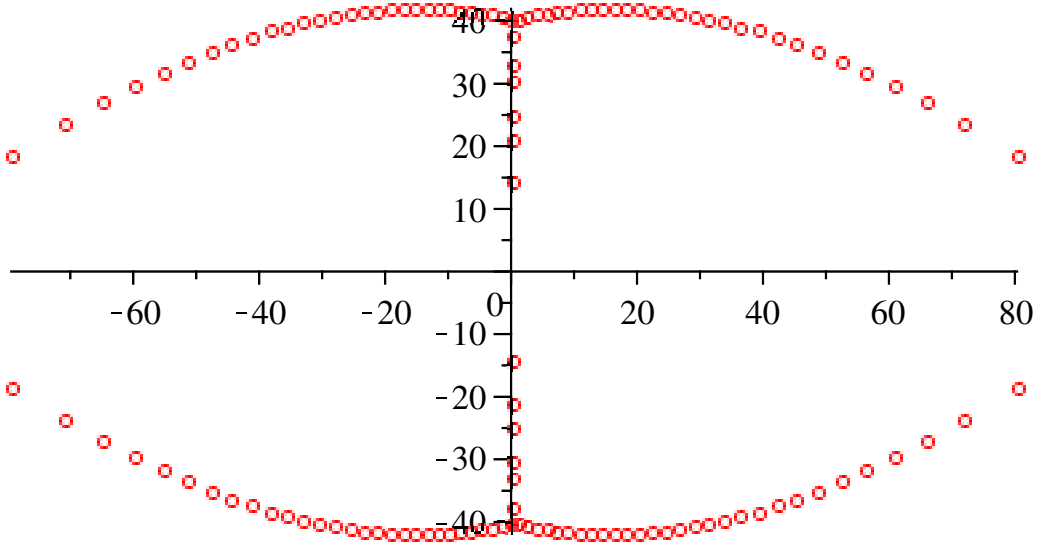
The extraneous zeros, those not on the line $\Re e(s) = 0.5$, seem to lie on conics.

Use the **SVD** (**S**ingular **V**alue **D**ecomposition) to compute a total least-squares conic (total degree 2 curve) which best fits the zeros with $\Re e(s) > 0.5$.

Taylor polynomial degree=128

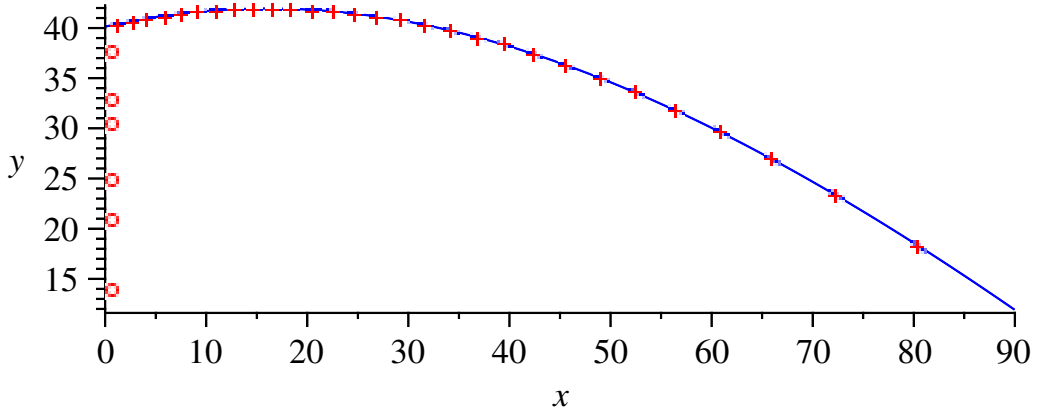
The zeros (circles) of a Taylor polynomial.

red zeros of the Taylor poly. to the symmetric Zeta function



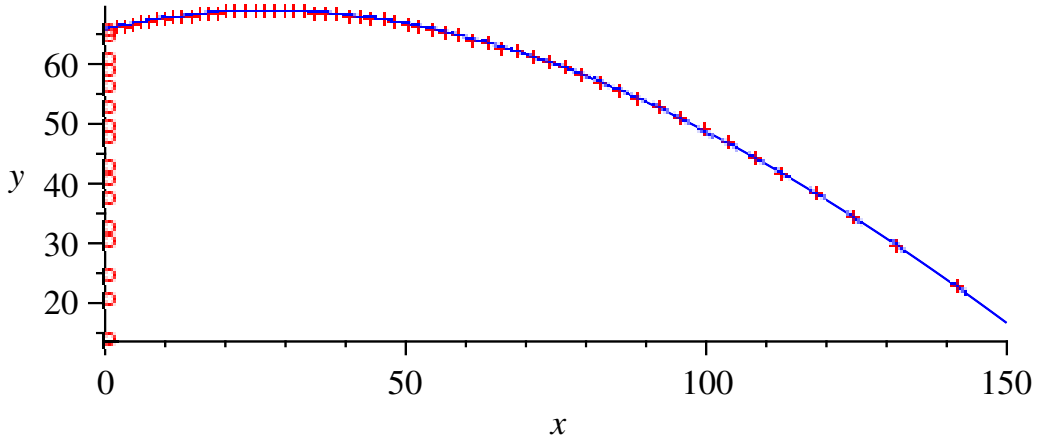
Since there are 2 symmetries, just view the first quadrant.
Fit a conic (hyperbola) to the extraneous zeros.

Zeta zeros (circles), other Taylor zeros (crosses)



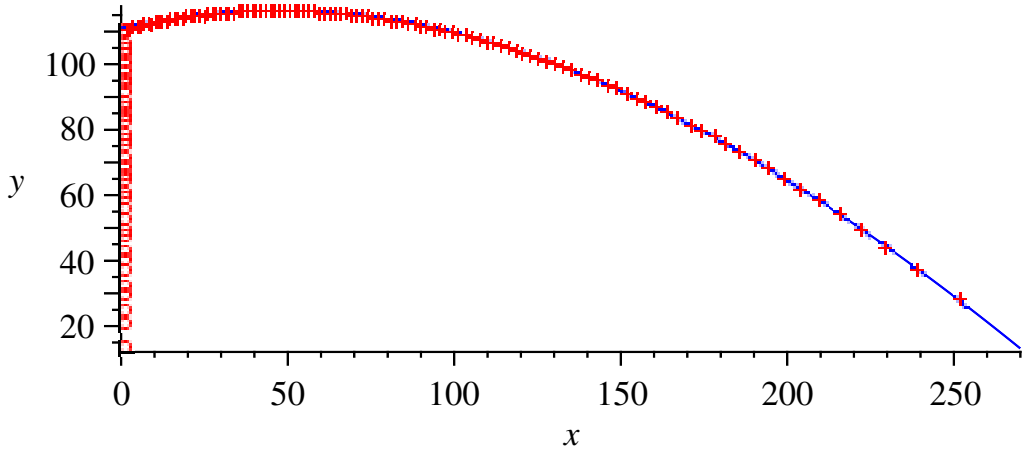
Taylor polynomial degree=256

Zeta zeros (circles), other Taylor zeros (crosses)



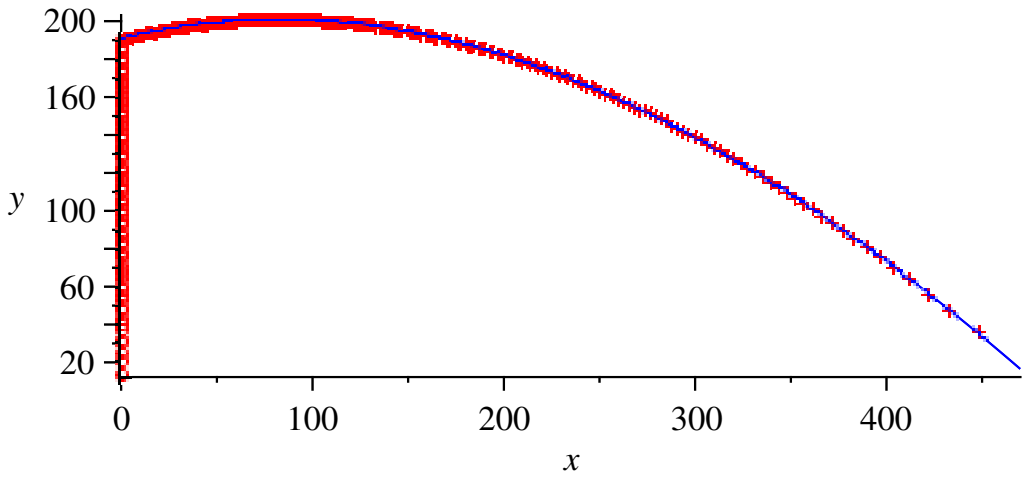
Taylor polynomial degree=512

Zeta zeros (circles), other Taylor zeros (crosses)



Taylor polynomial degree=1024

Zeta zeros (circles), other Taylor zeros (crosses)



References

Keith. O. Geddes, Stephen R. Czapor, George Labahn
Algorithms for Computer Algebra,
Kluwer Academic Publishers,1992.

Kaj Madsen
A root-finding algorithm based on Newton's method,
B.I.T. Numerical Mathematics, Vol. 13, Number 1, Mar. 1973
pages 71-75, Springer Netherlands,

M. Abramowitz and I. Stegun
Handbook of Mathematical Functions,
National Bureau of Standards,
Applied Mathematics Series - 55, Tenth printing, Dec. 1972.

Yves Nievergelt
Total Least Squares: State-Of-The-Art Regression in Numerical Analysis,
SIAM Review, Vol. 36, No. 2, pages 258-264, June 1994.

Peter Borwein
An Efficient Algorithm for the Riemann Zeta Function,
preprint, Jan. 1995